Magnetic monopoles in a gauge field theory from vortex strings

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A realization of the antisymmetric tensor and vector potentials, having the open dual vortex string as a source, is shown to be given by the fields of the 't Hooft-Polyakov model. The non-Abelian gauge transformations map into Abelian transformations characterized by a Minkoski vector and scalar. Magnetic current conservation is assured.

I. INTRODUCTION

If the various models for strong interactions are to be sensible, it should be possible to exhibit some of their similar behavior and predictions in certain domains. In this paper we plan to show an interesting connection between two different approaches to the study of hadronic processes. The two systems with which we propose to deal are motivated by the dual, relativistic string[1] and an SO(3) Higgs-type field theory. We shall exploit an analogy that strings display to charged point particles, to produce a new gauge field. Then, by making an ansatz involving unit Higgs fields and usual non-Abelian SO(3) gauge potentials, the string gauge fields will be shown to lead to 't Hooft-Polyakov[2] magnetic monopoles.

The gauge field theory associated with the string was originally seen in the context of direct interstring action.[3] It was intended there, however, to look at the field in its own right, rather than as an "adjunct" quantity. The gauge *potential* which was introduced is designated by $\phi_{\mu\nu}(x)=-\phi_{\nu\mu}(x)$ $(\mu,\nu=0,1,2,3)$; it is a second-rank antisymmetric tensor in Minkowski space.

One can build up and study Lagrangians involving this object and its couplings to matter of various kinds.[3,4] It is also possible to motivate $\phi_{\mu\nu}$ and a definite coupling by discussing relativistic vortices in a superfluid.[5] An overview of the current status and a collection of calculations for this field will be forthcoming in the near future. Here, however, we will concentrate on $\phi_{\mu\nu}$ as it relates to the magnetic-monopole structure of a particular non-Abelian gauge theory.

II. MOTIVATIONS, ANSATZ, AND MONOPOLES

If a charged particle is placed in an electromagnetic field, the interaction will be of the form

$$s_I=q\int_{\tau_i}^{\tau_f} d\tau\,\dot{x}^{\mu}(\tau)A_{\mu}(x(\tau)) \tag{1}$$

Here q is the charge of the particle, and τ is a monotonic variable which parametrizes the temporal flow over the domain (τ_i,τ_f). The four-vector $x_\mu(\tau)$ gives the instantaneous position so that $\dot{x}_\mu$ $(\equiv dx_\mu/d\tau)$ is the velocity or tangent vector to the world line of the particle. Finally, $A_\mu(x(\tau))$ is the vector potential of the electromagnetic field evaluated at the point in Minkowski space which coincides with the particle world line at "time" τ.

The gauge invariance of this coupling is exhibited by making the change

$$\delta A_\mu=\partial_\mu\Lambda\,, \tag{2}$$

where the space-time-dependent variable Λ is the arbitrary gauge parameter. Under the transformation (2), the interaction (1) varies by

$$\delta s_I=\Lambda\big|_{\tau_i}^{\tau_f}\,. \tag{3}$$

It is usually assumed without loss that the system evolves from and to the same gauge. Hence, δs_I vanishes. The interaction plus kinetic terms form a realistic model action for electrodynamics.

We would like to carry the above arguments over to the Nambu string. It sweeps out a two-dimensional world sheet whose points in Minkowski space are given by $x_\mu(\tau,\lambda)$. The variables τ and λ represent an arbitrary parametrization.

Let us generalize the line coupling (1) to the surface integral

$$S_I=g\int_{\tau_i}^{\tau_f}\int_{\lambda_0}^{\lambda_1} d\tau d\lambda\,\sigma^{\mu\nu}(\tau,\lambda)\phi_{\mu\nu}(x(\tau,\lambda))\,. \tag{4}$$

The tangent vector of the point particle has mapped into the surface tensor of the world sheet:

$$\sigma_{\mu\nu}\equiv\dot{x}_\mu x'_\nu-x'_\mu\dot{x}_\nu=-\sigma_{\nu\mu}\,,$$

where

$$\dot{x}_\mu\equiv\partial x_\mu/\partial\tau,\quad x'_\mu\equiv\partial x_\mu/\partial\lambda\,. \tag{5}$$

We have introduced the antisymmetric tensor potential $\phi_{\mu\nu}(x)$ and evaluated it at $x_\mu(\tau,\lambda)$. The limits of integration give the (finite in λ) domain of the parametrization, while the coupling constant g indicates the strength of the interaction.

Incidentally, if S_I is to have units of action, g must be a mass.

There is a gauge invariance associated with the quantity (4). If we make the change

$$\delta\phi_{\mu\nu} = \partial_\mu\Lambda_\nu - \partial_\nu\Lambda_\mu \,, \tag{6}$$

with the space-time-dependent four-vector gauge parameter Λ_μ, we obtain the variation

$$\delta S_I = 2g \int_{\lambda_0}^{\lambda_1} d\lambda\, x'_\mu \Lambda^\mu \Big|_{\tau_i}^{\tau_f} - 2g \int_{\tau_i}^{\tau_f} d\tau \dot{x}_\mu \Lambda^\mu \Big|_{\lambda_0}^{\lambda_1} . \tag{7}$$

While the first term is eliminated in a similar manner to (3), the second integral does not vanish in general. It is identically zero if periodic boundary conditions are imposed on the λ dependence. Doing so, however, means we are discussing only *closed* strings.

If the string is *open*, we can also invoke an extended form of gauge invariance. Notice that the second term in (7) is reminiscent of the electromagnetic coupling considered previously. Simply replace $2g\Lambda_\mu$ by gA_μ and view two point particles of equal and opposite charge. We are therefore led to modify the interaction by adding another term:

$$\begin{aligned} S_I \to S_I + e\int d\tau d\lambda\, \sigma^{\mu\nu} G_{\mu\nu} \\ = \int d\tau d\lambda\, \sigma^{\mu\nu}(g\,\phi_{\mu\nu} + eG_{\mu\nu}) \,, \end{aligned} \tag{8}$$

with

$$G_{\mu\nu} \equiv \partial_\mu B_\nu - \partial_\nu B_\mu \,. \tag{9}$$

We have introduced a vector field $B_\mu(x)$, its field strength $G_{\mu\nu}$, and a unitless constant e, which couple to the string in the way shown. It is easy to show that

$$\begin{aligned} e\int d\tau d\lambda\, \sigma^{\mu\nu} G_{\mu\nu} \\ = 2e\int d\lambda\, x'_\mu B^\mu \Big|_{\tau_i}^{\tau_f} - 2e\int d\tau\, \dot{x}_\mu B^\mu \Big|_{\lambda_0}^{\lambda_1} . \end{aligned} \tag{10}$$

Ignoring the first term we have added only an endpoint coupling to the interaction.

Now, from Eqs. (8) and (9), we see that the joint transformation

$$\begin{aligned} \delta\phi_{\mu\nu} &= \partial_\mu\Lambda_\nu - \partial_\nu\Lambda_\mu \,, \\ \delta B_\mu &= \partial_\mu\Lambda - (g/e)\Lambda_\mu \end{aligned} \tag{11}$$

leaves the interaction invariant even when the string is open. However, the transformation of the vector field is singular, since Λ_μ must have a nongradient part in order that $\delta\phi_{\mu\nu}$ be nontrivial.

At this point it is a simple matter to define a field strength for the potential $\phi_{\mu\nu}$. In electrodynamics the charged particles move according to the Lorentz force law, whose field dependence is only on the components of $F_{\mu\nu}$ ($\equiv \partial_\mu A_\nu - \partial_\nu A_\mu$). This law is therefore gauge invariant. In the analogous calculation for the string, the gauge-invariant field strength which appears in the generalized Lorentz force law[3] is given by

$$\begin{aligned} F_{\mu\nu\rho} &\equiv \partial_\mu\phi_{\nu\rho} + \partial_\nu\phi_{\rho\mu} + \partial_\rho\phi_{\mu\nu} \\ &= (e/g)(\partial_\mu\psi_{\nu\rho} + \partial_\nu\psi_{\rho\mu} + \partial_\rho\psi_{\mu\nu}) \,, \end{aligned} \tag{12}$$

with

$$\psi_{\mu\nu} \equiv (g/e)\phi_{\mu\nu} + G_{\mu\nu} \,.$$

Both $F_{\mu\nu\rho}$ and $\psi_{\mu\nu}$ are invariant under the transformation (11). Also, $F_{\mu\nu\rho}$, which is antisymmetric under index exchange, does not have sensitivity to whether the string is open or closed.

Another quantity of interest is the dual vector of $F_{\mu\nu\rho}$,

$$\begin{aligned} \tilde{F}_\mu &\equiv \tfrac{1}{6}\,\epsilon_{\mu\nu\rho\sigma} F^{\nu\rho\sigma} \\ &= \tfrac{1}{2}\,\epsilon_{\mu\nu\rho\sigma}\partial^\nu\phi^{\rho\sigma} = (e/2g)\epsilon_{\mu\nu\rho\sigma}\partial^\nu\psi^{\rho\sigma} . \end{aligned} \tag{13}$$

It is *a fortiori* gauge invariant and satisfies

$$\partial^\mu \tilde{F}_\mu = 0 \,. \tag{14}$$

The behavior of $\tilde{F}_\mu$ is reminiscent of the dual of the electromagnetic field, $\tilde{F}^{\mu\nu}$ ($\equiv \tfrac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$).

We now pause to contemplate these elements and their significance. It is not yet known whether the long-range $\phi_{\mu\nu}$ is a new fundamental object in field theory. However, the consideration of charged particles, which may be understood as phenomenological manifestations, leads to the concept of the electromagnetic field, a more basic physical entity. Is it therefore also true that the phenomenological, composite system represented by the string leads to a new fundamental field theory? Or, is the $\phi_{\mu\nu}$ a collective-mode field excitation already present in some gauge theory? Up to this point it has not been possible, using standard methods, to incorporate $\phi_{\mu\nu}$ into an enlarged color-gauge field theory.[4] The usual generalizations and likely new quantities have not led to a viable system.

On the other hand, if we look for $\phi_{\mu\nu}$ within the context of color-gauge theory, we find some positive results. Say that $\phi_{\mu\nu}$ is an entity within an SO(3) Higgs-type field theory. An ansatz is

$$\phi_{\mu\nu} = -(1/g)\epsilon_{abc}\hat{\phi}_a\partial_\mu\hat{\phi}_b\partial_\nu\hat{\phi}_c = -\phi_{\nu\mu} \,, \tag{15}$$

where

$$\hat{\phi}_a \equiv \phi_a/(\phi_a\phi_a)^{1/2} \quad (a,b,c = 1,2,3)$$

is the unit Higgs field. Also, we give the vector field in terms of the gauge potential $A_{a\mu}$ and $\hat{\phi}_a$,

$$B_\mu = \hat{\phi}_a A_{a\mu}\,, \tag{16}$$

with

$$G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu\,.$$

Now define

$$\begin{aligned}\mathfrak{F}_{\mu\nu} &\equiv \psi_{\mu\nu}\\ &= G_{\mu\nu} + (g/e)\phi_{\mu\nu}\\ &= \hat{\phi}_a G_{a\mu\nu} - (1/e)\epsilon_{abc}\hat{\phi}_a D_\mu \hat{\phi}_b D_\nu \hat{\phi}_c\,,\end{aligned} \tag{17}$$

where

$$G_{a\mu\nu} = \partial_\mu A_{a\nu} - \partial_\nu A_{a\mu} + e\epsilon_{abc}A_{b\mu}A_{c\nu}\,,$$

and

$$D_\mu \hat{\phi}_a = \partial_\mu \hat{\phi}_a + e\epsilon_{abc}A_{b\mu}\hat{\phi}_c\,.$$

We have therefore the gauge-invariant "electromagnetic" field strength of 't Hooft and Polyakov. The associated Maxwell's equations have magnetic four-vector current density as a source for $\tilde{\mathfrak{F}}_{\mu\nu}$. Explicit static solutions have already been constructed.[6] In addition, the conservation of magnetic charge is topological in character, rather than coming about through a Nöether-type symmetry principle.[7]

It is interesting to see how the gauge symmetry generated by the transformations

$$\begin{aligned}\delta A_{a\mu} &= \partial_\mu \alpha_a + e\epsilon_{abc}A_{b\mu}\alpha_c\,,\\ \delta\phi_a &= e\epsilon_{abc}\phi_b\alpha_c\end{aligned} \tag{18}$$

map into those of (11), given (15) and (16). The result is

$$\delta\phi_{\mu\nu} = \partial_\mu[(e/g)\alpha_a\partial_\nu\hat{\phi}_a] - \partial_\nu[(e/g)\alpha_a\partial_\mu\hat{\phi}_a]$$

and

$$\delta B_\mu = \partial_\mu(\alpha_a\hat{\phi}_a) - \alpha_a\partial_\mu\hat{\phi}_a\,. \tag{19}$$

Thus

$$\Lambda = \alpha_a\hat{\phi}_a$$

and

$$\Lambda_\mu = (e/g)\alpha_a\partial_\mu\hat{\phi}_a\,. \tag{20}$$

The Abelian scalar gauge parameter Λ is just the projection of the non-Abelian parameter α_a along the special direction in field space given by that of the Higgs field. The Abelian vector gauge parameter Λ_μ, however, is given by the projection of the non-Abelian parameter along the *Minkowski gradient* of the unit Higgs field. The four directions in field space given by $\partial_\mu\hat{\phi}_a$ are orthogonal to that given by $\hat{\phi}_a$, but cannot all be orthogonal to each other for SO(3). Thus, for this group the Λ_μ are not all independent.

Now recall the dual of the field strength (13). This quantity is conserved (14). If we put in the ansatz (15) and (16), the result is

$$\begin{aligned}\tilde{F}_\mu &= (e/2g)\epsilon_{\mu\nu\rho\sigma}\partial^\nu\mathfrak{F}^{\rho\sigma}\\ &= (e/g)\partial^\nu\tilde{\mathfrak{F}}_{\mu\nu}\\ &= -(1/2g)\epsilon_{\mu\nu\rho\sigma}\epsilon_{abc}\partial^\nu\hat{\phi}_a\partial^\rho\hat{\phi}_b\partial^\sigma\hat{\phi}_c\,.\end{aligned} \tag{21}$$

This shows that $-(g/4\pi e)\tilde{F}_\mu$ is the magnetic current density,[8] and the magnetic charge appears as

$$\begin{aligned}M &= -\frac{g}{4\pi e}\int d^3x\,\tilde{F}_0\\ &= \frac{1}{8\pi e}\int d^3x\,\epsilon_{abc}\epsilon_{ijk}\partial_i\hat{\phi}_a\partial_j\hat{\phi}_b\partial_k\hat{\phi}_c\\ &= \lim_{R\to\infty} -\frac{g}{8\pi e}\int_{S_R} d^2\xi\,\Sigma_{ij}\phi^{ij}\,,\end{aligned} \tag{22}$$

with

$$\Sigma_{ij} = \epsilon_{mn}\frac{\partial x_i}{\partial\xi_m}\frac{\partial x_j}{\partial\xi_n}\quad (m,n=1,2\,;\,i,j,k=1,2,3)\,.$$

The ordered pair ξ_n parametrizes the surface of the three-sphere S_R, while Σ_{ij} is its three-space surface tensor. In the limit as the radius $R\to\infty$, S_R is just the sphere in configuration space which goes into the unit sphere $\hat{\phi}_a\hat{\phi}_a = 1$, in field space, by a topological mapping. We have therefore a situation in which dM/dt vanishes, and for our ansatz M is the Kronecker index.

III. CONCLUSION

We have seen that by exploiting the analogy between charged particles and the electromagnetic field on the one hand, and strings and the field $\phi_{\mu\nu}$ on the other, that some well-known physics emerges. It is because $\phi_{\mu\nu}$ has, up to now, only permitted a trivial non-Abelian treatment that we were led to consider it as a composite field entity. This point of view applied to an SO(3) Higgs model produced an ansatz for $\phi_{\mu\nu}$ and a vector field B_μ (necessary to consider open strings) which led in turn to an "electromagnetic" field strength which is gauge invariant and has magnetic charge as a source. The conserved magnetic-current density is simply the dual vector of the field strength $F_{\mu\nu\rho}$ associated with the potential $\phi_{\mu\nu}$.

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