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Mass divergences in annihilation processes. II. Cancellation of divergences in cut vacuum polarization diagrams

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The cancellation of mass divergences in cut vacuum polarization diagrams is investigated. Ensembles of states are identified, the summation over which produces transition probabilities which are free of mass divergences. States included in these ensembles can be characterized as having a "jetlike" structure. The reasoning is extended to a large class of renormalizable field theories, including non-Abelian gauge theories.

I. INTRODUCTION

In the single-photon approximation, perturbation-theory cross sections for the production of elementary hadrons from e^+e^- annihilation are computed from cut vacuum polarization diagrams. In this paper, which is a continuation of Ref. 1 (referred to below as I), the cancellation of mass divergences in these cross sections is investigated.[2]

An immediate result of Kinoshita's theorem[3] for two-point functions in renormalizable field theories without superrenormalizable couplings is that the total cross section in e^+e^- annihilation in the one-photon approximation is free of mass divergences when the mass of one or more species of hadrons vanishes. On the other hand, exclusive processes are almost always divergent in the zero-mass limit in any order of perturbation theory. Kinoshita[3] and Lee and Nauenberg[4] proposed that a quantity which is finite in the zero-mass limit can be obtained by summing only over states in which massless particles are replaced by "jets" of parallel-moving massless particles with the same total energy.[5] I have formalized this proposition for one-photon annihilation processes and, subject to assumptions given below, proved it to all orders in perturbation theory for four-dimensional renormalizable theories with ϕ^4, Yukawa, or gauge vector couplings. As in I, the reasoning can be extended to $(\phi^3)_6$ and gravitational field theories.

The sum over states which produce the relevant ensembles are a straightforward generalization of the sum over soft-photon emission familiar from the infrared problem in QED.[6] In QED, mass divergences arise from momentum regions where on-shell massive fermions couple to soft massless photons (which are not self-coupled), and they cancel in the sum. In a theory in which massless particles couple to one another directly, we get additional mass divergences when on-shell massless lines with finite energy and parallel spatial momenta couple at single vertices. The cancellation of mass divergences in such a theory requires a sum over ensembles of states related by the action of vertices of this type.

Define two states as being "jet-related" if they differ by the emission or absorption of a number of zero-energy particles, or by the transformation of one set of parallel-moving particles into another. The ensembles will be specified in terms of sets of jet-related states. To make this idea more quantitative, define for any state a, and "angular energy current" in the e^+e^- c.m. frame

$$j_a(\Omega) = \sum_{i=1}^{n_a} \eta_i \,\delta(\Omega - \omega_i)\,, \tag{1.1}$$

where the sum is over the n_a massless particles in a, with energies $\{\eta_i\}$ and momentum directions $\{\omega_i\}$ (ω_i stands for angles θ_i and ϕ_i). Jet-related states have the same $j(\Omega)$. Each group of particles with collinear momenta may be described as a jet, and any set of jet-related states is characterized by the number of jets, as well as their energies and directions.

Just as it is not possible to measure exactly the total energy carried by soft massless particles, it is also not possible to determine whether two particles have exactly parallel momenta. Thus the "energy resolution" of QED generalizes to a whole set of resolutions which describe possible experimental acceptances. Each jet is defined not only by its direction and total energy E_i but also by an energy resolution δE_i and a fixed angular region $\Delta\Omega_i$. Any massless particle directed into $\Delta\Omega_i$ is counted as part of the jet. An additional energy resolution δE_0 is associated with the emission of soft particles into the region Ω_0, outside of all the jet regions $\Delta\Omega_i$.

Next, define a "jet ensemble" S, which includes states satisfying

$$\begin{aligned} &E_i - \delta E_i \leq \int_{\Delta\Omega_i} d^2\Omega\, j_a(\Omega) \leq E_i + \delta E_i\,, \\ &0 \leq \int_{\Omega_0} d^2\Omega\, j_a(\Omega) \leq \delta E_0\,. \end{aligned} \tag{1.2}$$

Let G be a vacuum polarization graph or, in a gauge theory, some gauge-invariant set of vacuum polarization graphs. The following proposition will be shown below. *The total contribution* $P^G(A_S, q^2/M^2, g_R)$ *to the cross section from states in S generated by cutting G in all possible ways is free of mass divergences.* Here A_S stands for the collection of dimensionless quantities E_i^2/q^2, $\delta E_i^2/q^2$, and $\Delta\Omega_i$. q is the (timelike) external momentum of G. g_R is the renormalized coupling defined with respect to the reference mass M. M, of course, is nonzero (i.e., renormalization is performed off shell). This proposition appeared as a conjecture in Ref. 7, where it was invoked to derive two-jet structure for annihilation final states at high energy in all orders of perturbation theory, and was verified to lowest nontrivial order.

The verification of the proposition outlined below requires certain assumptions. It is assumed that regularization and renormalization of ultraviolet behavior can be carried out without introducing mass divergences, and also that the power counting procedure developed in I is a valid measure of the behavior of Feynman integrals.

There are two results of I which are essential to the proof of the proposition. Suppose G_α is a vacuum polarization graph, cut to give a physical state α. Mass divergences come from points in the loop momentum spaces of the vertex functions $\Gamma^{(\alpha)}$ into which G is cut where loop momentum contours are trapped on shell, or from points in α phase space where tree subgraphs of the $\Gamma^{(\alpha)}$ are forced on shell. The former are called "pinch" singular points (SP's). With any SP is associated a reduced graph of G formed by contracting lines which are off shell.

In I it was shown: (i) That every internal vertex of the reduced graph of any pinch SP is "soft". That is, it connects only parallel-moving finite-momentum lines and/or zero-momentum lines. Therefore *any* physical state found by cutting the reduced diagram of G_α at a pinch SP is *jet-related* to state α; (ii) that divergences in exclusive cross sections smeared over finite regions of phase space are at worst logarithmic. (ii) is actually what limits the applicability of the proposition to theories without super-renormalizable couplings. As we will see below, the basic cancellation mechanism has little to do with the details of the theory. It operates in any theory, but can only eliminate logarithmic divergences in sums over physical states. Since massless theories with super-renormalizable couplings in general have power divergences even in unphysical Green's functions, they are not covered by these arguments.

The essential idea is that it is natural to expect the mass divergence associated with any cut α of G to be canceled by divergences in precisely those other cuts β which appear in the reduced diagrams of pinch SPs of G_α. Since such states β are all jet-related to α, they will all contribute to any P^G to which α contributes.

To implement this expectation, it is useful to notice the following fact: Let P be a pinch SP in the loop momentum space of G_α, and suppose that state β is found by cutting the reduced diagram of G_α at P. Then state α will also be found by cutting the reduced diagram of G_β at just the same point P in its loop momentum space. Indeed, the reduced diagrams of G_α and G_β corresponding to P are identical. This suggests that we might try to combine the *integrands* of sets of cut diagrams, and then integrate over neighborhoods of their common pinch SPs.

Actually, it turns out to be more convenient to do the combining after the "minus" component, $l^- = l^0 - l^3$, of each loop integral has been done. This leads, as was shown by Chang and Ma,[8] to a set of terms suggestive of infinite-momentum time-ordered perturbation theory.[9] After this integration, the combination of terms is easy. Some manipulations put the resulting expression into a form which can be shown to be finite after integration over a region in the remaining loop momenta of G. The contour integration reasoning used in I plays an important role here as well. The necessity for integration translates into the presence of phase-space resolutions in the definition of the finite quantities P^G.

The argument is organized as follows: In Sec. II the method of Chang and Ma[8] is reviewed and used to show how contributions from different cuts can be combined to produce expressions from which mass divergence has been eliminated.

In Sec. III a set of regions in the loop momentum space of an arbitrary vacuum polarization graph is identified. We show how these regions make it possible to organize the combination of cut graphs so as to sum over states in P^G only.

In Sec. IV, P^G is calculated for an arbitrary graph G in a scalar theory, and is shown to be finite. Section V extends the results of Section IV to theories with spin. Certain technical difficulties resulting from the use of infinite-momentum variables at intermediate stages of the calculation are dealt with in this section.

II. METHOD FOR COMBINING CUTS

In this section we will describe the kind of manipulation by which contributions from different cuts can be combined. This process, coupled with the power counting techniques developed in Sec. V of I, will enable us to express each quantity $P^G(A_S, \ldots)$ as a finite sum of terms, each of which is free of mass divergence.

Consider a specific cut of a vacuum polarization graph G. In the following, "cut" will always refer to a cut of G in the q channel. Also, a cut will be referred to as "on shell" when all its lines are on shell with physical momenta. Define

$$g^{(\alpha)} = \int \Gamma_L^{(\alpha)} d\tau^{(\alpha)} \Gamma_R^{(\alpha)*}. \tag{2.1}$$

$d\tau^{(\alpha)}$ represents the phase-space element, including any numerator momenta and matrix factors. For now, however, we will specialize to the case of all scalar lines. $\Gamma_{L(R)}^{(\alpha)}$ is the left (right) vertex function formed from G by cut α.

We have, schematically,

$$\Gamma_L^{(\alpha)} = \int_{-\infty}^{\infty} \prod dk^+ d^2\vec{k} \left(\int_{-\infty}^{\infty} \prod dk^- I_L \right), \tag{2.2}$$

where I_L is the Feynman integrand and the products run over all loop momenta. We will perform the k^- integrations first, employing the method described by Chang and Ma.[8]

Let there be n lines, v vertices, and l loops in $\Gamma_L^{(\alpha)}$. The prescription of Ref. (8) for scalar lines is as follows: Factors of π, coupling constants, and combinatoric factors, which do not affect the argument, will be suppressed below. Change variables from loop momenta k^- to line momenta l^- by substituting a δ function at each vertex:

$$\delta\Big(q^- - \sum_{i=1}^{m} p_i^-\Big)\Gamma_L^{(\alpha)} = i^n(-i)^{v+1}\int_{-\infty}^{\infty} \prod_{i=1}^{l} dk_i^+\, d^2\vec{k}_i \int_{-\infty}^{\infty} \prod_{j=1}^{n} dl_j^-(l^2+i\epsilon)^{-1} \prod_{k=1}^{v} \delta(I_k^- - O_k^-). \tag{2.3}$$

Here I_k^- and O_k^- represent the sum of minus momenta entering the kth vertex and leaving the kth vertex, respectively. $\sum_{i=1}^m p^-$ is the sum of minus momenta of the outgoing lines. Next, apply the Fourier transforms

$$\begin{aligned} \frac{i}{l^2+i\epsilon} &= \int_{-\infty}^{\infty} dt\, e^{it(l^- - |\vec{l}|^2/l^+ + i\epsilon/l^+)} \frac{\theta(l^+ t)}{|l^+|}\,, \\ \delta(I_k^- - O_k^-) &= \int_{-\infty}^{\infty} \frac{d\tau_k}{(2\pi)} e^{-i(I_k^- - O_k^-)\tau_k}\,. \end{aligned} \tag{2.4}$$

The l^- and t integrations can be done explicitly to give

$$\delta\Big(q^- - \sum_{i'=1}^{m} p_{i'}^-\Big)\Gamma_L^{(\alpha)} = (-i)^{v+1}\int_{-\infty}^{\infty} \prod_{i=1}^{l} dk_i^+ d^2\vec{k}_i \int_{-\infty}^{\infty} \prod_{j=1}^{v} d\tau_j \prod_{k=1}^{n} \frac{\theta(l_k^+(\tau_{f(k)} - \tau_{i(k)}))}{|l_k^+|} \exp\Big(-i(\tau_{f(k)} - \tau_{i(k)})\frac{\vec{l}_k^{\,2}}{l_k^+}\Big)$$
$$\times \exp\Big(-iq^-\tau_1 + i\sum_{i'=1}^{m} p_{i'}^-\tau_{i(i')}\Big). \tag{2.5}$$

In (2.5), $\tau_{f(k)}$ and $\tau_{i(k)}$ are the Fourier transform parameters associated with the vertices at which line k arrives and leaves, respectively. τ_1 is associated with the vertex at which the off-shell momentum q attaches.

Finally, the τ integrations can be performed, leading to a set of "τ-ordered" diagrams which can be written down according to the usual rules.[9] State denominators are of the form $(q^- - S_\beta + i\epsilon)$, with

$$S_\beta = \sum_{j=1}^{m_\beta} \frac{|\vec{l}_j|^2}{l_j^+}\,, \tag{2.6}$$

where the sum goes over the m_β lines in state β. Because of the θ functions in (2.5), only graphs in which positive energy flows forward in τ occur, in accordance with the rules of infinite-momentum perturbation theory.[9] Letting $T(\Gamma)$ denote the set of permissible τ ordering of Γ we find

$$\Gamma_L^{(\alpha)} = -\sum_{T(\Gamma_L^{(\alpha)})} \int \prod_{i=1}^{l} dk_i^+ d^2\vec{k}_i \prod_{j=1}^{n} \frac{\theta(l_j^+)}{|l_j^+|} \prod_{\beta\in T(\Gamma_L^{(\alpha)})} (q^- - S_\beta + i\epsilon)^{-1}. \tag{2.7}$$

We can go through the same process with $\Gamma_R^{(\alpha)*}$ and derive a result of the same form as (2.7). Substituting into (2.1), and expressing $d\tau^{(\alpha)}$ in an explicit form we find

$$g^{(\alpha)} = \int\Big(\prod_{i=1}^{L} dk_i^+ d^2\vec{k}_i\Big)\prod_{j=1}^{N} \frac{\theta(l_j^+)}{|l_j^+|} \sum_{T(\Gamma_R^{(\alpha)})} \sum_{T(\Gamma_L^{(\alpha)})} \prod_{\beta\in T(\Gamma_L^{(\alpha)})} (q^- - S_\beta + i\epsilon)^{-1}\delta(q^- - S_\alpha)$$
$$\times \prod_{\beta'\in T(\Gamma_R^{(\alpha)})} (q^- - S_{\beta'} - i\epsilon)^{-1}, \tag{2.8}$$

where N is now the number of lines in the vacuum polarization graph G, and L is the number of loops in G.

Next we can sum over all possible cuts α of G. Grouping the terms by τ orderings of G,

$$\sum_\alpha g^{(\alpha)} = \int\left(\prod_{i=1}^{L} dk_i^+ d^2\vec{k}_i\right)\prod_{j=1}^{N}\frac{\theta(l_j^+)}{|l_j^+|}\sum_{T(G)}\sum_\alpha\prod_\beta (q^- - S_\beta + i\epsilon)^{-1}\delta(q^- - S_\alpha)\prod_{\beta'}(q^- - S_{\beta'} - i\epsilon)^{-1}, \quad (2.9)$$

$$\sum_\alpha g^{(\alpha)} = i\int\left(\prod_{i=1}^{L} dk_i^+ d^2\vec{k}_i\right)\prod_{j=1}^{N}\frac{\theta(l_j^+)}{|l_j^+|}\sum_{T(G)}\left[\prod_\gamma (q^- - S_\gamma + i\epsilon)^{-1} - \prod_\gamma (q^- - S_\gamma - i\epsilon)^{-1}\right]. \quad (2.10)$$

The second form follows easily from a repeated application of the distribution identity

$$(x + i\epsilon)^{-1} - (x - i\epsilon)^{-1} = -2\pi i\,\delta(x). \quad (2.11)$$

Equation (2.10) is immediately recognizable as a restatement of the Cutkosky rules[10] as applied to the graph G: The discontinuity of the diagram is found by summing over the relevant cut diagrams. The reasoning leading up to (2.10) can be reversed to derive the Feynman integral form:

$$\sum_\alpha g^{(\alpha)} = -i\int\left(\prod_{i=1}^{L} dk_i^+ d^2\vec{k}_i\right) \times \int\prod_{j=1}^{L} dk_j^-(I_G - I_G^*). \quad (2.12)$$

The significance of this rederivation of a standard result lies in the fact that the whole calculation can be carried out at fixed values of all the plus and transverse momenta in G. That is, once all minus integrals have been evaluated, the discontinuity of a subintegral over any finite region in the plus and transverse momenta of G is equal to the sum of the corresponding subintegrals in the cuts of G. Notice that these subintegrals involve phase space, as well as the internal loops of the cut graphs.

That this result is relevant to the cancellation of mass divergences can be seen once we observe that the four-momentum of an on-shell line is uniquely determined by its plus and transverse momenta alone, unless they all vanish. Thus a singular point is specified by giving these three components for all the loops of the corresponding reduced diagram. Integrating over a small region in plus and transverse momenta, we will encounter only a limited class of singular points, even when the minus momenta are fully integrated over. In the limit of vanishing plus and transverse momenta for some subset of the lines, the minus integrals sum over singular points related by a redistribution of energy among those on-shell lines moving in the $-z$ direction.

Let R denote a region of this type. Let α be a cut of graph G, and let r_α denote the projection of R onto the phase space of cut α. Then the contribution of region R to the cut diagram ("cut subintegral") is

$$h^{(\alpha)}(R) = \int_{r_\alpha}\gamma_L^{(\alpha)}(R)\,d\tau^{(\alpha)}\gamma_R^{(\alpha)*}(R), \quad (2.13)$$

where

$$\gamma_L^{(\alpha)}(R) = \int_{r_L}\left(\prod d^4 l_i\right) I_L. \quad (2.14)$$

r_L denotes the region of internal integration of Γ_L specified by R. A similar definition holds for $\gamma_R^{*(\alpha)}$.

Adding together all the cut subintegrals we find the analog of (2.12),

$$\sum_\alpha h^{(\alpha)}(R) = -i\int_R\prod_{i=1}^{L} d^4 l_i\,(I_G - I_G^*). \quad (2.15)$$

Now each $h^{(\alpha)}$ in (2.13) is in general mass divergent; these quantities will not even be defined unless they are regulated in some way: by adding a small mass to propagators, or by dimensional regularization, for instance. On the other hand, the sum (2.15) is less singular than the individual cut subintegrals. The multiple integrals in both terms of (2.15) are contour integrations and can be deformed in just the same way as in the full Feynman integral, except that now the plus and transverse momentum contours must be fixed on the real axis at the boundary of the region R.

Mass divergences could arise in (2.15) either from the plus and transverse momentum end points, or from pinch singular points in the interior of R. But the reasoning of I shows that the only pinch singular points in the interior of R are those whose on-shell lines have zero momentum, and such singular points cannot lead to mass divergences in the theories being considered.[11] Thus mass divergences can only arise from the boundary of R, a region of lower dimension.

In the appendix of I it was shown that power counting at an arbitrary SP (not necessarily a pinch SP) indicates no worse than logarithmic divergence, except for contributions in gauge theories which decouple from cross sections. The power-counting estimate in the neighborhood of a surface of SP's results from the behavior of both the integrand and the volume element for variables normal to the surface. In particular, if the integral is not trapped everywhere on the surface, but only on a submanifold of lower dimension, the effect is to increase the dimension of the normal variable volume element, and correspondingly suppress scaling behavior by a power.

To be more precise, divergence was shown in I to be at worst logarithmic for SP's where plus momentum (equivalently energy) does not flow in the same direction as the loop momentum in every line of any loop of the corresponding reduced diagram. But for any reduced diagram containing such a loop, the minus momentum integral of that particular loop has poles in only one half-plane and is not pinched at the corresponding SP. If the regions R extend to $\pm\infty$ in all minus momenta, any such SP's can always be avoided in expressions such as (2.15). The exceptional case where the plus and transverse momenta of every line in the loop vanishes (so that, at the SP, the integrand is independent of the minus momentum) is automatically restricted to a submanifold of reduced dimension by rotational invariance.

Consider a specific surface Q of SP's, where a given set of lines, not all with zero momentum, goes on shell. Suppose Q has dimension D_Q. If a region like R in (2.15) is chosen so that its boundary intersects Q in a submanifold whose dimension is $D_Q - 1$ or less, then the power counting associated with surface Q is suppressed by a power. If R is chosen this way for every Q, (2.15) is finite, and the sum of the $h^{(\alpha)}$'s is calculable in this form without ever introducing any of the regularization necessary to define each $h^{(\alpha)}$ individually.

The shortcoming of (2.15) is that we have not introduced information about any jet ensemble S into it, so that the intermediate states encountered in the sum are not limited to any subregion of phase space in general. Below, we derive a set of subintegrals along the same lines as (2.15), but which are more directly related to the quantities $P^G(A_S, \ldots)$.

III. IDENTIFICATION OF REGIONS

Let G be an arbitrary vacuum polarization graph, and S some jet ensemble. In this section we identify a set of regions y_g in the loop momentum space of G, each associated with a subgraph g of G. The subgraphs g are formed from G by contracting some set of lines into the two vertices of G at which the external momentum q attaches. They are constructed to contain no tadpole subdiagrams.

For any such subgraph g we define, for each cut α of g,

$$i(y_g, \alpha) = \int_{s^{(\alpha)}(g)} d\tau_\alpha \left[\int_{k_L^{(\alpha)}(g,\tau)} \left(\prod d^4 l\right) I_{\Gamma_L^{(\alpha)}} \right] \times \left[\int_{k_R^{(\alpha)}(g,\tau)} \left(\prod d^4 l'\right) I^*_{\Gamma_R^{(\alpha)}} \right] , \tag{3.1}$$

where $s^{(\alpha)}(g)$ is the projection of y_g on α phase space, and $k_L^{(\alpha)}(g,\tau)$ and $k_R^{(\alpha)}(g,\tau)$ are the projections of y_g on the internal loop momentum spaces of $\Gamma_L^{(\alpha)}$ and $\Gamma_R^{(\alpha)}$ at point τ. $i(y_g, \alpha)$ is defined to vanish if α is not a cut of g.

Roughly speaking, each y_g is constructed so that the mass divergences in any $i(y_g, \alpha)$ come only from pinch SPs whose reduced diagrams are subgraphs of g. It will be shown that, in accordance with comments made in the Introduction, summing over cuts of α of g gives a quantity

$$I(g) = \sum_\alpha i(y_g, \alpha) , \tag{3.2}$$

which is free of mass divergences. As mentioned above, for this purpose each y_g must be chosen to intersect surfaces of SP's only in manifolds of lower dimension. At the same time, the y_g's are to be defined in such a way that

$$\sum_g I(g) = P^G(A_S, \ldots) , \tag{3.3}$$

so that the finiteness of each $I(g)$ ensures that $P^G(A_S, \ldots)$ is also free of mass divergence.

The following are the conditions to be imposed on the regions y_g:

(a) In (3.1), $s^{(\alpha)}(g)$ equals the projection of S on α phase space for every α, and at each point τ in α phase space the collection of regions $k_{L(R)}^{(\alpha)}(g,\tau)$ covers the complete loop momentum spaces of $\Gamma_{L(R)}^{(\alpha)}$ without overlap. This gives (3.3).

(b) If P is a pinch SP of any $i(y_g, \alpha)$, and R is the reduced diagram of P, then the finite-energy lines of R form a subgraph of g. This means that lines not in g are never trapped on shell with finite energy in any $i(y_g, \alpha)$.

(c) Let Q be a manifold of SP's in the momentum space of G. If Q is of dimension D_Q, the intersection of Q with the boundary of any y_g is of dimension $D_Q - 1$ or less.

(d) The definition of y_g involves restrictions on plus and transverse momenta only; minus momenta are always integrated from minus infinity to plus infinity, as in the last section.

The construction of a set of regions satisfying (a)–(d) is in principle easy. For a given g, consider the union x_g of all SP's consistent with (a) and (b). The y_g's are formed by simply expanding the x_g's into the full $4L$-dimensional space of G loop momenta, subject to the additional constraints. (c) presents no particular difficulties, because for any graph there are only a finite number of manifolds of SP's. Otherwise, the details of the construction are free.

IV. CANCELLATION OF DIVERGENCES: SCALARS

We now proceed to calculate $i(y_g, \alpha)$ by integrating the minus momenta in (3.1) using methods described in Sec. II. For the time being we restrict ourselves to all scalar lines.

We get

$$i(y_g,\alpha)=\int_{y_g}\left(\prod_{i=1}^{L} dk_i^+ d^2\vec{k}_i\right)\left(\prod_{j=1}^{N}\frac{\theta(l_j^+)}{|l_j^+|}\right)\sum_{T(\Gamma_L^{(\alpha)})}\left[\prod_{\beta}(q^- - S_\beta+i\epsilon)^{-1}\right]\delta(q^- - S_\alpha)\times\sum_{T(\Gamma_R^{(\alpha)})}\left[\prod_{\beta'}(q^- - S_{\beta'}-i\epsilon)^{-1}\right]. \tag{4.1}$$

The next step will be to reorganize the sums over vertex orderings in (4.1). Let f_L and f_R be the left- and right-hand subgraphs which are contracted to form g from G. We group each vertex ordering according to the right-most vertex μ in f_L and the left-most vertex μ' in f_R. $T_\mu(\Gamma_L^{(\alpha)})$ will stand for the set of all vertex orders of graph $\Gamma_L^{(\alpha)}$ in which μ is the right-most vertex in f_L. Similarly, ${}_{\mu'}T(\Gamma_R^{(\alpha)})$ will stand for the set of orderings with μ' the left-most vertex in f_R. Then

$$i(y_g,\alpha)=\int_{y_g}\left(\prod_{i=1}^{L} dk_i^+ d^2\vec{k}_i\right)\left(\prod_{j=1}^{N}\frac{\theta(l_j^+)}{|l_j^+|}\right)\left[\sum_{\mu}\sum_{T_\mu(\Gamma_L^{(\alpha)})}\prod_{\beta}(q^- - S_\beta+i\epsilon)^{-1}\right]\delta(q^- - S_\alpha)\times\left[\sum_{\mu'}\sum_{{}_{\mu'}T(\Gamma_R^{(\alpha)})}\prod_{\beta'}(q^- - S_{\beta'}-i\epsilon)^{-1}\right]. \tag{4.2}$$

In each term in the μ sum, the vertex ordering $T_\mu(\Gamma_L^{(\alpha)})$ determines the state λ which appears just to the right of vertex μ, and in the same way ${}_{\mu'}T(\Gamma_R^{(\alpha)})$ determines the state λ' appearing just to the left of μ'. Now λ splits $\Gamma_L^{(\alpha)}$ into two subgraphs, which we will call γ_{LL}^{λ} and $\gamma_{LR}^{\lambda,\alpha}$, and λ' splits $\Gamma_R^{(\alpha)}$ into subgraphs $\gamma_{RL}^{\alpha,\lambda'}$ and $\gamma_{RR}^{\lambda'}$. By definition, vertex μ will be included in both γ_{LL}^{λ} and $\gamma_{LR}^{\lambda,\alpha}$, and vertex μ' in both $\gamma_{RL}^{\lambda',\alpha}$ and $\gamma_{RR}^{\lambda'}$. The "dissection" of G according to this procedure is illustrated schematically in Fig. 1. In terms of this regrouping,

$$i(y_g,\alpha)=\int_{y_g}\left(\prod_{i=1}^{L} dk_i^+ d^2\vec{k}_i\right)\left(\prod_{j=1}^{N}\frac{\theta(l_j^+)}{|l_j^+|}\right)\left[\sum_{\mu}\sum_{\lambda}\left(\sum_{t_\mu(\gamma_{LL}^{\lambda})}\prod_{\delta}(q^- - S_\delta+i\epsilon)^{-1}\right)\times\left(\sum_{{}_\mu t(\gamma_{LR}^{\lambda,\alpha})}\prod_{\beta}(q^- - S_\beta+i\epsilon)^{-1}\right)\right]\times\delta(q^- - \delta_\alpha)\left[\sum_{\mu'}\sum_{\lambda'}\left(\sum_{t_{\mu'}(\gamma_{RL}^{\alpha,\lambda'})}\prod_{\beta'}(q^- - S_{\beta'}-i\epsilon)^{-1}\right)\left(\sum_{{}_{\mu'}t(\gamma_{RR}^{\lambda'})}\prod_{\delta'}(q^- - S_{\delta'}-i\epsilon)^{-1}\right)\right]. \tag{4.3}$$

In (4.3), $t_\nu(h)$ and ${}_\nu t(h')$ denote the set of vertex orderings in which ν and ν' are the right-most and left-most vertices in graphs h and h', respectively.

The point of the reorganization leading to (4.3) is that all the states β and β' of $\gamma_{LR}^{\lambda,\alpha}$ and $\gamma_{RL}^{\alpha,\lambda'}$ must be cuts of g, and therefore, because of requirement (a) on y_g, can be on shell only in region S. States δ and δ' of γ_{LL}^{λ} and $\gamma_{RR}^{\lambda'}$, on the other hand, must be cuts of G which involve one or more lines in f_L and f_R, respectively. The consequence of this restriction will become clear below.

We are now ready to sum over cuts α for which $i(y_g,\alpha)$ is nonzero. Because cuts to the left of μ or the right of μ' involve lines not in g, cuts which contribute to the sum for a given vertex ordering always appear between λ and λ'.

Performing the summation over α, we find

$$I(g)=\int_{y_g}\left(\prod_{i=1}^{L} dk_i^+ d^2\vec{k}_i\right)\left(\prod_{j=1}^{N}\frac{\theta(l_j^+)}{|l_j^+|}\right)\sum_{\lambda}\sum_{\lambda'}\left\{\left(\sum_{\mu}\sum_{t_\mu(\gamma_{LL}^{\lambda})}\prod_{\delta}(q^- - S_\delta+i\epsilon)^{-1}\right)\times\left(\sum_{T(\mathcal{G}_{\lambda,\lambda'})}\sum_{\alpha}\left[\prod_{\beta}(q^- - S_\beta+i\epsilon)^{-1}\delta(q^- - S_\alpha)\prod_{\beta'}(q^- - S_{\beta'}-i\epsilon)^{-1}\right]\right)\times\left(\sum_{\mu'}\sum_{{}_{\mu'}t(\gamma_{RR}^{\lambda'})}\prod_{\delta'}(q^- - S_{\delta'}-i\epsilon)^{-1}\right)\right\}, \tag{4.4}$$

where $\mathcal{G}_{\lambda,\lambda'}$ is the vacuum polarization graph formed by contracting everything in G to the left of μ and to the right of μ' to a point.

We can now apply (2.11) to give

$$I(g)=i\int_{\mathcal{Y}_g}\left(\prod_{i=1}^{L} dk_i^+ d^2\vec{k}_i\right)\left(\prod_{j=1}^{N}\frac{\theta(l_j^+)}{|l_j^+|}\right)\sum_{\lambda,\lambda'}\left[\left(\sum_{\mu}\sum_{t_\mu(\gamma_{LL}^\lambda)}\prod_{\delta}(q^- - S_\delta + i\epsilon)^{-1}\right)\right.$$
$$\left.\times\sum_{T(\mathcal{G}_{\lambda,\lambda'})}\left(\prod_{\gamma}(q^- - S_\gamma + i\epsilon)^{-1} - \prod_{\gamma}(q^- - S_\gamma - i\epsilon)^{-1}\right)\left(\sum_{\mu'}\sum_{\mu',t(\gamma_{RR}^{\lambda'})}\prod_{\delta'}(q^- - S_{\delta'} - i\epsilon)^{-1}\right)\right]. \quad (4.5)$$

(4.5) includes all the vertex-ordered state denominators corresponding to the vacuum polarization graph $\mathcal{G}_{\lambda,\lambda'}$. This suggests that we might undo some of the minus integrals, as we did in going from (2.10) to (2.12), to get an integral in which there is more freedom to deform contours.

This process would leave us with the question of what to do with the state denominators for the graphs γ_{LL}^λ and $\gamma_{RR}^{\lambda'}$. Notice that, even after μ and μ' are summed over, not all the vertex orderings of these graphs appear in (4.5). We can, however, relate these state denominators to Feynman integrals as follows:

Consider, for instance, the quantity

$$\xi(\gamma_{LL}^\lambda)=\sum_{t_\mu(\gamma_{LL}^\lambda)}\prod_{\delta}(q^- - S_\delta + i\epsilon)^{-1}\prod_{j\in\gamma_{LL}^\lambda}\frac{\theta(l_j^+)}{|l_j^+|}. \quad (4.6)$$

In this sum, all vertex orderings have μ as their last vertex, and the vertex of G at which q attaches as their initial vertex. As a result, the first and last state denominators are always the same. If there are V vertices in γ_{LL}^λ, then there are $V-1$ state denominators, and the last one is

$$S_{V-1}=q^- - \sum_{i\in Q}\frac{\vec{l}_i^{\,2}}{l_i^+} - \sum_{j\in N}\frac{\vec{l}_j^{\,2}}{l_j^+}, \quad (4.7)$$

where the set Q consists of all lines in γ_{LL}^λ attached to vertex μ, and the set N of all lines which are shared by γ_{LL}^λ and $\mathcal{G}_{\lambda,\lambda'}$. We can now recognize $\xi(\gamma_{LL}^\lambda)$ as the result of integrating over the minus momenta of the vacuum polarization graph H_L formed by "pinching" the lines in set N at vertex μ, up to plus-momentum factors for lines in N. Thus

$$\left(\prod_{j\in N}\frac{\theta(l_j^+)}{|l_j^+|}\right)\xi(\gamma_{LL}^\lambda) = -\int_{-\infty}^{\infty}\left(\prod_{\substack{\text{loops}\\ \text{in } H_L}} dl^-\right)\times\prod_{\substack{\text{lines}\\ \text{in } H_L}}(l^2+i\epsilon)^{-1} i^{n(H_L)-V-1}. \quad (4.8)$$

The graphical relation of H_L to γ_{LL}^λ and N is illustrated in Fig. 2. $n(H_L)$ is the number of lines in H_L.

Now we can rewrite $I(g)$ as

$$I(g)=\int_{\mathcal{Y}_g}\left(\prod_{i=1}^{L} dk_i^+ d^2\vec{k}_i\right)\sum_{\mu,\mu',\lambda,\lambda'}\left(\prod_{j\in N\cup N'} l_j^+\right)\left(\int_{-\infty}^{\infty}\prod_{\substack{\text{loops of}\\ H_L}} I_{H_L}^{(\mu)}\right)\left(\int_{-\infty}^{\infty}\prod_{\substack{\text{loops of}\\ \mathcal{G}_{\lambda\lambda'}}}(I_{\mathcal{G}_{\lambda\lambda'}} - I^*_{\mathcal{G}_{\lambda\lambda'}})\right)$$
$$\times\left(\int_{-\infty}^{\infty}\prod_{\substack{\text{loops of}\\ H_R}} I_{H_R}^{(\mu')*}\right). \quad (4.9)$$

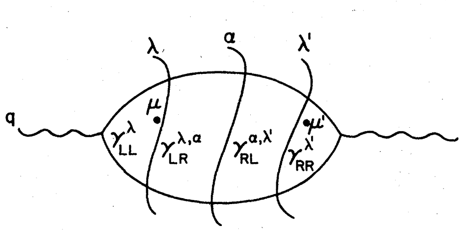

FIG. 1. Schematic "dissection" of a vertex-ordered cut vacuum polarization graph, discussed in Sec. IV. λ, α, and λ' are cuts. The points labeled μ and μ' represent the vertices just to the left of cut λ and the right of cut λ', respectively.

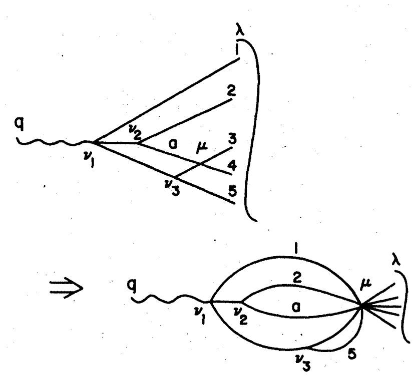

FIG. 2. Illustration of "pinching" procedure discussed in Sec. IV. In this example, lines 1, 2, and 5 make up the set N of Eq. (4.9).

In (4.9), we see that each of the lines in N and N' is "doubled" in the sense that each one in N contributes a denominator to both I_{H_L} and $I_{\mathcal{G}_{\lambda,\lambda'}}$, for instance. The extra units in the denominator are made up by an extra plus-momentum factor in the numerator and an extra minus loop momentum relative to the G integrand.

Contour integration arguments can now be applied to $I(g)$ in the form (4.9) to show that it is free of mass divergences.

From the point of view of the integrands and minus loop momenta, (4.9) looks like the product of integrals for three independent vacuum polarization graphs: H_L, $\mathcal{G}_{\lambda,\lambda'}$, and H_R, as shown schematically in Fig. 3(a). The difference is that the plus and transverse loop momenta are still those of the original graph G. Nevertheless, as in (2.15), the set of points at which momentum contour integrals are trapped is drastically reduced in (4.9) relative to the individual quantities [the $i(y_g, \alpha)$'s] which go to make it up.

The first thing to realize is that $f_{L(R)}$ is a subgraph of $H_{L(R)}$, and because of condition (b) of Sec. III, all the lines of $f_{L(R)}$ can be deformed off shell everywhere in the interior of y_g. (We neglect here pinch SP's with only zero-momentum lines, since they never give rise to mass divergences in the theories we are considering.[11]) Therefore all the lines in $f_{L(R)}$ are contracted at any pinch SP of (4.9) in y_g. This reduces the possible form of the reduced diagram of a pinch SP of (4.9) to that shown in Fig. 3(b).

Next, we can apply the reasoning of I to show that the contour may be further deformed to take additional lines in H_L/f_L and H_R/f_R off shell. This is straightforward for those lines not in N or N': Any loop with nonzero momentum flowing in H_L/f_L cannot satisfy the Landau equations [Eq. (2.3) of I] at the SP. The situation is more complicated for the loops which flow through lines in N or N'. By construction, their plus and transverse momenta flow both in H_L/f_L or H_R/f_R and $\mathcal{G}_{\lambda,\lambda'}$, so there is the possibility of their contours being trapped between singularities from separate diagrams. In any case, contours can be deformed so that only these lines are on shell in H_L or H_R, as shown in Fig. 3(c).

Suppose for the time being that none of the lines l_i in N or N' have precisely $l_i^+ = |\vec{l}_i| = 0$. Then the contours of the minus loop integrals which pass through the lines in N and N' (and do not appear in $\mathcal{G}_{\lambda,\lambda'}$) can be deformed to take the N and N' lines off shell, giving a reduced diagram like Fig. 3(d). This is just a vacuum polarization graph, and we know that its SP's can all be avoided. It can be concluded that, barring the case of some subset of lines in N or N' moving in exactly the

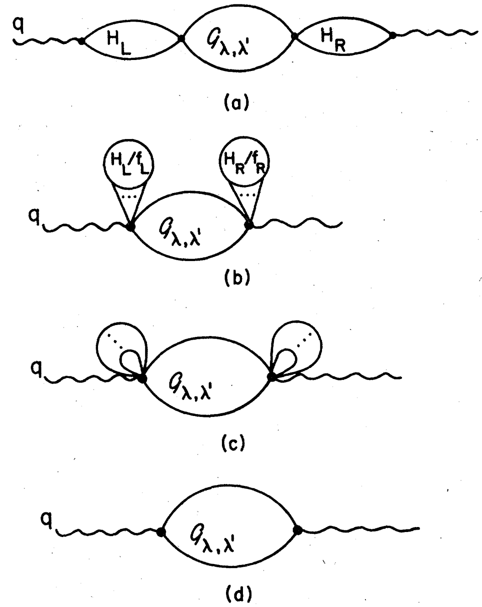

FIG. 3. Reduced diagrams related to the discussion of Eq. (4.9).

$-z$ direction, no pinch SP's with on-shell finite-energy lines are encountered in the interior of y_g.

What happens when lines in N or N' do move into the $-z$ direction? Here our ability to deform the minus loop momenta passing through these lines does not help us, since l_i^- is multiplied by l_i^+ in each Feynman denominator. But these SP's are eliminated by the factors of l_i^+ in the numerator associated with each such line. The contour integrals of (4.9) are thus trapped only at SP's encountered on the boundary of y_g.

We are now in the situation discussed for the integral Eq. (2.15). Condition (c) of Sec. III implies that power counting from any surface of singularity points encountered on the boundary is suppressed by a power relative to the estimate found using the methods of I. But, as observed at the end of Sec. II above, power counting indicates that divergences associated with any relevant manifold of SP's are never worse than logarithmic. This is true of each of the $i(y_g, \alpha)$ which are summed to give (4.9). It can be concluded that $I(g)$ is free of mass divergences, at least for the scalar theories which we have considered so far. By (3.3), P^G is also finite, and the proposition is verified for scalar theories.

V. INCLUSION OF SPIN

Spin requires a modification of the reasoning of Sec. IV because of the presence of factors of minus momentum in Feynman integral numerators. Factors of plus and transverse momentum, of course, do not affect the procedure, since only minus integrals are actually evaluated. For gauge theories, we work in the Feynman gauge, which is chosen to avoid having to deal with gauge denominators which would complicate the minus integrations. In treating non-Abelian gauge theories, ghosts are treated on the same footing as other lines, since their propagators are defined with the same $i\epsilon$ prescription as for normal particles. Since gauge-invariant cross sections are to be computed, those contributions which cancel ghosts will automatically be included.

On a formal level, the problem of numerator momenta is easily dealt with. Use momentum conservation to express each factor of minus momentum as a sum of the minus momenta of individual lines, and then apply the simple algebraic identity

$$\frac{l^-}{l^2}=\frac{1}{l^+}\left(\frac{\vec{l}^2}{l^2}+1\right) \tag{5.1}$$

to eliminate explicit minus-momentum dependence in the numerator. This process has a simple graphical interpretation: In the first term the l^- factor of a given line is simply replaced by $\vec{l}^2/l^+$, and in the second, the line is contracted, and a factor $1/l^+$ associated with the vertex into which that line is merged. For each of these new graphs the analog of (4.9) can certainly be derived.

In general, singularities are introduced by (5.1) at $l^+=0$ in individual terms, but they are spurious and cancel in the sum. For this reason, no $i\epsilon$ prescription need be given for the l^+ denominators, although for purposes of definition any prescription will do, since, for $\vec{l}^2\neq 0$,

$$\delta(l^+)\left(\frac{\vec{l}^2}{l^2}+1\right)=0. \tag{5.2}$$

In addition, the substitution (5.1) can lead to diagrams with "tadpole" subgraphs, connected to the rest of the diagram by only a single vertex. No lines from such a subgraph can be cut, and its minus integrals can be evaluated without reference to the rest of the graph. Situations of this type do not modify the reasoning of the last section, and will be ignored below.

The substitution (5.1) does not solve all the problems associated with numerator momenta. This can be seen by considering a loop each of whose lines has a minus momentum factor in the numerator. The product of the $(1/l^+)$ terms for all the lines of such a loop is completely independent of the minus loop momentum, whose integral then diverges linearly. Such large k^- divergences are related to the fact that, in the presence of numerator momenta, the plus and minus integrals do not always commute. This point is discussed below. What is important here is simply to realize that problems of this sort are all associated with regions of infinite minus loop momenta and have nothing to do with mass divergences. They are present as well for diagrams with only massive lines.

If we delay consideration of these problems, we can continue with the procedure of the last section and arrive at the same result; $P^G(A_S,\ldots)$ remains free of mass divergences when the effects of spin are taken into account. It should be emphasized, however, that in gauge theories finiteness is a property only of gauge-invariant combinations of graphs, where the power divergences identified in Sec. V of I are eliminated.

Having formally extended the argument to theories with spin, we return to the question of the large k^- divergences found after the substitution (5.1). In particular, I will argue that they will not contribute to the full integral when account is taken of the correct integration procedure.

The problem to be dealt with is that integrating over k^- in a loop all of whose lines carry minus numerator factors seems to give a linear divergence. The first thing to notice is that a connected subgraph, all of whose lines have minus-momentum factors in the numerator, can contain at most a single closed loop. This is due in part to the fact that if every line in a subgraph is to have a minus-momentum factor, no invariants—which require plus momentum factors as well—from that subgraph alone can occur in the numerator. Thus no vector polarization sum can both begin and end at internal vertices of the subgraph, since such a situation would certainly generate an invariant numerator factor. As a result, there can be no loops involving vector lines beginning and ending on scalar or fermion lines.

It is also easy to see that no fermion loops with external scalars are possible, because the fermion-scalar vertex is proportional to the identity (or γ_5) and the numerator of one fermion can be commuted (or anticommuted) past the vertex to give a product with the numerator of the other. But in $\not{p}\not{p}'$ the coefficient of $p^-p'^-$ is zero because $\gamma_+{}^2=0$. In fact, the only possible loops are pure scalar, fermion, or vector loops, connected to external vector lines. Connecting two such loops by a vector line, however, always results in a polarization sum, and therefore a plus factor in the numerator. Loops that give problems of this type are therefore isolated, and can be dealt with one at a time.

The main point of this discussion is that any

Feynman integral is *defined* with the prescription that the energies be integrated over first. An operation such as Wick rotation, which makes symmetric integration possible, depends precisely on the convention that energies be dealt with first. A simple example will help to illustrate the significance of this rule to our case.

Consider a fermion loop with n lines, and in particular that monomial of numerator momenta with a factor $(k^-)^m$, $m \leq n$, where k is the loop momentum. Other numerator factors do not enter into the argument, and can be ignored. The correct definition of the diagram thus involves the integral

$$I_{n,m} = \int_{-\infty}^{\infty} dk_3 \int_{-\infty}^{\infty} dk_0 \frac{(k_0 - k_3)^m}{\prod_{i=1}^{n} [(k_0 + E_i)^2 - (k_3 + P_i)^2 - \vec{K}_i^2 + i\epsilon]} \tag{5.3}$$

where $(E_i, P_i, \vec{K}_i)$ is the momentum of the ith line when $k_0 = k_3 = 0$.

Suppose we now decide to change variables to $k^\pm = k_0 \pm k_3$. As a first step, we find

$$I_{n,m} = \int_{-\infty}^{\infty} dk_3 \int_{-\infty}^{\infty} dk_- \frac{k_-^m}{\prod_{i=1}^{n} \{[k_- + (E_i - P_i)][k_- + 2k_3 + (E_i + P_i)] - \vec{K}^2 + i\epsilon\}} . \tag{5.4}$$

There is no problem in exchanging the orders of integration and then changing from k_3 to k_+. So,

$$I_{n,m} = \tfrac{1}{2} \int_{-\infty}^{\infty} dk_- k_-^m \int_{-\infty}^{\infty} dk_+ \frac{1}{\prod_{i=1}^{n} \{[k_- + (E_i - P_i)][k_+ + (E_i + P_i)] - \vec{K}_i^2 + i\epsilon\}} , \tag{5.5}$$

We now have the right variables, but with the prescription that k_+ be integrated over first. If, as in Sec. IV, we want to evaluate the k^- integral first, we have to be able to exchange the orders of integration. But if $m = n$, this is not possible. In fact, for $m = n$, (5.8) is linearly divergent if k^- is integrated first. When the k^+ integral is done first, on the other hand, the resulting integral vanishes for $k^- < -\max(E_i - P_i)$ or $k^- > -\min(E_i - P_i)$, and (5.8) is finite. It is easy to see that first changing variables to k_+ does not solve the problem.

Using (5.1), we always organize numerator momenta to replace minus momenta with factors of plus and transverse momentum, so that we can apply the reasoning for scalar diagrams directly. This simple substitution, of course, does not solve the problem of interchanging the integrals for the case $m = n$. We can, however, easily use it to see that the difference between integrating k^- and k^+ first, although formally infinite, has two simplifying properties.

The first results from the fact that after the substitution (5.1), the two-dimensional regions of integration which lead to problems are given by k^- very large and $k^+ \sim -(E_i + P_i)$ for some i. For finite k^- and nonvanishing $k^+ + (E_i + P_i)$ the order of integration is irrelevant. As a result the difference is independent of the minus components of momenta which are external to the loop (i.e., the quantities $E_i - P_i$).

Second, the difference in an integral like $I_{n,n}$ is purely imaginary. Using (5.1) and defining the k^+ denominators by any consistent $i\epsilon$ convention, either the k^+ or k^- integral can be evaluated first by Cauchy's theorem. In either case, the result is of the form $2\pi i$ times a real number. The remaining integral has both pole terms, which give an overall real contribution, and principal values, which give an overall imaginary contribution. The real part thus comes from a product of δ functions. But the integrals commute for δ functions, so the difference has no real part.

The result is that, if we insist on integrating minus momenta first, the extra infinite contributions which are generated can be canceled by the addition of special vertices, in which the loop in question has been replaced by a formally infinite pure imaginary function of plus and transverse momenta. Such vertices do not affect the reasoning leading to Eq. (4.9).

It is worth reemphasizing that the problems with noncommuting integrals are completely an artifact of the infinite momentum variables that have been chosen. The trouble would have been avoided had energies been integrated over first. In this case, however, the resulting perturbation expansion has many more terms, corresponding to arbitrary orderings of vertices, and the mechanism by which the different terms combine to eliminate divergences is more complex.

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