Adler-Weisberger relation and the quark model*

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We domonstrate consistencies between two seemingly different methods of calculating the axial-vector coupling constant g_A.

The axial-vector coupling constant g_A is defined by

$$\langle p(p')|A_\mu(0)|n(p)\rangle = \bar{u}(p')(g_A\gamma_\mu\gamma_5 + \cdots)u(p) \tag{1}$$

and experimentally has the value $g_A = 1.250 \pm 0.009$.[1] This value can be understood theoretically via the Adler-Weisberger relation[2] which uses the partial conservation of the axial-vector current hypothesis (PCAC) to obtain g_A in terms of the π^+p and π^-p total cross sections

$$1 - \frac{1}{g_A{}^2} = \frac{4}{\pi}\frac{m_N{}^2}{g_{\pi NN}{}^2}\int_{m_N+m_\pi} dW \frac{W}{W^2 - m_N{}^2} \times [\sigma_{\pi^+p}(W) - \sigma_{\pi^-p}(W)], \tag{2}$$

where $g_{\pi NN}{}^2/4\pi = 14.6$. Use of the experimental cross sections yields $g_A = 1.16$, while attempts to correct for nonzero mass pions yields $g_A = 1.24$.[2]

Another way in which g_A may be calculated is through the quark model where the axial-vector current is written in terms of quark fields

$$A_\mu(x) = \bar{u}(x)\gamma_\mu\gamma_5 d(x). \tag{3}$$

Taking the matrix elements of this between nucleon states yields

$$g_A = \tfrac{5}{3}(1 - \tfrac{4}{3}L), \tag{4}$$

where L is related, in a way that will be made precise later, to the lower components in the four-component Dirac wave function for the bound quarks.[3] The precise value of g_A is, of course, model dependent, but published values generally lie between $g_A = 1.1$ and $g_A = 1.3$.

Both of these techniques are appealing ways of understanding the axial-vector coupling. However, at first sight there appears to be no connection between them. It is the purpose of this paper to describe a manner in which they are related. We do this by first showing that in a particular limit, that of nonrelativistic SU(6), both methods yield the value $g_A = \frac{5}{3}$. We then show how the physics of quark binding produces deviations from this limit in both models. Additional deviations due to interactions which provide mass splittings within SU(6) multiplets are not included in Eq. (4), and we discuss these effects within the quark-model framework.

It is clear that Eq.(4) is compatible with the SU(6) value, as the lower components vanish in such a theory. To study Eq. (2) we must be more definite about the nature of this limit. The main contribution to the Adler-Weisberger relation comes from the resonance region where the particle states that couple to $\pi^\pm p$ are the nucleon and Δ and their excited states, the N^* and Δ resonances families. The various members of these families are generally spaced several hundred MeV apart. This is clearly not a nonrelativistic situation since the binding energy and the energy splittings between radial and orbital excitations are not small compared to the ground-state energy. Another way to see that the real world does include relativistic effects is to consider the momentum of a bound quark via the uncertainty principle

$$p = (p_x{}^2 + p_y{}^2 + p_z{}^2)^{1/2} \approx \sqrt{3}/R,$$

where R is a measure of the confinement volume. For $R \approx 1$–1.5 fm, $p \approx 250$–350 MeV for each quark. Relativistic effects will be significant unless the confinement radius is much larger than the Compton wavelength of a quark. The nonrelativistic limit then implies weak binding and a spectrum of excited states with $m_{N^*} \approx m_N$. We also require that SU(6) be a good symmetry in this limit (e.g., $m_\Delta = m_N$).

We must investigate πN scattering in this limit. We write the quark-pion interaction as

$$\mathcal{H}_{\text{int}}(x) = g[\psi^*(x)\vec{\sigma}\tau_i\psi(x)] \cdot \vec{\nabla}\phi^i(x). \tag{5}$$

Each quark in the nucleon is in a state of positive parity. Equation (5) will only connect it to other states of positive parity, so that πN will not couple to odd-parity states (e.g., D_{13}, $S_{11}, \ldots$ using standard notation). Angular momentum may change by at most one unit. Only $J^P = \frac{1}{2}^+, \frac{3}{2}^+$ states will have the correct quantum numbers. These include the Δ and radial excitations of the N and Δ (e.g., P_{11}'). If we write

$$\phi(\vec{x}) = \frac{1}{\sqrt{2\omega}} e^{i\vec{k}\cdot\vec{x}}, \tag{6}$$

the relevant matrix elements will have the form

$$\langle N_f^* | \mathcal{H}_{\text{int.}} | \pi N \rangle = g\vec{\sigma}\cdot\vec{k} \int d^3x\, e^{i\vec{k}\cdot\vec{x}} \psi_f^*(x)\psi_i(x)\,. \quad (7)$$

In the small-k limit, which is relevant when $m_{N^*} \approx m_N$, $e^{i\vec{k}\cdot\vec{x}} \approx 1$, and the above matrix element vanishes by the orthogonality of the wave functions unless $\psi_i = \psi_f$. This latter condition is only satisfied for the Δ. Thus we see that all resonances except Δ decouple from πN in the nonrelativistic limit. A similar effect occurs in the derivation of the Adler-Weisberger relation. In the SU(6) limit the axial charge is a generator of SU(6) and hence only connects states within the same SU(6) multiplet.

To study the effect of the Δ we can calculate, using SU(6), the $\pi^+ p \Delta^{++}$ coupling, defined as

$$\langle \Delta^{++} | T | \pi^+ p \rangle = \frac{g_{\pi^+ p \Delta^{++}}}{m_p} \bar{u}(p) P^\mu u_\mu(\Delta)\,. \quad (8)$$

In terms of the $\pi^+ np$ coupling we get

$$g_{\pi^+ p \Delta^{++}} = \tfrac{2}{5}\sqrt{6}\, g_{\pi^+ np} \quad (9)$$

As we take $m_\Delta \to m_N$ (keeping m_π small enough that the Δ is always above threshold), both the numerator and denominator in Eq. (2) vanish equally fast. The integral may be done using a Breit-Wigner form for the Δ pole with the result $g_A = \frac{5}{3}$.

In the real world the analysis of the Adler-Weisberger relation is considerably different. The $\pi^\pm p$ total cross sections are reproduced in Fig. 1. We note two main effects that distinguish the nonrelativistic SU(6) situation from the physical world.[4] The first is that the baryon resonances other than the Δ are considerably heavier than the nucleon and do couple to πN. Since this effect is primarily in the $\pi^- p$ cross section, it lowers the value of g_A. The second effect is mass splitting within the SU(6) multiplets. Numerical study reveals that it is only the p-Δ splitting that is significant; shifting the position of the resonances has comparitively little effect. Being more precise, in the real world the Δ by itself will yield $g_A = 1.44$ instead of $g_A = \frac{5}{3}$, a 14% difference. The remaining decrease, of slightly larger magnitude (17%), is due to the presence of the baryon resonances.[4]

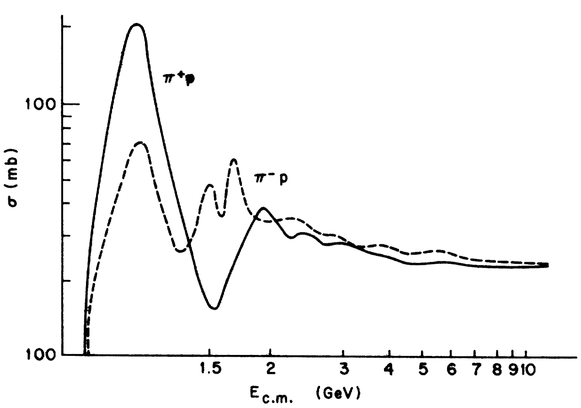

FIG. 1. The total $\pi^+ p$ (solid line) and $\pi^- p$ (dotted line) cross sections (see Ref. 1).

In translating from the SU(6) world to the real world we must bind the quarks more tightly. In accord with the uncertainty principle this makes them more relativistic, and the lower component of the wave function becomes more important. This in turn lowers the value of g_A derived via the quark model, Eq. (4). For this to be consistent with the Adler-Weisberger relation the binding must induce effects that also lower g_A in the latter method. It does so by coupling the resonance states to πN. This is accomplished in two ways. The binding will increase the mass of the various excited states. (It is well known that the quantum numbers and approximate positions of the resonances can be reasonably accounted for in various quark models.) Also, our coupling constant argument given above fails as the binding becomes stronger. Equation (5) is no longer adequate to describe the interaction. States of odd parity may couple in, and lower components in the wave function will change the orthogonality conditions.

These effects are general and not limited to any particular quark model. In order to have an explicit demonstration we will calculate them in a crude quark model consisting of an infinite square well[5] of radius R. This has been chosen as the simplest way to study quark binding. In general quark models the S-wave quark state may be written

$$\psi(x) = \begin{pmatrix} iu(r)\chi \\ l(r)\vec{\sigma}\cdot\hat{r}\chi \end{pmatrix} e^{-iEt} \quad (10)$$

with $u(r)$ and $l(r)$ the upper and lower radial wave functions and χ the two-component Pauli spinor. As a measure of the upper and lower components let us define

$$U \equiv \int d^3x\, u^2(r)\,, \qquad L \equiv \int d^3x\, l^2(r) \quad (11)$$

with the normalization condition

$$\int d^3x\, \psi^+(x)\psi(x) = U + L = 1\,. \quad (12)$$

Matrix elements of the axial-vector current are of the form

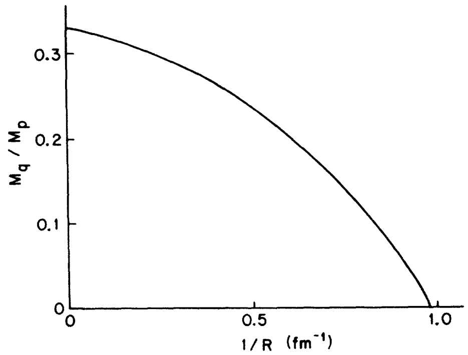

FIG. 2. Quark mass as function of the confinement radius R.

$$g_A=\left\langle P\right|\int d^3x\,\bar{\psi}(x)\gamma_\mu\gamma_5\tau_3\psi(x)\left|P\right\rangle$$
$$=\tfrac{5}{3}\int d^3x[u^2(r)-\tfrac{1}{3}l^2(r)] \tag{13}$$
$$=\tfrac{5}{3}(1-\tfrac{4}{3}L)\,.$$

The quantities L in Eqs. (4) and (11) are identical. In the square-well model

$$u(r)=Nj_0(pr)\,,$$
$$l(r)=-N\left(\frac{E-m}{E+m}\right)^{1/2}j_1(pr)\,, \tag{14}$$

with a boundary condition at the edge of the well

$$\tan pR=\frac{-pR}{ER+mR-1}\,. \tag{15}$$

There is also a $J=\frac{1}{2}$ P-wave state with wave function

$$\psi(x)=\begin{pmatrix} i\tilde{l}(r)\vec{\sigma}\cdot\hat{r}\chi \\ \tilde{u}(r)\chi \end{pmatrix} e^{-iE't}\,, \tag{16}$$

$$\tilde{u}(r)=N'j_0(p'r)\,,$$
$$\tilde{l}(r)=-N'\left(\frac{E'+m}{E'-m}\right)^{1/2}j_1(p'r) \tag{17}$$

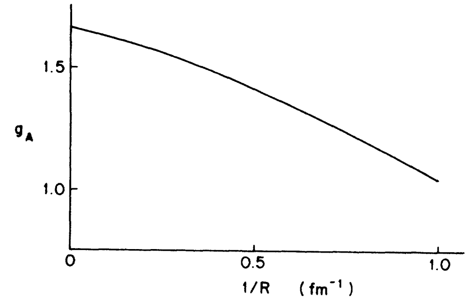

FIG. 4. g_A as function of the confinement radius R.

and boundary condition

$$\tan p'R=\frac{p'R}{E'R-mR+1}\,. \tag{18}$$

As a constraint on the model we require that the total energy of the three quarks be equal to the average mass of the proton and the Δ, $\bar{m}_p\approx 1.18$ GeV. When the binding is weak ($R\to\infty$) the quark mass will be $\frac{1}{3}\bar{m}_p$. As the quarks are more tightly bound and the kinetic energy becomes more important the quark mass must decrease in order to hold $\bar{m}_p$ constant, until with $R\approx 1$ fm we need massless quarks. The variation of m_q with binding is given in Fig. 2.

Let us look at each of the processes in turn. We may study the relative importance of the upper and lower components by plotting the ratio L/U as a function of the binding radius in Fig. 3. As expected, the lower component is negligible in weak-binding situations but becomes important in tightly bound configurations. The quark-model value of g_A is given in Fig. 4. As advertised it decreases from $\frac{5}{3}$ when $R\to\infty$ to 1.09 in the most tightly bound state.

The energy of the excited states can be found by studying the solutions to Eqs. (15) and (18). As examples, the mass of the first radial excitation of the nucleon (the state P_{11}') and the first orbital

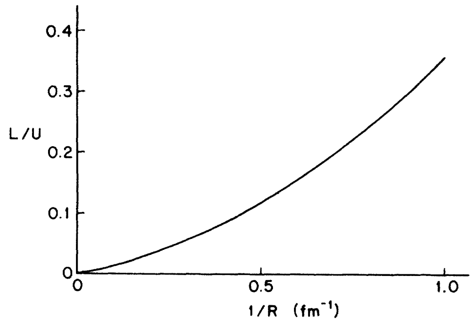

FIG. 3. The ratio L/U, defined in Eq. (11) as function of the confinement radius R.

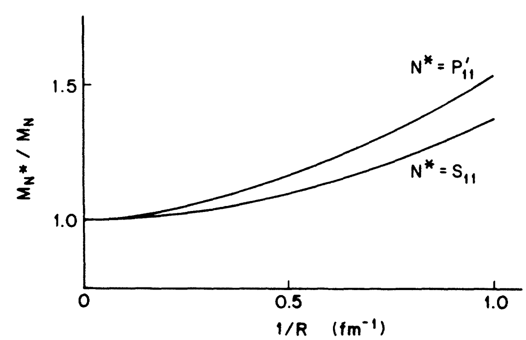

FIG. 5. The mass of the excited states as function of the confinement radius R.

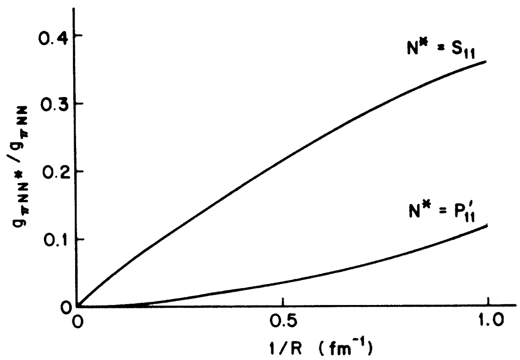

FIG. 6. The πNN^* coupling constant, defined in Eqs. (20) and (21) as function of the confinement radius R.

excitation (S_{11}) is given as a function of binding in Fig. 5.[6] Finally, we need to comment on the coupling of the resonances to πN. If we take as our interaction

$$\mathcal{H}_{\text{int.}}(x) = g[\bar{\psi}(x)\vec{\tau}\gamma_\mu\gamma_5\psi(x)]\cdot\partial^\mu\vec{\phi}(x) \tag{19}$$

we can directly calculate the coupling constants in terms of the quark wave functions. In the no-recoil low-k approximation

$$\langle N_f^*|\mathcal{H}_{\text{int.}}|\pi N_i\rangle = g\vec{\sigma}\cdot\vec{k}\left(\delta_{if} - \frac{4}{3}\int d^3x\, l_f(x) l_i(x)\right) \tag{20}$$

for N_f^* a $J=\frac{1}{2}^+, \frac{3}{2}^+$ state. Here $\delta_{if}=1$ only for $N_f^* = p, \Delta$, and we can see explicitly how the lower components enter. For a transition to a state of odd parity (e.g., S_{11})

$$\langle N_f^*|\mathcal{H}_{\text{int.}}|\pi N_i\rangle = 2g(k^2+m_\pi^2)^{1/2}\int d^3x\, l_f(x) l_i(x)\,. \tag{21}$$

Examples of these are given in Fig. 6.[6] Including these states in $\pi^- p$ scattering will decrease g_A. Inclusion of all the other states would require a more elaborate theory, which we do not attempt as we are only interested in the general features of these processes.

The quark-model value of g_A is derived in theories that in general have a nucleon-Δ mass degeneracy. Since the effects of the interactions which remove this degeneracy are important in the Adler-Weisberger calculation of g_A, it is worthwhile to study the problem in the quark model. Again we attempt to extract the general features of such interactions by performing a crude calculation.

The origins of SU(6) breaking are not well understood. However, an attractive possibility is that it results from the interaction of quarks with vector particles, presumably the gluons of quantum chromodynamics. The spin-dependent forces in such a theory do provide a nonzero $m_\Delta - m_N$[7] and also account for the sign and magnitude of the neutron's charge radius.[8] In addition, new components are now present in the particle wave function; these may consist of three quarks and a gluon, three quarks and a quark-antiquark pair, or simply three quarks in excited states. These will alter the value of g_A.

Since our previous analysis suggests that roughly half of the decrease in g_A is due to binding, we use for our estimate the square-well parameters that in the last section gave $g_A = 1.44$. Owing to the masses of the quarks in this situation, states with an additional quark pair are suppressed by the energy denominator in perturbation theory relative to those with quarks and gluons. Our estimate of the effects of mixing in higher-energy three-quark states finds that they are negligible due to the smallness of the matrix element of the axial-vector current connecting the excited and ground states. The dominant contributions are those in Fig. 7. A final approximation, which allows for considerable technical simplification, is to consider only the lowest-energy intermediate state. This should be adequate for the sign and approximate magnitude of the effect.[9] In this approximation the N-Δ mass splitting is

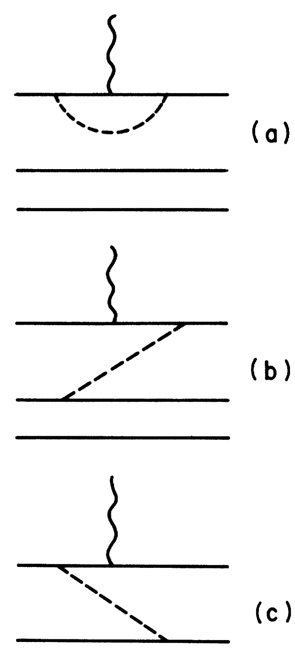

FIG. 7. The lowest-order gluon correction diagrams. Solid lines represent quarks, dotted lines gluons, and wavy lines W bosons.

$$(m_N - m_\Delta) = 2\langle p\,|g^2 \int d^3x\, [\bar{\psi}(x)\vec{\gamma}\tfrac{1}{2}\lambda^A\psi(x)]\cdot\vec{A}(x)\,(1/E_g) \int d^3y\, [\bar{\psi}(y)\vec{\gamma}\tfrac{1}{2}\lambda^A\psi(y)]\cdot\vec{A}(y)\,|p\rangle\,, \tag{22}$$

where $\vec{A}(y)$ is the gluon vector potential in the lowest-energy mode, and E_g is the gluon energy in this mode. The change in g_A is

$$\Delta g_A = \langle p\,|g^2 \int d^3x\, [\bar{\psi}(x)\vec{\gamma}\tfrac{1}{2}\lambda^A\psi(x)]\cdot\vec{A}(x)\,(1/E_g{}^2) \int d^3z\, \bar{\psi}(z)\gamma_3\gamma_5\tau_3\psi(z) \int d^3y\, [\bar{\psi}(y)\vec{\gamma}\tfrac{1}{2}\lambda^A\psi(y)]\cdot\vec{A}(y)\,|p\rangle\,. \tag{23}$$

Performing the appropriate spin summations and normalizing our interaction to the mass difference, Eq. (22), allows us to rewrite Eq. (2) as

$$g_A = \tfrac{5}{3}(1 - \tfrac{4}{3}L) + \frac{(m_N - m_\Delta)}{3E_g}(1 - \tfrac{4}{3}L)\,. \tag{24}$$

The sign of the effect is given by the sign of the mass difference. In the square well the lowest gluon energy is $E_g = 2.7/R$, yielding a magnitude of

$$\Delta g_A = -0.25 \tag{25}$$

or a 15% effect. More tightly bound models give a smaller result. It is reassuring that the sign falls out naturally and that the size is approximately correct. Again there is a consistency between the two methods.

In conclusion, we have shown how two seemingly different methods of calculating the axial-vector coupling constant agree in the limit of nonrelativistic SU(6), and how the same physics produces similar deviations in the two frameworks. We attribute this to a consistency between the quark model and PCAC.

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