Form of parity-covariant relativistic two-particle forces

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For two-particle Newtonian equations of motion that are parity-invariant as well as Poincaré-invariant, the accelerations are in the plane of the relative position and relative velocity. This simplifies the Currie-Hill invariance conditions, which then show that in the relativistic forces (the proper-time derivatives of the relativistic momenta) the scalar coefficients depend on the momenta of the two particles only through their Lorentz-invariant dot product and their dot products with the relative position. These properties, which were discovered in the manifestly invariant formalism, are shown here to follow also very simply from the Newtonian formulation.

Consider a classical-mechanical system of two particles ($n=1,2$) described by positions $\vec{x}_n$, velocities $\vec{v}_n = d\vec{x}_n/dt$, and Newtonian equations of motion that give the accelerations as functions of the positions and velocities at one time,

$$d\vec{v}_n/dt = \vec{f}_n(\vec{x}, \vec{v}_1, \vec{v}_2) .$$

We make these equations invariant for time translations by not letting $\vec{f}_n$ depend explicitly on time, for space translations by letting $\vec{f}_n$ depend on the positions only through the relative position $\vec{x} = \vec{x}_1 - \vec{x}_2$, for rotations and parity by letting $\vec{f}_n$ be a vector function of $\vec{x}, \vec{v}_1, \vec{v}_2$ (that rotates as a vector when $\vec{x}, \vec{v}_1, \vec{v}_2$ rotate and changes sign when $\vec{x}, \vec{v}_1, \vec{v}_2$ change sign). Our real interest is in Poincaré-invariant equations of motion, but let us consider Galilei invariance first.

Galilei invariance implies that $\vec{f}_n$ depends on the velocities only through the relative velocity $\vec{v} = \vec{v}_1 - \vec{v}_2$. Then we have

$$\vec{f}_n = a_n \vec{x} + b_n \vec{v} , \qquad (1)$$

with a_n, b_n functions of the Galilei-invariant scalars $\vec{x}^2$, $\vec{v}^2$, $\vec{x}\cdot\vec{v}$.

For Lorentz-invariant equations of motion, $\vec{f}_n$ depends on the velocities in a complicated way to satisfy the Currie-Hill invariance conditions.[1-5] Nevertheless, for equations of motion that are parity-invariant as well as Poincaré-invariant, $\vec{f}_n$ is of the form (1); the accelerations are in the plane of the relative position and relative velocity. This simplifies the Currie-Hill invariance conditions, which then show that in the relativistic forces (the proper-time derivatives of the relativistic momenta) the scalar coefficients (to which a_n, b_n are simply related) depend on the two relativistic momenta only through their Lorentz-invariant dot product and their dot products with the relative position.

These properties were discovered in the manifestly invariant formalism of predictive relativistic mechanics.[6-10] Here it is shown that they also follow very simply from the Newtonian formulation. This is a correction to my earlier misunderstanding of one of the equations of the manifestly invariant formalism.[11] It completes a direct proof that the two formalisms are equivalent in this case.[12]

Let us first consider the implications of parity invariance. There are certainly reference frames for which $\vec{x}, \vec{v}_1, \vec{v}_2$ are all in a plane at some time (for example, a frame in which one particle has zero velocity at some time). Then the accelerations $\vec{f}_n$ also are in that plane, because they are vector functions of $\vec{x}, \vec{v}_1, \vec{v}_2$ for rotations and parity. From this and the equations of motion it follows that $\vec{x}, \vec{v}_1, \vec{v}_2, \vec{f}_1, \vec{f}_2$ are in that plane for all time. This means that the particle paths lie in a fixed plane in space, for this reference frame.

The other reference frames are obtained from the Poincaré group of transformations. Each transformation in the Poincaré group is a product of space and time translations followed by a rotation followed by a Lorentz transformation.[13] For a space or time translation, or a rotation, a fixed plane is transformed into a fixed plane. For a Lorentz transformation, a fixed plane is transformed into a plane moving with constant velocity; if we consider (without loss of generality) the Lorentz transformation to a frame moving with velocity β in the x direction, and the fixed plane

$$y = ax + b ,$$

we get

$$y' = y = a(x' + \beta t')(1-\beta^2)^{-1/2} + b$$
$$= a(1-\beta^2)^{-1/2}x' + b + a\beta(1-\beta^2)^{-1/2}t' .$$

(We use units such that $c=1$ throughout the paper.)

Thus, for any reference frame, the particle paths lie in a plane moving with constant velocity. This means the accelerations are in the plane of the relative position and relative velocity; the $\vec{f}_n$ are of the form (1) with a_n, b_n scalars for rotation and parity.

An algebraic statement of this is that $\vec{x}\times\vec{v}\cdot\vec{f}_n$ is zero. This is certainly true for a frame where $\vec{x}, \vec{v}_1, \vec{v}_2, \vec{f}_1, \vec{f}_2$ are all in a fixed plane. To show that it is true for every frame, we just need to show that it is not changed by a Lorentz transformation from a frame where $\vec{x}, \vec{v}_1, \vec{v}_2, \vec{f}_1, \vec{f}_2$ are in a fixed plane. For a Lorentz transformation to a frame moving with velocity $\vec{\beta}$ we have[5]

$$\vec{x}_n' = \vec{x}_n - [1-(1-\beta^2)^{1/2}]\beta^{-2}(\vec{x}_n\cdot\vec{\beta})\vec{\beta},$$

$$(1-\vec{v}_n\cdot\vec{\beta})\vec{v}_n' = (1-\beta^2)^{1/2}\vec{v}_n + [1-(1-\beta^2)^{1/2}]\beta^{-2}(\vec{v}_n\cdot\vec{\beta})\vec{\beta} - \vec{\beta},$$

$$(1-\beta^2)^{-1}(1-\vec{v}_n\cdot\vec{\beta})^3\vec{f}_n' = (1-\vec{v}_n\cdot\vec{\beta})\vec{f}_n + (\vec{f}_n\cdot\vec{\beta})\vec{v}_n - [1-(1-\beta^2)^{1/2}]\beta^{-2}(\vec{f}_n\cdot\vec{\beta})\vec{\beta},$$

where for $t'=0$ we take $\vec{x}_n, \vec{v}_n, \vec{f}_n$ at $t=\vec{x}_n(t)\cdot\vec{\beta}$. Since $\vec{x}, \vec{v}_1, \vec{v}_2, \vec{f}_1, \vec{f}_2$ are in a fixed plane for all time, which implies that $\vec{x}_1(t=\vec{x}_1(t)\cdot\vec{\beta}) - \vec{x}_2(t=\vec{x}_2(t)\cdot\vec{\beta})$ also is in that plane, it is straightforward to calculate $\vec{x}'\times\vec{v}'\cdot\vec{f}_n'$ and see that is zero.

Let

$$\vec{u}_n = m_n d\vec{x}_n/d\tau_n = m_n\vec{v}_n(1-\vec{v}_n^{\,2})^{-1/2},$$

$$\vec{F}_n = d\vec{u}_n/d\tau_n = m_n^{-1}(\vec{u}_n^{\,2}+m_n^2)^{1/2}d\vec{u}_n/dt.$$

In terms of these, the Currie-Hill invariance conditions are[14]

$$(x_{nj}-x_{n'j})m_{n'}(\vec{u}_{n'}^{\,2}+m_{n'}^2)^{-1/2}\sum_{l=1}^{3}F_{n'l}\partial F_{nk}/\partial u_{n'l} + (x_{nj}-x_{n'j})(\vec{u}_{n'}^{\,2}+m_{n'}^2)^{-1/2}\sum_{l=1}^{3}u_{n'l}\partial F_{nk}/\partial x_{n'l} + (\vec{u}_n^{\,2}+m_n^2)^{1/2}\partial F_{nk}/\partial u_{nj} + (\vec{u}_{n'}^{\,2}+m_{n'}^2)^{1/2}\partial F_{nk}/\partial u_{n'j} - \delta_{jk}(\vec{u}_n^{\,2}+m_n^2)^{-1/2}(\vec{u}_n\cdot\vec{F}_n) = 0, \quad (2)$$

for $j,k=1,2,3$ with $n'=2,1$ for $n=1,2$. Let

$$\vec{F}_n = A_n(\vec{x}_n-\vec{x}_{n'}) + B_n\vec{u}_n + C_n\vec{u}_{n'}.$$

In the invariance conditions (2), the tensor δ_{jk} is multiplied by

$$(\vec{u}_n^{\,2}+m_n^2)^{1/2}B_n + (\vec{u}_{n'}^{\,2}+m_{n'}^2)^{1/2}C_n - (\vec{u}_n^{\,2}+m_n^2)^{-1/2}(\vec{u}_n\cdot\vec{F}_n).$$

This is zero for accelerations of the form (1) because

$$\vec{f}_n = m_n^2(\vec{u}_n^{\,2}+m_n^2)^{-1}[\vec{F}_n - (\vec{u}_n^{\,2}+m_n^2)^{-1}(\vec{u}_n\cdot\vec{F}_n)\vec{u}_n]$$
$$= m_n^2(\vec{u}_n^{\,2}+m_n^2)^{-1}[A_n(\vec{x}_n-\vec{x}_{n'}) + B_n(\vec{u}_n^{\,2}+m_n^2)^{1/2}\vec{v}_n + C_n(\vec{u}_{n'}^{\,2}+m_{n'}^2)^{1/2}\vec{v}_{n'} - (\vec{u}_n\cdot\vec{F}_n)(\vec{u}_n^{\,2}+m_n^2)^{-1/2}\vec{v}_n].$$

In other words, we have

$$m_n^2 B_n = [(\vec{x}_n-\vec{x}_{n'})\cdot\vec{u}_n]A_n + [\vec{u}_n\cdot\vec{u}_{n'} - (\vec{u}_n^{\,2}+m_n^2)^{1/2}(\vec{u}_{n'}^{\,2}+m_{n'}^2)^{1/2}]C_n$$

for two-particle equations of motion that are parity-invariant as well as Poincaré-invariant.

Then the other terms of the invariance conditions (2), proportional to the nine tensors $u_{nj}(x_{nk}-x_{n'k})$, $u_{nj}u_{nk}$, $u_{nj}u_{n'k}$, $u_{n'j}(x_{nk}-x_{n'k})$, $u_{n'j}u_{nk}$, $u_{n'j}u_{n'k}$, $(x_{nj}-x_{n'j})u_{nk}$, $(x_{nj}-x_{n'j})u_{n'k}$, $(x_{nj}-x_{n'j})(x_{nk}-x_{n'k})$ are separately zero. The first six of these simply tell us we get zero when we operate on A_n, B_n, or C_n with D_n or $D_{n'}$, where

$$D_n = 2(\vec{u}_n^{\,2}+m_n^2)^{1/2}\frac{\partial}{\partial\vec{u}_n^{\,2}} + (\vec{u}_{n'}^{\,2}+m_{n'}^2)^{1/2}\frac{\partial}{\partial\vec{u}_1\cdot\vec{u}_2}.$$

If we make a change of variables from $\vec{u}_1^{\,2}$, $\vec{u}_2^{\,2}$, $\vec{u}_1\cdot\vec{u}_2$ to $\beta_1=\vec{u}_1^{\,2}$, $\beta_2=\vec{u}_2^{\,2}$, and

$$\gamma = \vec{u}_1\cdot\vec{u}_2 - (\vec{u}_1^{\,2}+m_1^2)^{1/2}(\vec{u}_2^{\,2}+m_2^2)^{1/2}, \quad (3)$$

we find that

$$\frac{\partial}{\partial\beta_n} = \frac{\partial}{\partial\vec{u}_n^{\,2}} + \tfrac{1}{2}(\vec{u}_n^{\,2}+m_n^2)^{-1/2}(\vec{u}_{n'}^{\,2}+m_{n'}^2)^{1/2}\frac{\partial}{\partial\vec{u}_1\cdot\vec{u}_2},$$

$$D_n = 2(\vec{u}_n^{\,2}+m_n^2)^{1/2}\frac{\partial}{\partial\beta_n},$$

so A_n, B_n, C_n are independent of β_1 and β_2. In other words, for two-particle equations of motion that are parity-invariant as well as Poincaré-invariant, A_n, B_n, C_n depend on $\vec{u}_1^{\,2}$, $\vec{u}_2^{\,2}$, $\vec{u}_1\cdot\vec{u}_2$ only through the Lorentz-invariant dot product (3).

Of course A_n, B_n, C_n are also functions of $\vec{x}^2$, $(\vec{x}_n-\vec{x}_{n'})\cdot\vec{u}_n$, $(\vec{x}_n-\vec{x}_{n'})\cdot\vec{u}_{n'}$. The remaining terms of the invariance conditions (2) are

$$(\vec{u}_{n'}^{\,2}+m_{n'}^{\,2})^{-1/2}\left[m_{n'}\sum_{l=1}^{3} F_{n'l}\partial F_{nk}/\partial u_{n'l}+\sum_{l=1}^{3} u_{n'l}\partial F_{nk}/\partial x_{n'l}\right]$$

$$+(\vec{u}_n^{\,2}+m_n^{\,2})^{1/2}[(x_{nk}-x_{n'k})\partial A_n/\partial(\vec{x}_n-\vec{x}_{n'})\circ\vec{u}_n+u_{nk}\partial B_n/\partial(\vec{x}_n-\vec{x}_{n'})\cdot\vec{u}_n+u_{n'k}\partial C_n/\partial(\vec{x}_n-\vec{x}_{n'})\cdot\vec{u}_n]$$

$$+(\vec{u}_{n'}^{\,2}+m_{n'}^{\,2})^{1/2}[(x_{nk}-x_{n'k})\partial A_n/\partial(\vec{x}_n-\vec{x}_{n'})\circ\vec{u}_{n'}+u_{nk}\partial B_n/\partial(\vec{x}_n-\vec{x}_{n'})\cdot\vec{u}_{n'}+u_{n'k}\partial C_n/\partial(\vec{x}_n-\vec{x}_{n'})\circ\vec{u}_{n'}]=0\,,$$

for $k=1,2,3$ with $n'=2,1$ for $n=1,2$. An overall factor of $(x_{nj}-x_{n'j})$ has been dropped.

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[1]D. G. Currie, Phys. Rev. 142, 817 (1966).

[2]R. N. Hill, J. Math. Phys. 8, 201 (1967).

[3]D. G. Currie and T. F. Jordan, in *Lectures in Theoretical Physics*, edited by A. O. Barut and W. E. Brittin (Gordon and Breach, New York, 1968), Vol. XA, p. 91.

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[5]T. F. Jordan, Phys. Rev. D 11, 2807 (1975); 11, 3035 (1975).

[6]L. Bel, A. Salas, and J. M. Sánchez-Ron, Phys. Rev. D 7, 1099 (1973).

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[9]J. M. Sánchez-Ron, J. Phys. A 9, 1877 (1976).

[10]R. Lapiedra and L. Mas, Phys. Rev. D 13, 2805 (1976).

[11]T. F. Jordan, Phys. Rev. D 16, 313 (1977). Equation (13) of that paper does not follow from the manifestly invariant formalism; it does not follow from Eqs. (6) and (10). The reason is that the ξ_n of the manifestly invariant formalism depend on the relative time $x_0=x_{10}-x_{20}$ and there is a derivative with respect to x_0 in Eq (6) which is not included in Eq. (13). I regret any difficulties this misunderstanding may have caused.

[12]This paper provides a complete proof for Eq. (19) of X. F. Fustero and R. Lapiedra, preceding paper, Phys. Rev. D 17, 2821 (1978). It is not sufficient to prove it for a frame in which one of the velocities is zero and then say, as they do, that form invariance establishes it for all frames. Form invariance means the acceleration is the same function of the positions and velocities in all frames, but their argument only establishes the form of that function when one of the velocities is zero, not $\vec{f}_n(\vec{x},\vec{v}_1,\vec{v}_2)$ but only $\vec{f}_n(\vec{x},\vec{v}_1=0,\vec{v}_2)$. I am grateful to Fustero and Lapiedra for sending me a copy of their paper.

[13]F. R. Halpern, *Special Relativity and Quantum Mechanics* (Prentice-Hall, Englewood Cliffs, New Jersey, 1968), p. 12, gives a proof that every transformation in the homogeneous Lorentz group is a product of a rotation followed by a pure Lorentz transformation. It is easy to see that any transformation in the Poincaré group is a product of a space-time translation followed by a transformation in the homogeneous Lorentz group; if $x'=\Lambda x+a$, then also $x'=\Lambda(x+\Lambda^{-1}a)$.

[14]D. G. Currie and T. F. Jordan, Phys. Rev. 167, 1178 (1968) give the invariance conditions for $d\vec{u}_n/dt$.