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Semisimple unified gauge theories of strong, weak, and electromagnetic interactions

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Unified theories of strong, weak, and electromagnetic interactions, which are based on simple groups, require large unification energies on the order of 10^9 to 10^{17} GeV. This paper describes a new type of semisimple model in which the unification occurs at energies as low as 10^3 GeV. Most gauge models for a group $C \times F$ have two coupling constants which cannot be related without destroying renormalizability. In this work a computer is used to look at millions of different models for various $C \times F$ and fermion representations. The machine picks out those models which are compatible with the existence of any renormalizable coupling-constant relation at the two-loop level. A small number of such theories is found. In every case the coupling-constant relation is linear and fixes the strengths of the C and F interactions so that they are inversely proportional to the size of the respective groups. Therefore, a semisimple model, which is constructed from a small C group and a large F group, will have a natural hierarchy of interaction strengths. If we embed color SU(3) in C and SU(2) $\times$ U(1) in F, this group-theoretical effect can account for much of the observed difference between strong and weak interactions. Because the remaining difference can be explained by a relatively small renormalization effect, the unification energy can be small. We exhibit an SU(3) $\times$ O(11) model of this sort, in which the unification energy is 10^3 or 10^4 GeV. The theory is in agreement with the usual low-energy phenomenology and particle spectra. As the energy approaches the unification energy from below, many new leptons and O(11) gauge bosons will be produced suddenly. In this model the great strength of the strong interactions can be attributed to the fact that color SU(3) is one of the smallest Lie groups.

I. INTRODUCTION

Most people believe that the divergences in quantum field theory indicate that there is a cutoff energy above which the principles of quantum field theory are no longer valid. For example, if "point" particles are really the excitations of tiny strings, locality will break down as the energy approaches the inverse of the string length. New microscopic structures of this kind may be encountered well below the Planck energy ($G^{-1/2} \simeq 10^{19}$ GeV) at which the quantum corrections to gravity become important. In this case there is no way of understanding quantum gravity in terms of quantum field theory. However, we can still hope to unify the strong, weak, and electromagnetic interactions in a cutoff-independent manner, i.e., in terms of a renormalizable field theory. All unified theories, which have been constructed up to this time,[1] are based on some simple group, in which the "observed" groups [SU(3) and SU(2) $\times$ U(1)] are embedded. Above a unification energy M all interactions are characterized by the same strength. Below M all of the observed difference in strength of strong and weak interactions ($\alpha_S/\alpha \sim 50$) is attributed to a logarithmic renormalization effect. Since the renormalization effect must be large, the unification energy must also be large; typically M ranges from 10^9 to 10^{17} GeV. This kind of model is speculative since it requires the extrapolation of quantum field theory into a high-energy domain where it may not be valid. Such theories are also suspect since they require two disparate mass scales: $M \simeq 10^9$ GeV and $M_W \simeq 10^2$ GeV. This is at variance with Gildener's[2] finding that, if all masses have a common origin in field theory, their ratio is bounded by $\alpha^{-1/2} \simeq 12$.

The purpose of this paper is to construct unified theories of strong, weak, and electromagnetic interactions, which are based on semisimple groups with one coupling constant. Such models have a natural scale, the ratio of the dimensions of groups, which serves to explain part of the difference in the strengths of strong and weak interactions. Consider a gauge theory for the semisimple group $C \times F$. The Lagrangian for such a group usually contains two independent coupling constants, c and f. A coupling-constant relation, $c(f)$, can be imposed only if it is renormalizable, i.e., only if the constrained coupling constants can still absorb all divergences. This means that $c(f)$ must be a trajectory of the Callan-Symanzik function which is stable under perturbations; in other words, $c(f)$ must be a trajectory of the one-loop β function which is also a trajectory of the two-loop form of β. If C and F are isomorphic and the fermion representations are chosen symmetrically, there is a well-known example of such a coupling-constant relation: $c = f$. This relation is renormalizable to all orders since the constrained Lagrangian is invariant under a discrete exchange symmetry which lies outside the gauge group. In the present investigation,[3] a computer is used to scan millions of semisimple theories for nonisomorphic groups and to pick out those groups and representations which are compatible with the existence of any renormalizable coupling-constant relation. The computer finds a handful of isolated models which admit a renormalizable coupling-constant relation at the two-loop level. In each case the relation turns out to be linear and specifies C and F interaction strengths which are inversely proportional to the dimensions of the respective groups.

It is not known if these models are renormalizable at the three-loop level. This issue can be resolved by computing the three-loop divergences (i.e., by computing the three-loop β function). However, this requires an elaborate calculation and has not been done. Of course, renormalizability will be proved to all orders if we can find a global symmetry of the constrained Lagrangian. There is some reason to believe that such a symmetry exists. For spin-0–spin-$\frac{1}{2}$ Lagrangians almost all coupling-constant relations, which are renormalizable at the one- and two-loop levels, are associated with global invariance of the Lagrangian under symmetry or supersymmetry transformations.[4] Therefore these coupling-constant relations are almost always found to be renormalizable to all orders. This suggests an important connection between renormalizability and symmetry: a coupling-constant relation is renormalizable *if and only if* it is globally symmetric.[5] Perhaps the coupling-constant relations, described in this paper, are invariant under some type of symmetry transformation on the space of all boson and fermion fields.

Semisimple models with one coupling constant can be used to construct unified field theories with a low unification mass M. Consider a semisimple model of bosons and fermions in which the dimension of C is much smaller than the dimension of F, $D(C) \ll D(F)$. For energies above M, the strength of the C interactions will exceed the strength of the F interactions by a factor $D(F)/D(C)$. Below M the symmetry is dynamically broken[6] from $C \times F$ to $SU(3) \times SU(2) \times U(1)$. The strong interactions are described by an SU(3) subgroup[7] of C, and weak-electromagnetic interactions are associated with an $SU(2) \times U(1)$ subgroup[8] of F. Therefore, at an energy just below M, the strong-interaction strength will be greater than the weak-interaction strength by the factor $D(F)/D(C)$. This accounts for some of the observed difference between the two interactions. At the much lower energies, which are explored at current particle accelerators, the strong-interaction strength is even greater due to a renormalization effect. This renormalization effect can be small since it need explain only part of the difference between strong and weak interactions; therefore the unification mass M can be low. As is evident from the following formula, the value of M is dramatically (exponentially) reduced by the fact that the difference between strong and weak interactions persists at the unification energy

$$M \simeq 10^{17 g(M)^2/g_S(M)^2}\ \text{GeV}.$$

Here $g(M)^2/g_S(M)^2$, the asymptotic ratio of weak and strong coupling constants, is a small "group-theoretical" number which is approximately scaled by $D(C)/D(F)$.

These ideas are illustrated in an $SU(3) \times O(11)$ model. The unification energy M is 10^3 to 10^4 GeV. At energies below M there is the usual particle spectrum of the "known" quarks and leptons. The fine-structure constant and Weinberg angle are nearly independent of energy and have the usual values ($\sin^2\theta \simeq 0.39$). The strength of the strong interaction decreases by a factor of 3 as the energy rises from 3 GeV to M. As the energy approaches M many new O(11) bosons and many new leptons appear. Above M the full unified group, $SU(3) \times O(11)$, is observed; the strength of the SU(3) interactions exceeds the strength of the O(11) interactions by a factor $D(F)/D(C) = \frac{55}{8}$. In this type of model it can be said that the strong interactions are strong *because* they are described by one of the smallest Lie groups, SU(3).

This paper is organized in the following manner: Section II contains the form of the two-loop β function for semisimple gauge theories. In Sec. III the β function is used to find the necessary conditions for the renormalizability of coupling-constant relations. A computerized search for renormalizable coupling-constant relations is described in Sec. IV. Semisimple models of strong, weak, and electromagnetic interactions are discussed in Sec. V. Conclusions are outlined in Sec. VI. The last section also contains a discussion of the following speculation: Semisimple gauge models with one coupling constant may be the low-energy approximation of spontaneously compactified Einstein-Yang-Mills theories in $4+n$ dimensions.

II. CALLAN-SYMANZIK FUNCTION FOR SEMISIMPLE GROUPS

Let C and F be compact simple groups with dimensions $D(C)$ and $D(F)$ and with totally antisymmetric structure constants g^C_{abc} and g^F_{klm} normalized so that

$$g^C_{ade} g^C_{bde} = \delta_{ab}\,, \qquad (1)$$
$$g^F_{kmn} g^F_{lmn} = \delta_{kl}\,.$$

Denote the corresponding gauge and ghost fields by $A_{a\mu}$, $A_{k\mu}$ and ω_a, ω_k. Take the fermion fields to be left-handed and denote them by ψ (all indices suppressed). Any fermion representation (possibly reducible) will be generated by Hermitian matrices C_a, F_k which are normalized so that

$$[C_a, C_b] = i g^{C}_{abc} C_c,\quad [F_k, F_l] = i g^{F}_{klm} F_m,\quad [C_a, F_k] = 0. \tag{2}$$

The corresponding Lagrangian for the semisimple group $C \times F$ is[9]

$$\begin{aligned}\mathcal{L} = & -\tfrac{1}{4}(\partial_\mu A_{a\nu} - \partial_\nu A_{a\mu} - c_B g^{C}_{abc} A_{b\mu} A_{c\nu})^2 \\ & -\tfrac{1}{4}(\partial_\mu A_{k\nu} - \partial_\nu A_{k\mu} - f_B g^{F}_{klm} A_{l\mu} A_{m\nu})^2 \\ & + \bar{\psi}(i\not\partial - c_B \not A_a C_a - f_B \not A_k F_k)\left(\frac{1-\gamma_5}{2}\right)\psi \\ & + \partial_\mu \omega_a^* \partial^\mu \omega_a + c_B g^{C}_{abc} \partial_\mu \omega_a^* \omega_b A^\mu_c \\ & + \partial_\mu \omega_k^* \partial^\mu \omega_k + f_B g^{F}_{klm} \partial_\mu \omega_k^* \omega_l A^\mu_m \\ & - \frac{1}{2\alpha_{BC}}(\partial A_a)^2 - \frac{1}{2\alpha_{BF}}(\partial A_k)^2 + \text{mass terms}. \end{aligned} \tag{3}$$

Note that the fermion fields can be taken to be left-handed without loss of generality since any terms with right-handed spinors can be written in a left-handed form for the corresponding complex-conjugate representation.

Following 't Hooft,[10] we define the renormalization mass μ and the renormalized coupling constants c and f by

$$\begin{aligned} c_B &= \mu^{(4-d)/2}\left[c + \frac{a_C(c,f)}{d-4} + O\left(\left(\frac{1}{d-4}\right)^2\right)\right], \\ f_B &= \mu^{(4-d)/2}\left[f + \frac{a_F(c,f)}{d-4} + O\left(\left(\frac{1}{d-4}\right)^2\right)\right]. \end{aligned} \tag{4}$$

The Callan-Symanzik functions for these coupling constants are given by[11]

$$\begin{aligned} \beta_C &= \tfrac{1}{2}\left(a_C - c\frac{\partial a_C}{\partial c} - f\frac{\partial a_C}{\partial f}\right), \\ \beta_F &= \tfrac{1}{2}\left(a_F - c\frac{\partial a_F}{\partial c} - f\frac{\partial a_F}{\partial f}\right). \end{aligned} \tag{5}$$

The one-loop and two-loop contributions to these functions have the form

$$\begin{aligned} \beta_{(1)C} &= \frac{c^3}{48\pi^2} V_C, \\ \beta_{(1)F} &= \frac{f^3}{48\pi^2} V_F, \\ \beta_{(2)C} &= \frac{c^3}{384\pi^4}(B_{CC}c^2 + B_{CF}f^2), \\ \beta_{(2)F} &= \frac{f^3}{384\pi^4}(B_{FC}c^2 + B_{FF}f^2), \end{aligned} \tag{6}$$

where

$$\begin{aligned} V_C &= 2T_C - 11, \\ V_F &= 2T_F - 11, \\ B_{CC} &= 5T_C - 17 + 3Q_{CC}, \\ B_{CF} &= 3Q_{CF}, \\ B_{FC} &= 3Q_{FC}, \\ B_{FF} &= 5T_F - 17 + 3Q_{FF}, \\ T_C\delta_{ab} &= \mathrm{Tr}(C_aC_b), \\ T_F\delta_{kl} &= \mathrm{Tr}(F_kF_l), \\ Q_{CC}\delta_{ab} &= \mathrm{Tr}(C_aC_bC_cC_c), \\ Q_{CF}\delta_{ab} &= \mathrm{Tr}(C_aC_bF_kF_k), \\ Q_{FC}\delta_{kl} &= \mathrm{Tr}(F_kF_lC_aC_a), \\ Q_{FF}\delta_{kl} &= \mathrm{Tr}(F_kF_lF_mF_m). \end{aligned} \tag{7}$$

This form of the β function can be derived in the following way: Caswell[12] and Jones[13] have calculated the two-loop β function for a simple group. For a semisimple group the calculation is the same except for the diagrams in Fig. 1, in which the gauge bosons of one group (say, F) affect the wave-function renormalization of the gauge bosons of the other group (say, C). The important part of the group-theoretical weight of these diagrams is

$$\mathrm{Tr}[C_aC_b(c^2C_cC_c + f^2F_kF_k)] = (c^2Q_{CC} + f^2Q_{CF})\delta_{ab}. \tag{8}$$

For a simple group (say, C) and a single irreducible fermion representation this weight reduces to c^2Q_{CC}, which is the same as the term $g^2C_2(R)T(R)$ in Caswell's Eq. (5). Conversely, to obtain the semisimple β function of Eqs. (6) and (7) from the simple β function of Caswell's Eq. (5), it suffices to replace the term $g^2C_2(R)T(R)$ in the latter by the group-theoretical weight in Eq. (8).

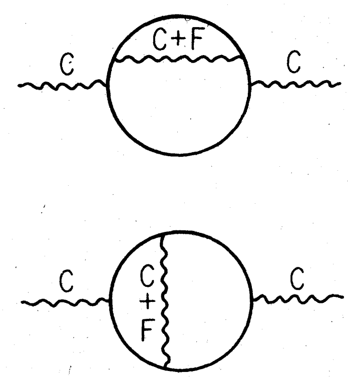

FIG. 1. Fermion-loop diagrams in which gauge bosons of one group (say, F) affect the Callan-Symanzik function of the other group (C).

The fermion representation is the sum of irreducible blocks, labeled by the index N and generated by the Hermitian matrices C_a^N, F_k^N [normalized as in Eq. (2)]. Each block is the product of an irreducible representation of C with generators $\tilde{C}_a^N$ and dimension D_C^N and an irreducible representation of F with generators $\tilde{F}_k^N$ and dimension D_F^N. In other words

$$(C_a^N)_{\alpha\lambda\alpha'\lambda'} = (\tilde{C}_a^N)_{\alpha\alpha'}\delta_{\lambda\lambda'}, \quad (F_k^N)_{\alpha\lambda\alpha'\lambda'} = \delta_{\alpha\alpha'}(\tilde{F}_k^N)_{\lambda\lambda'}. \tag{9}$$

where α, α' run from 1 to D_C^N and λ, λ' run from 1 to D_F^N.

Let T_C^N be the trace of an irreducible representation of C,

$$T_C^N \delta_{ab} = \mathrm{Tr}(\tilde{C}_a^N \tilde{C}_b^N). \tag{10}$$

Then the eigenvalue of the quadratic Casimir invariant of that representation is

$$I_C^N = \frac{D(C) T_C^N}{D_C^N}. \tag{11}$$

The following formulas express the T's and Q's of Eq. (7) in terms of these invariants:

$$T_C = \sum_N T_C^N D_F^N, \quad T_F = \sum_N D_C^N T_F^N,$$
$$Q_{CC} = \sum_N T_C^N I_C^N D_F^N, \quad Q_{CF} = \sum_N T_C^N D_F^N I_F^N, \tag{12}$$
$$Q_{FC} = \sum_N D_C^N I_C^N T_F^N, \quad Q_{FF} = \sum_N D_C^N T_F^N I_F^N.$$

It is now easy to compute the two-loop Callan-Symanzik functions for a wide variety of semisimple groups and fermion representations.[14] For the fundamental representation of any compact simple group G ($G = C$ or F) the values of T_G^N and D_G^N are listed in Table I.[15] For the adjoint representation of G, $T_G^N = 1$ and $D_G^N = D(G)$, which is also given in Table I. For $G = \mathrm{SU}(n)$ the dimension and trace of the completely summetric rth-rank tensor representation are

$$D_G^N = \frac{(r+1)(r+2)\cdots(r+n-1)}{(n-1)!}, \quad T_G^N = \frac{(n+2)(n+3)\cdots(n+r)}{2n(r-1)!}. \tag{13}$$

III. RENORMALIZABILITY OF COUPLING-CONSTANT RELATIONS

Consider a gauge theory for the group $C \times F$ and some fermion representations. In this section we assume the existence of an unspecified renormalizable coupling-constant relation and derive the necessary conditions which must be satisfied by $C \times F$, the fermion representations, and the coupling-constant relation itself.

TABLE I. $D(G)$ is the dimension of group G. D_G^N and T_G^N are the dimension and trace of the fundamental representation.

G	$D(G)$	D_G^N	T_G^N
SU(n)	n^2-1	$n \geq 2$	$\frac{1}{2n}$
SO(n)	$\frac{1}{2}n(n-1)$	$n \geq 7$	$\frac{1}{n-2}$
Sp(n)	$\frac{1}{2}n(n+1)$	$n = \text{even} \geq 4$	$\frac{1}{n+2}$
G_2	14	7	$\frac{1}{4}$
F_4	52	26	$\frac{1}{3}$
E_6	78	27	$\frac{1}{4}$
E_7	133	56	$\frac{1}{3}$
E_8	248	248	1

Any coupling-constant relation is a line in coupling-constant space $c(t), f(t)$ where t is the single remaining coupling constant in the constrained Lagrangian. Because we want the theory to be perturbative, $c(t)$ and $f(t)$ are assumed to be analytic functions; in every other way they are unspecified. Since the constrained theory is renormalizable, in each order of perturbation theory it must be possible to absorb all divergences into wave-function renormalization and into a single bare coupling constant t_B. In other words, there must be a function $A(t)$ such that the divergences can be absorbed by[16]

$$t_B = t + \frac{A(t)}{d-4} + O\left(\left(\frac{1}{d-4}\right)^2\right). \tag{14}$$

Therefore the divergences in the Lagrangian must have the form

$$c(t_B) = c\left[t + \frac{A(t)}{d-4} + O\left(\left(\frac{1}{d-4}\right)^2\right)\right] = c(t) + \frac{A(t)}{d-4}\frac{dc(t)}{dt} + O\left(\left(\frac{1}{d-4}\right)^2\right),$$
$$f(t_B) = f(t) + \frac{A(t)}{d-4}\frac{df(t)}{dt} + O\left(\left(\frac{1}{d-4}\right)^2\right). \tag{15}$$

Since these divergences must also have the form of Eq. (4), it is necessary that

$$A(t)\,\frac{dc(t)}{dt}=a_C[c(t),f(t)],$$
$$A(t)\,\frac{df(t)}{dt}=a_F[c(t),f(t)]. \tag{16}$$

These equations must be satisfied order by order in perturbation theory; therefore at the two-loop level we have

$$A_1(t)\,\frac{dc(t)}{dt}=a_{(1)C}[c(t),f(t)],$$
$$A_1(t)\,\frac{df(t)}{dt}=a_{(1)F}[c(t),f(t)],$$
$$A_2(t)\,\frac{dc(t)}{dt}=a_{(2)C}[c(t),f(t)], \tag{17}$$
$$A_2(t)\,\frac{df(t)}{dt}=a_{(2)F}[c(t),f(t)],$$

where the perturbative expansions of $A(t)$ and a_G ($G=C$ or F) are

$$A(t)=A_{(1)}(t)+A_{(2)}(t),$$
$$a_G=a_{(1)G}+a_{(2)G}.$$

Equation (5) implies that the one- and two-loop parts of $\beta_G=\beta_{(1)G}+\beta_{(2)G}$ are given by $\beta_{(1)G}=-a_{(1)G}$ and $\beta_{(2)G}=-2a_{(2)G}$. Therefore Eqs. (17) can be cast into the form

$$\bar{A}_1(t)\,\frac{dc(t)}{dt}=\beta_{(1)C}[c(t),f(t)],$$
$$\bar{A}_1(t)\,\frac{df(t)}{dt}=\beta_{(1)F}[c(t),f(t)],$$
$$\bar{A}_2(t)\,\frac{dc(t)}{dt}=\beta_{(2)C}[c(t),f(t)], \tag{18}$$
$$\bar{A}_2(t)\,\frac{df(t)}{dt}=\beta_{(2)F}[c(t),f(t)],$$

where $\bar{A}_1(t)=-A_1(t)$ and $\bar{A}_2(t)=-2A_2(t)$. These are necessary conditions which must be satisfied by any renormalizable coupling-constant relation at the two-loop level. Since $\bar{A}_1(t)$ can be absorbed by a redefinition of the parameter t, the first two lines of Eq. (18) say that $c(t),f(t)$ is a trajectory of the one-loop β function. The sum of the first- and second-order parts of Eq. (18) implies that the same functions describe a trajectory of the two-loop form of the β function. In other words, renormalizability requires that coupling-constant relations describe a trajectory of the β function which is *stable* under perturbation. Note that this condition is independent of the gauge-fixing parameters and the mass terms.

This stability requirement is a kind of generalized symmetry condition. For scalar-spinor models almost all trajectories, which are stable at the one- and two-loop levels, are stable to all orders because they are associated with an internal symmetry or supersymmetry of the constrained Lagrangian.[4] As shown in Sec. IV, most semisimple gauge models do not have any stable trajectory and, therefore, cannot admit any higher symmetries. It is hoped that the small number of semisimple theories, which do have stable coupling-constant relations, are invariant under some set of transformations (possibly discrete) of the gauge and spinor fields.

Equation (6) can be used to rewrite Eq. (18) as

$$\tilde{A}_1(t)\,\frac{dc}{dt}=c^3V_C,$$
$$\tilde{A}_1(t)\,\frac{df}{dt}=f^3V_F,$$
$$\tilde{A}_2(t)\,\frac{dc}{dt}=c^3(B_{CC}c^2+B_{CF}f^2), \tag{19}$$
$$\tilde{A}_2(t)\,\frac{df}{dt}=f^3(B_{FC}c^2+B_{FF}f^2),$$

where $\tilde{A}_1$ and $\tilde{A}_2$ are proportional to $\bar{A}_1$ and $\bar{A}_2$. It is a matter of algebra to show that models which admit solutions to the above equations must satisfy one of the following sets of conditions on the β function coefficients:

Case 1:

$$V_C\neq 0,\quad V_F\neq 0,$$
$$B_{CC}B_{FF}-B_{CF}B_{FC}\neq 0, \tag{20}$$
$$V_CV_F>0,$$
$$B_{FF}V_C^2-B_{CC}V_F^2+V_CV_F(B_{FC}-B_{CF})=0.$$

This turns out to be the only case of interest. Any model which satisfies these conditions admits a renormalizable linear coupling-constant relation[17]:

$$\frac{c^2}{f^2}=\frac{V_F}{V_C}.$$

Solutions of this type are found in Sec. IV and used to construct unified field theories in Sec. V.

Case 2:

$$V_C=V_F=0. \tag{21}$$

In models of this kind $\beta_{(1)C}=0=\beta_{(1)F}$; in other words, there are no one-loop divergences to be absorbed. Therefore any two-loop trajectory (linear or nonlinear) is a possible coupling-constant relation. There are a number of models of this type, but they are not interesting from a practical point of view since they do not pick out a specific coupling-constant relation at the two-loop level.

TABLE II. The range of groups and fermion representations which were tested for the existence of a renormalizable coupling-constant relation. Example: The second line means that for $C = \mathrm{SU}(2)$ and $F = \mathrm{SU}(3\text{–}5)$ the fermion representation was allowed to contain up to two blocks which are singlets under C and adjoint representations under F, up to two blocks which are singlets under C and fundamental representations under $F, \ldots,$ and up to two blocks which are completely symmetric tensors under C and fundamental representations under F. Only second- and third-rank symmetric tensors are considered.

F	SING(C) × ADJ(F)	SING(C) × FUND(F)	ADJ(C) × SING(F)	ADJ(C) × ADJ(F)	ADJ(C) × FUND(F)	FUND(C) × SING(F)	FUND(C) × ADJ(F)	FUND(C) × FUND(F)	TENS × SING(F)	TENS × ADJ(F)	TENS × FUND(F)
						$C = \mathrm{SU}(2)$					
SU(3–5)	4	4	4	1	2	4	2	6	0	0	0
SU(3–5)	2	2	2	0	2	2	2	4	2	0	2
O(7–10)	4	4	4	0	0	4	2	6	2	0	1
Sp(4–10)	4	4	2	0	1	4	2	6	2	0	1
G_2	4	4	4	0	0	4	2	6	2	0	1
F_4	5	6	5	0	0	6	1	2	4	0	0
E_6	5	6	5	0	0	6	1	2	4	0	0
						$C = \mathrm{SU}(3)$					
SU(2)	2	2	2	0	2	2	2	4	2	0	2
SU(3–8)	4	4	4	1	2	4	2	6	0	0	0
SU(3–8)	2	2	2	0	2	2	2	4	2	0	2
O(7–14)	4	4	4	0	0	4	2	6	2	0	1
Sp(4–10)	4	4	4	0	1	4	2	6	2	0	1
Sp(12–16)	4	4	2	0	1	4	2	6	2	0	1
G_2	4	4	4	0	0	4	2	6	2	0	1
F_4	5	6	5	0	0	6	1	2	4	0	0
E_6	5	6	5	0	0	6	1	2	4	0	0
						$C = G_2$					
SU(2–10)	4	4	4	0	2	4	0	6	0	0	0
O(7–15)	4	4	1	4	4	2	4	4	0	0	0
Sp(4–16)	4	4	1	4	4	2	4	4	0	0	0
F_4	4	4	1	4	4	2	4	4	0	0	0
E_6	4	4	1	4	4	2	4	4	0	0	0
E_7	4	4	1	4	4	2	4	4	0	0	0
						$C = \mathrm{SU}(4)$					
SU(2–3)	2	2	2	0	2	2	2	4	2	0	2
SU(4–5)	4	4	4	1	2	4	2	6	0	0	0
SU(4–5)	2	2	2	0	2	2	2	4	2	0	2
O(7–10)	4	4	4	0	0	4	2	6	2	0	1
Sp(4–10)	4	4	2	0	1	4	2	6	2	0	1
G_2	4	4	4	0	0	4	2	6	2	0	1
F_4	5	6	5	0	0	6	1	2	4	0	0
E_6	5	6	5	0	0	6	1	2	4	0	0

TABLE II. (*continued*).

F	SING(C) × ADJ(F)	SING(C) × FUND(F)	ADJ(C) × SING(F)	ADJ(C) × ADJ(F)	ADJ(C) × FUND(F)	FUND(C) × SING(F)	FUND(C) × ADJ(F)	FUND(C) × FUND(F)	TENS × SING(F)	TENS × ADJ(F)	TENS × FUND(F)
$C = SU(5)$											
SU(2–4)	2	2	2	0	2	2	2	4	2	0	2
SU(5)	4	4	4	1	2	4	2	6	0	0	0
SU(5)	2	2	2	0	2	2	2	4	2	0	2
O(7–10)	4	4	4	0	0	4	2	6	2	0	1
Sp(4–10)	4	4	2	0	1	4	2	6	2	0	1
G_2	4	4	4	0	0	4	2	6	2	0	1
F_4	5	6	5	0	0	6	1	2	4	0	0
E_6	5	6	5	0	0	6	1	2	4	0	0

Case 3:

$$V_C \neq 0 \neq V_F,$$
$$B_{CC}B_{FF} - B_{CF}B_{FC} = 0\,, \tag{22}$$
$$V_C B_{CF} + V_F B_{CC} = 0.$$

This sort of theory can have a coupling-constant relation which is a one-loop trajectory along which there are no two-loop divergences. No examples of this phenomenon have been found.

Case 4:

$$V_C \neq 0 \neq V_F,\qquad B_{CC}B_{FF} - B_{CF}B_{FC} = 0, \tag{23}$$

and one of the following sets of conditions:

$$\frac{V_C}{B_{CF}} = \frac{V_F}{B_{FF}}\,,\qquad \frac{V_C}{B_{CC}} = \frac{V_F}{B_{FC}}\,, \tag{23a}$$

or

$$B_{CC} = 0 = B_{FC}\,,\qquad \frac{V_C}{B_{CF}} = \frac{V_F}{B_{FF}}\,, \tag{23b}$$

or

$$B_{CF} = 0 = B_{FF},\qquad \frac{V_C}{B_{CC}} = \frac{V_F}{B_{FC}}\,. \tag{23c}$$

Solutions of this type can also admit nonlinear coupling-constant relations, but none have been discovered.

IV. SEMISIMPLE GAUGE MODELS WITH ONE COUPLING CONSTANT

Equations (20)–(23) represent necessary conditions to be satisfied by any group $C \times F$ and fermion representation, which are compatible with the existence of a renormalizable coupling-constant relation. A computer is used to scan the groups and representations in Table II and to pick out those which satisfy any set of necessary conditions. C and F must be chosen from the list (Table I) of all compact simple Lie groups. The search is focused on groups for which $D(C) \ll D(F)$. As mentioned before, this type of theory is expected to provide a natural explanation of the relative strengths of strong and weak interactions. The case in which C (or F) is SU(2) is included for completeness even though it is not of practical interest. The most intensive search is conducted for models in which C is the smallest possible group, namely, SU(3). We also consider the pos-

TABLE III. Semisimple gauge theories which admit a renormalizable coupling-constant relation. As an example, the eighth line should be read as follows: the coupling-constant relation $c^2/f^2=1$ is renormalizable if $C\times F=\mathrm{SU}(3)\times\mathrm{Sp}(4)$ and the fermion representation contains p_1+p_2 blocks which are singlets under C and adjoint representations under F, p_3 blocks which are singlets under C and fundamental representations under F, . . . , and $1-p_3$ blocks which are completely symmetric rth-rank tensors ($r=2$) under C and fundamental representations under F. The letters p_1, p_2, p_3 can be zero or any positive integers.

$C\times F$	$\frac{c^2}{f^2}$	SING(C) × ADJ(F)	SING(C) × FUND(F)	ADJ(C) × SING(F)	ADJ(C) × ADJ(F)	ADJ(C) × FUND(F)	FUND(C) × SING(F)	FUND(C) × ADJ(F)	FUND(C) × FUND(F)	TENS(C) × SING(F)	TENS(C) × ADJ(F)	TENS(C) × FUND(F)	r
SU(2) × SU(3)	1	p_1	0	p_1	0	0	0	p_2	0	0	0	0	...
SU(2) × O(8)	1	$3+p$	2	p	0	0	0	0	2	0	0	0	...
SU(2) × Sp(4)	1	$2+p$	4	p	0	0	4	2	1	0	0	0	...
SU(3) × SU(2)	1	p_1+p_2	0	p_2	0	p_3	p_1	0	0	p_1	0	0	2
SU(3) × SU(3)	1	p_1+p_2	p_3	p_2	0	p_4	$p_1+p_3+p_5$	p_4+p_5	p_6	p_1	0	p_5	2
SU(3) × O(7)	1	$1+p$	4	p	0	0	4	0	2	0	0	0	...
SU(3) × O(11)	1	$3+p_1$	0	p_2	0	0	p_1-p_2	0	2	p_1-p_2	0	0	2
SU(3) × Sp(4)	1	p_1+p_2	p_3	p_2	0	p_3	p_1	2	$2-p_3$	p_1	0	$1-p_3$	2
SU(3) × F_4	1	$4+p_1+p_2$	0	p_2	0	0	$4+p_1$	0	1	p_1	0	0	2
G_2 × O(7)	1	p_1	0	p_1	0	1	0	3	3	0	0	0	...
G_2 × O(7)	1	0	4	1	0	1	2	3	1	0	0	0	...
G_2 × O(8)	1	p_1	0	p_1	0	0	0	p_2	0	0	0	0	...
G_2 × Sp(4)	1	p_1	4	p_1	0	3	2	1	2	0	0	0	...
G_2 × Sp(6)	1	$1+p_1$	2	p_1	0	1	2	2	0	0	0	0	...
SU(4) × SU(2)	1	p_1	2	p_1	0	0	0	0	0	2	0	1	2
SU(4) × O(8)	1	$3+p_1$	3	p_1	0	0	4	1	6	2	0	0	2
SU(5) × Sp(4)	1	p_1	0	p_1	0	p_2	0	0	0	0	0	0	...
SU(5) × Sp(4)	$\frac{5}{6}$	3	2	1	0	0	1	0	4	0	0	1	3
SU(5) × G_2	$\frac{5}{6}$	3	4	2	0	0	1	0	5	2	0	0	3
SU(5) × G_2	$\frac{5}{6}$	3	4	3	0	0	2	1	5	0	0	1	3

sibility that C is G_2, SU(4), or SU(5). In this case the symmetry is supposed to be broken dynamically down to SU(3).[18] Even larger choices for C can be envisioned, but they will be associated with enormous F groups.

The fermion representation is a sum of irreducible blocks, each of which is the product of irreducible representations of C and F. Singlet, fundamental, and adjoint representations of both C and F are considered. We also allow the completely symmetric tensor representation of C when C is SU(n).[19] For a given group $C\times F$ the fermion representation is typically allowed to contain up to eight different types of irreducible blocks; each type of block may occur up to four or five times in the overall representation. Table II gives the exact range of theories which have been examined. Approximately 100 semisimple groups have been considered; the fermion representation of each group has been allowed to have about $5^8 \sim 4\times 10^5$ different forms. Therefore over forty million models have been tested for a renormalizable coupling-constant relation.

The necessary conditions, Eqs. (20)–(23), are constraints on the invariants of the group and fermion representations. Since these invariants are characterized by integers, there are very few solutions of these equations. It turns out that all solutions are of types 1 and 2. Models of type 1 are listed in Table III, along with the corresponding linear renormalizable coupling-constant relation. Almost all solutions have coupling constants which are related by $c=f$. This means that the ratio of strengths of C and F interactions is of order $D(F)/D(C)$. To see this, note that the interaction strength of C gauge bosons is characterized by $|cg^C_{abc}|^2/4\pi$, where g^C_{abc} is a typical value of the C structure constants. For a fixed index a, $-ig^C_{abc}$ is a Hermitian matrix whose magnitude is characterized by the average of its $D(C)$ eigenvalues. Equation (1) shows that the sum of the squares of these eigenvalues is unity; therefore a typical squared eigenvalue has the magnitude $1/D(C)$. Thus the strength of the C interactions is of order[20] $(c^2/4\pi)[1/D(C)]$. Similarly, the F interaction strength is characterized by $(f^2/4\pi)\times[1/D(F)]=(c^2/4\pi)[1/D(F)]$. Therefore these models have a natural hierarchy of interactions which differ in strength by $D(F)/D(C)$. By embedding SU(3) in a small C and SU(2) × U(1) in a large F, this hierarchy can be used to account for the relative strength of strong and weak interactions.

The models of Table III have not been checked for anomalies. However, anomalies are no threat to the theories with $c=f$. Equation (20) shows that the fermion representations in this type of solution can always be doubled (or multiplied by any integer).[21] Since these representations can be divided evenly between right-handed and left-handed fermions,[14] anomalies can always be removed in this way. The few models with $c\neq f$ do not have this property; the number of fermion representations of each type is completely fixed by renormalizability and cannot be multiplied by an integer.

V. UNIFIED THEORIES OF STRONG, WEAK, AND ELECTROMAGNETIC INTERACTIONS

A. General procedure

The general method of constructing unified models is discussed in this subsection. Particular attention is focused on the renormalization-group formulas[22] which show that the unification mass M is very sensitive to the ratio $D(C)/D(F)$.

Consider any semisimple model with a renormalizable coupling-constant relation (see Table III), which is based on the group $C\times F$ and a fermion representation with generators C_a, F_k. For simplicity, suppose that the coupling-constant relation is $c=f$. The first step in the construction of a unified theory is to choose an SU(3) subgroup of C, generated by C_a ($a=1,\ldots,8$), and an SU(2) × U(1) subgroup of F, generated by F_k ($k=1,2,3$) and F_4. These groups are associated with the strong- and weak-electromagnetic interactions which are observed at low energy. The next step is to choose a U(1) subgroup [labeled $U(1)_Q$] of SU(2) × U(1), which will describe electromagnetism. The full symmetry group $C\times F$ is assumed to be broken dynamically according to the following hierarchy:

$$C\times F \to SU(3)\times SU(2)\times U(1) \to SU(3)\times U(1)_Q .$$

$SU(3)\times U(1)_Q$ is unbroken, and the corresponding massless gauge bosons are the gluons of quantum chromodynamics (QCD) and the photon. The breaking of SU(2) × U(1) down to $U(1)_Q$ is characterized by the mass, $M_W \sim 50$ GeV, of the weak-interaction bosons. The breaking of $C\times F$ down to SU(3) × SU(2) × U(1) is associated with the unification mass M, which is also the approximate mass of all other gauge bosons. At energies greater than M the S matrix is described by the full theory of Eq. (3). It was pointed out in Sec. IV that the ratio of strengths of the C and F interactions is of order $D(F)/D(C)$. At an energy $E<M$ the S matrix is described approximately (up to terms of order E/M) by an "effective" Lagrangian[23] for the group SU(3) × SU(2) × U(1). This Lagrangian is obtained from Eq. (3) by removing all bosons and fermions with masses of order M and greater; therefore, it has the form

$$\mathcal{L}_{\text{eff}}=-\tfrac{1}{4}(\partial_\mu A_{a\nu}-\partial_\nu A_{a\mu}-h_{BS}g^C_{abc}A_{b\mu}A_{c\nu})^2-\tfrac{1}{4}(\partial_\mu A_{k\nu}-\partial_\nu A_{k\mu}-h_B g^F_{klm}A_{l\mu}A_{m\nu})^2$$
$$-\tfrac{1}{4}(\partial_\mu B_\nu-\partial_\nu B_\mu)^2+\bar{\psi}(i\not\partial-h_{BS}\not A_a C_a-h_B\not A_k F_k-h'_B\not B F_4)\left(\frac{1-\gamma_5}{2}\right)\psi$$
$$+\text{ghost, gauge-fixing, and mass terms,} \tag{24}$$

where a,b,c are summed from 1 to 8; k,l,m are summed from 1 to 3, and only low-mass fermions are included. The renormalized coupling constants, h_S, h, and h', are constrained by the following symmetry condition: The coupling constant c of the full theory and the coupling constants of the corresponding effective theory are equal when the renormalization mass μ is chosen to be $\mu=M$:

$$h_S=h=h'=c. \tag{25}$$

The quantities h_S,h,h' are related to the usual coupling constants of QCD and the Weinberg-Salam model[8] (g_S,g,g') by three group-theoretical numbers, X_3,X_2,X_1. X_3 is defined by the embedding of SU(3) in C:

$$X_3 g^C_{abc}\equiv f_{abc}, \tag{26}$$

where $a,b,c=1,\ldots,8$ and the f_{abc} are the usual SU(3) structure constants with the normalization

$$f_{ade}f_{bde}=3\,\delta_{ab}. \tag{27}$$

An argument of the kind given in Sec. IV shows that a typical embedding choice implies

$$X_3{}^2\sim\tfrac{3}{8}D(C). \tag{28}$$

It is convenient to define SU(3) generators of the fermion representation which are normalized according to

$$\begin{aligned}&C'_a=X_3C_a,\\&[C'_a,C'_b]=if_{abc}C'_c.\end{aligned} \tag{29}$$

Therefore C'_3 is the usual generator of "color isospin." The easiest way to calculate X_3 is to consider any irreducible representation of C generated by matrices $\tilde{C}_a$ with the normalization of Eq. (2). The SU(3) subgroup is generated by matrices $\tilde{C}'_a$ normalized as in Eq. (29). Obviously,

$$X_3{}^2=\frac{\mathrm{Tr}(\tilde{C}'_3\tilde{C}'_3)}{\mathrm{Tr}(\tilde{C}_3\tilde{C}_3)}. \tag{30}$$

The denominator is just the T^N_C, given in Table I and Eq. (13). The numerator can be computed directly from the "color isospin" decomposition of C's representation.

X_2 is defined by the embedding of SU(2) in F:

$$X_2 g^F_{klm}=\epsilon_{klm}, \tag{31}$$

where k,l,m range from 1 to 3. For a typical embedding choice we expect

$$X_2{}^2\sim\tfrac{2}{3}D(F). \tag{32}$$

Let F'_k be the following SU(2) generators of the fermion representation:

$$\begin{aligned}&F'_k=X_2F_k,\\&[F'_k,F'_l]=i\epsilon_{klm}F'_m.\end{aligned} \tag{33}$$

In other words, F'_3 is the usual "weak isospin" generator. To calculate X_2, consider any irreducible representation of F with generators $\tilde{F}_k$, normalized as in Eq. (2). If the $\tilde{F}'_k$ are the corresponding SU(2) generators normalized as in Eq. (33), then

$$X_2{}^2=\frac{\mathrm{Tr}(\tilde{F}'_3\tilde{F}'_3)}{\mathrm{Tr}(\tilde{F}_3\tilde{F}_3)}. \tag{34}$$

As before, the denominator is given by the T^N_F of Table I and Eq. (13); the numerator is determined by the "weak isospin" decomposition of F's representation. Notice that there is only a finite number of ways of embedding SU(3) [or SU(2)] in C (or F); therefore X_3 and X_2 must be chosen from a discrete set of possible values.

X_1 is determined by the relationship of $\mathrm{U}(1)_Q$ and SU(2)×U(1). The electric charge Q, which generates $\mathrm{U}(1)_Q$, must be a linear combination of F_3 and F_4. Since we want the W bosons to have charges ±1, Q must have the form

$$Q=X_2F_3+X_1F_4. \tag{35}$$

It is useful to define $F'_4\equiv X_1F_4$ so that

$$Q=F'_3+F'_4. \tag{36}$$

Thus F'_4 is one-half the usual "weak hypercharge." To calculate X_1, consider any irreducible representation of F, generated by matrices $\tilde{F}_k$ with the normalization of Eq. (2). Then

$$X_1{}^2+X_2{}^2=\frac{\mathrm{Tr}Q^2}{\mathrm{Tr}(\tilde{F}_3\tilde{F}_3)}. \tag{37}$$

The denominator is just T^N_F, and the numerator is determined by the charge assignments in the representation.

The coupling constant of QCD (g_S) and the coupling constants of the Weinberg-Salam model (g and g') are related to the h's by

$$g_S=\frac{h_S}{X_3},\quad g=\frac{h}{X_2},\quad g'=\frac{h'}{X_1}. \tag{38}$$

The effective Lagrangian can now be written in a more familiar form,

$$\mathcal{L}_{\text{eff}} = -\tfrac{1}{4}(\partial_\mu A_{a\nu} - \partial_\nu A_{a\mu} - g_{BS} f_{abc} A_{b\mu} A_{c\nu})^2 - \tfrac{1}{4}(\partial_\mu A_{k\nu} - \partial_\nu A_{k\mu} - g_B \epsilon_{klm} A_{l\mu} A_{m\nu})^2$$
$$- \tfrac{1}{4}(\partial_\mu B_\nu - \partial_\nu B_\mu)^2 + \bar\psi(i\partial\!\!\!/ - g_{BS} A\!\!\!/_a C'_a - g_B A\!\!\!/_k F'_k - g'_B B\!\!\!/ F'_4)\left(\frac{1-\gamma_5}{2}\right)\psi$$
$$+\text{ghost, gauge-fixing, and mass terms,} \qquad (39)$$

where a,b,c and k,l,m run from 1 to 8 and 1 to 3, respectively. This Lagrangian describes the low-energy domain of the full $C\times F$ theory if the coupling constants lie on a trajectory which satisfies Eq. (25); i.e., at $\mu = M$,

$$X_3 g_S = X_2 g = X_1 g' = c. \qquad (40)$$

Equations (28) and (32) indicate that a typical embedding of SU(3) and SU(2) in C and F will give

$$\frac{g_S{}^2}{g^2} = \frac{X_2{}^2}{X_3{}^2} \sim \frac{16}{9}\frac{D(F)}{D(C)} \qquad (41)$$

at $\mu = M$. Therefore, by choosing $D(C) \ll D(F)$, we can account for part of the great relative strength of the strong interactions.

The electric charge and the Weinberg angle are defined in the usual way:

$$e = \frac{gg'}{(g^2+g'^2)^{1/2}}, \qquad \tan\theta = \frac{g'}{g}. \qquad (42)$$

In this language the "boundary conditions" of Eq. (40) are

$$\frac{g_S{}^2}{e^2} = \frac{X_1{}^2 + X_2{}^2}{X_3{}^2}, \qquad \sin^2\theta = \frac{X_2{}^2}{X_1{}^2+X_2{}^2}, \qquad g_S{}^2 = \frac{c^2}{X_3{}^2} \qquad (43)$$

at $\mu = M$.

At a renormalization point $\mu \ll M$ the coupling constants $g_S(\mu)$, $g(\mu)$, and $g'(\mu)$ will differ from the values in Eq. (40) by a power series in $c^2\ln(1+M^2/\mu^2)$. The power series can be added up[22] by computing the coupling-constant trajectories of the renormalization group. These trajectories have the following form:

$$\frac{1}{g_S(\mu)^2} = \frac{1}{g_S(M)^2} - \frac{1}{8\pi^2}(2S_3 - 11)\ln\frac{\mu}{M},$$
$$\frac{1}{g(\mu)^2} = \frac{1}{g(M)^2} - \frac{1}{12\pi^2}(2S_2 - 11)\ln\frac{\mu}{M}, \qquad (44)$$
$$\frac{1}{g'(\mu)^2} = \frac{1}{g'(M)^2} - \frac{1}{12\pi^2} S_1 \ln\frac{\mu}{M}.$$

S_1, S_2, S_3 are proportional to traces of the generators of the fermion representation; for example, if $\mathcal{L}_{\text{eff}}$ contains only the (1) and (3) representations of SU(3) (i.e., leptons and quarks) and the (1), (2), and (3) representations of SU(2), then

$S_3 = \frac{1}{18}$ (number of quarks in $\mathcal{L}_{\text{eff}}$),

$S_2 = \frac{1}{4}$ (number of lepton doublets + number of quark doublets in $\mathcal{L}_{\text{eff}}$) + number of lepton and quark triplets in $\mathcal{L}_{\text{eff}}$, (45)

S_1 = sum of $F_4'^2$ over all fermions in $\mathcal{L}_{\text{eff}}$.

With the aid of the "boundary conditions" of Eq. (40) and the definitions in Eq. (42), Eq. (44) can be solved for

$$\ln\frac{M}{\mu} = \frac{6\pi\{[1/\alpha(\mu)]X_3{}^2 - [1/\alpha_S(\mu)](X_1{}^2+X_2{}^2)\}}{3(X_1{}^2+X_2{}^2)(11-2S_3) + 2X_3{}^2(S_1+2S_2-11)},$$
$$\sin^2\theta(\mu) = \frac{3X_2{}^2(11-2S_3) - 2X_3{}^2(11-2S_2) + [\alpha(\mu)/\alpha_S(\mu)][2X_2{}^2S_1 + 2X_1{}^2(11-2S_2)]}{3(X_1{}^2+X_2{}^2)(11-2S_3) + 2X_3{}^2(S_1+2S_2-11)}, \qquad (46)$$
$$\left(\frac{c^2}{4\pi}\right)_{\mu=M} = \frac{X_1{}^2}{[\cos^2\theta(\mu)/\alpha(\mu)] - (S_1/3\pi)\ln(M/\mu)},$$

where $\alpha = e^2/4\pi$ and $\alpha_S = g_S{}^2/4\pi$.

These equations are important constraints on the following procedure for constructing unified theories. The first step is to take a model from Table III and choose an embedding of SU(3), SU(2), and $U(1)_Q$ in the full group $C\times F$. This determines X_1, X_2, X_3. Next, the full particle spectrum as given in Table III is divided into low ($<M$) and high ($\geq M$) mass sectors. S_1, S_2, S_3 are determined by the quantum numbers of the low-mass particles. At low energies (say, $\mu^2 = 8\ \text{GeV}^2$) we have the experimental results[24]

$$\alpha \simeq \tfrac{1}{137}\,, \tag{47}$$

$$0.2 \lesssim \alpha_S \lesssim 0.4 \tag{48}$$

Finally, the X's, S's, Eqs. (47)–(48), and Eq. (46) are used to compute M, $\sin^2\theta$ at $\mu=\sqrt{8}$ GeV, and $c^2/4\pi$ at $\mu=M$. In a satisfactory model these output numbers should satisfy the following constraints: M is expected to be low (say, 10^3 or 10^4 GeV)[25] for reasons outlined in Sec. I, $\sin^2\theta$ at $\mu=\sqrt{8}$ GeV should match the experimental value[26]

$$\sin^2\theta \simeq 0.35\,, \tag{49}$$

and $c^2/4\pi$ at $\mu=M$ should be reasonably small if perturbation theory is to be trusted.

Before constructing such a theory, it is useful to invert the above logic and demonstrate that a low value of M can be achieved only in models with large ratios $D(F)/D(C)$. First, notice that Eq. (46) implies

$$\frac{X_2{}^2}{X_3{}^2}=\frac{[\sin^2\theta(\mu)]/\alpha(\mu)+(1/3\pi)(11-2S_2)\ln(M/\mu)}{1/\alpha_S(\mu)+(1/2\pi)(11-2S_3)\ln(M/\mu)}\,. \tag{50}$$

Next, assume that there are not too many particles in the low-mass spectrum. For example, suppose that hadrons are described by three left-handed and three right-handed quark doublets (each available in three colors); leptons are in three left-handed doublets, three right-handed doublets, and assorted singlets. Then Eq. (45) shows that

$$S_3=2\,,\quad S_2=6\,. \tag{51}$$

Now, substituting Eq. (51), Eqs. (47)–(49), and $\mu=\sqrt{8}$ GeV into Eq. (50),

$$\frac{X_2{}^2}{X_3{}^2}=\frac{48.1-0.106\ln M}{1.35+1.11\ln M}\,. \tag{52}$$

This curve is approximated by

$$M\simeq 10^{17/(X_2{}^2/X_3{}^2)}\ \text{GeV}. \tag{53}$$

Therefore, any model with a low unification mass, say $M\sim 10^3$ GeV, must have $X_2{}^2/X_3{}^2\sim 6$; then Eq. (41) shows that such a model must[27] have $D(F)/D(C)\sim 3$.

Equations (50) and (53) also hold for unified theories which are based on a simple group G. In these models a typical embedding of SU(3) and SU(2) in G gives [see Eq. (41)]

$$\frac{X_2{}^2}{X_3{}^2}\sim\frac{16}{9}\,\frac{D(G)}{D(G)}\sim 1.8.$$

Therefore the unification mass M is almost always large[28]: $M\sim 10^9$ GeV. This is easy to understand: If a simple group is used, the strong and weak interactions have the same strength at $\mu=M$. Therefore the great relative strength of the strong interaction is entirely due to a large renormalization effect; this requires a huge M. In a semisimple model with $D(F)/D(C)\sim 3$, the large relative strength of the C interactions at $\mu=M$ accounts for part of the strength of the strong interactions; therefore the renormalization effect can be reduced, and M is drastically smaller.

Table III contains several models with large $D(F)/D(C)$; the relevant groups are

SU(3) × O(7),

SU(3) × O(11),

SU(3) × F_4,

G_2 × O(8),

SU(4) × O(8).

For each group there is only a limited number of ways of embedding SU(3), SU(2), and $U(1)_Q$. Each embedding choice gives a definite value of $\sin^2\theta$ and a certain particle spectrum. The next subsection outlines a theory based on $C\times F=$ SU(3) × O(11) which is consistent with experimental information on $\sin^2\theta$ and the low-energy particle spectrum. Almost all other models, which can be embedded in the full groups listed above, are not compatible with low-energy phenomenology.

B. SU(3)×O(11) model

Consider the model in Table III with $C\times F=$ SU(3) × O(11). For $p_1=p_2=0$ the fermion representation contains two blocks of quarks, which transform as (3, 11) under SU(3) × O(11), and three blocks of leptons, which transform as (1, 55). It turns out that SU(3), SU(2), and $U(1)_Q$ cannot be embedded in this model in a way which is consistent with experiment. This subsection is a discussion of the "doubled" version of this theory with four (3,11) blocks and six (1, 55) blocks. The known low-energy phenomena can be embedded in the "doubled" model in a unique way which is described below.

X_3 is uniquely determined since $C=$ SU(3). To calculate X_3, consider the (3) representation of SU(3), generated by $\tilde{C}_a$ with the normalization of Eq. (2). Table I implies that the denominator of Eq. (30) is $\frac{1}{6}$. Since the (3) representation contains one doublet and one singlet of "color isospin," the numerator of Eq. (30) is $\frac{1}{2}$. Therefore

$$X_3{}^2=3. \tag{54}$$

To embed SU(2) × U(1) in O(11), consider the (11) representation of O(11). It is generated by imaginary, antisymmetric matrices $\tilde{F}_k$, normalized as in Eq. (2). Therefore Table I implies

$$\mathrm{Tr}\tilde{F}_3\tilde{F}_3=\tfrac{1}{9}\,. \tag{55}$$

Let J_k and L_k ($k=1,2,3$) be the 4×4 imaginary antisymmetric matrices which generate the $(\frac{1}{2},\frac{1}{2})$ representation of $\mathrm{SU}(2)\times\mathrm{SU}(2)$:

$$\begin{aligned}
&[J_k,J_l]=i\epsilon_{klm}J_m\,,\\
&[L_k,L_l]=i\epsilon_{klm}L_m\,,\\
&[J_k,L_l]=0\,,\\
&\mathrm{Tr}(J_kJ_l)=\delta_{kl}\,,\\
&\mathrm{Tr}(L_kL_l)=\delta_{kl}\,,\\
&\mathrm{Tr}(J_kL_l)=0\,.
\end{aligned} \tag{56}$$

Then a set of 11×11 imaginary antisymmetric matrices, which satisfy Eq. (55) and which generate an $\mathrm{SU}(2)\times\mathrm{U}(1)$ subgroup of O(11), are ($k=1,2,3$)

$$\begin{aligned}
\tilde{F}_k&=\frac{1}{\sqrt{18}}\begin{bmatrix} J_k & & & & \\ & J_k & & 0 & \\ & & 0 & & \\ & 0 & & 0 & \\ & & & & 0\end{bmatrix},\\
\tilde{F}_4&=\frac{1}{6\sqrt{7}}\begin{bmatrix} L_3 & & & \\ & 5L_3 & & 0 \\ & & \begin{pmatrix}0 & i\\ -i & 0\end{pmatrix} & \\ & 0 & & 0\end{bmatrix}.
\end{aligned} \tag{57}$$

To calculate X_2, notice that Eq. (55) means that the denominator of Eq. (34) is $\frac{1}{9}$. Because each $(\frac{1}{2},\frac{1}{2})$ representation of $\mathrm{SU}(2)\times\mathrm{SU}(2)$ contains two "weak isospin" doublets, the numerator of Eq. (34) is $4(\frac{1}{2})=2$. Thus

$$X_2{}^2=18. \tag{58}$$

Since $X_2{}^2/X_3{}^2=6$, this model will yield $M\simeq 10^3$ GeV.

X_1 is now determined by the embedding of $\mathrm{U}(1)_Q$ in $\mathrm{SU}(2)\times\mathrm{U}(1)$. Equations (35), (57), and (58) show that there will be quark doublets with charges $\frac{2}{3}$, $-\frac{1}{3}$ only if

$$X_1{}^2=28. \tag{59}$$

With this choice of embedding the (11) representation contains four "weak isospin" doublets with charges

$$\begin{pmatrix}\frac{2}{3}\\ -\frac{1}{3}\end{pmatrix},\ \begin{pmatrix}\frac{1}{3}\\ -\frac{2}{3}\end{pmatrix},\ \begin{pmatrix}\frac{4}{3}\\ \frac{1}{3}\end{pmatrix},\ \begin{pmatrix}-\frac{1}{3}\\ -\frac{4}{3}\end{pmatrix} \tag{60}$$

and three singlets with charges $\frac{1}{3}$, $-\frac{1}{3}$, 0. Evidently, the u,d doublet and c,s doublet will have to lie in different (3,11) blocks. The adjoint representation of O(11) decomposes into $\mathrm{SU}(2)\times\mathrm{U}(1)$ multiplets with quantum numbers $(2|\vec{\mathrm{F}}'|+1,F_4')$:

$$\begin{aligned}
(55)=&(1,\pm\tfrac{5}{3})+(1,\pm1)+(1,\pm\tfrac{2}{3})\\
&+2(1,\pm\tfrac{1}{3})+3(1,0)+(2,\pm\tfrac{7}{6})\\
&+(2,\pm\tfrac{5}{6})+2(2,\pm\tfrac{1}{2})+2(2,\pm\tfrac{1}{6})\\
&+(3,\pm1)+(3,\pm\tfrac{2}{3})+2(3,0).
\end{aligned} \tag{61}$$

Observe that the two blocks with quantum numbers $(2,-\frac{1}{2})$ are suitable[29] for describing the ν_e,e^- doublet since they have the charge content

$$\begin{pmatrix}0\\ -1\end{pmatrix}.$$

Substituting these values of X_1,X_2,X_3 into Eq. (43) gives the following "boundary conditions" on $g_S(\mu)$, $\theta(\mu)$, $e(\mu)$:

$$\begin{aligned}
&\frac{g_S(M)^2}{e(M)^2}=15.3,\\
&\sin^2\theta(M)=0.39,\\
&g_S(M)^2=\tfrac{1}{3}(c^2)_{\mu=M}.
\end{aligned} \tag{62}$$

Thus the strong interaction is many times larger than the electromagnetic interaction at the unification energy.

We can work backward from phenomenology to show that the above method of embedding SU(3), SU(2), and $\mathrm{U}(1)_Q$ in $\mathrm{SU}(3)\times\mathrm{O}(11)$ is almost unique. Since $C=\mathrm{SU}(3)$, there is obviously only one choice of X_3, $X_3{}^2=3$. Equation (53) demonstrates that a unification mass[25] of 10^3 GeV can be achieved only if $X_2{}^2/X_3{}^2\sim 6$. Therefore SU(2) must be embedded in O(11) so that $X_2{}^2\sim 18$. Equations (34) and (55) then imply that the (11) representation must have an SU(2) decomposition such that $\mathrm{Tr}\tilde{F}_3'\tilde{F}_3'\sim 2$. This means that the (11) representation must be decomposed into four doublets and three singlets or into one triplet and eight singlets. Since the u,d quark doublet must be contained in the (3,11) representation of $\mathrm{SU}(3)\times\mathrm{O}(11)$, the (11) must contain three singlets and four doublets, one of which has charges $\frac{2}{3},-\frac{1}{3}$. Because of the reality of the (11) representation, another of the doublets must have charges[30] $\frac{1}{3},-\frac{2}{3}$. The remaining two doublets and three singlets must have charge assignments of the form

$$\begin{pmatrix}q_1\\ q_1-1\end{pmatrix},\ \begin{pmatrix}1-q_1\\ -q_1\end{pmatrix},\ q_2,-q_2,0.$$

The charge q_1 is determined in terms of q_2 by the requirement that the Weinberg angle be near the experimental value, $\sin^2\theta \sim 0.4$ at low μ. Since the value of M is low, we expect the renormalization effect to be small; therefore we should require $\sin^2\theta(M) \sim 0.4$. Because $X_2{}^2 \sim 18$, Eq. (43) implies $X_1{}^2 \sim 27$. Then Eq. (37) shows that we must have $\mathrm{Tr}Q^2 \sim 5$. This means that q_1 and q_2 are related by

$$\mathrm{Tr}Q^2 = \tfrac{10}{9} + 2q_1{}^2 + 2(q_1 - 1)^2 + 2q_2{}^2 \sim 5. \tag{63}$$

A second constraint on q_1 and q_2 is provided by the requirement that the charge of ν_e turn out to be zero. The adjoint representation of O(11) must contain a "weak isospin" doublet with charges $0, \mp 1$, which describes the ν_e and e^- or e^+. Since the adjoint representation transforms as part of the direct product $(11) \times (11)$, it will contain a doublet, one component of which has vanishing charge, only if $q_2 = \pm\frac{1}{3}$ or $q_2 = \pm\frac{2}{3}$ or $q_2 = \pm q_1$ or $q_2 = \pm(q_1 - 1)$ or $q_1 = 0$ or $q_1 - 1 = 0$. Therefore the pair q_1, q_2 must be chosen from the following discrete set of values, which are dictated by the above choices in conjunction with Eq. (63):

$$\begin{aligned}
&q_2 = \pm\tfrac{1}{3}, \quad q_1 \sim \tfrac{4}{3} \text{ or } -\tfrac{1}{3},\\
&q_2 = \pm\tfrac{2}{3}, \quad q_1 \sim \tfrac{6}{5} \text{ or } -\tfrac{1}{5},\\
&q_2 = \pm q_1, \quad q_1 \sim 1 \text{ or } -\tfrac{1}{3},\\
&q_2 = \pm(q_1 - 1), \quad q_1 \sim \tfrac{4}{3} \text{ or } 0,\\
&q_1 = 0, \quad q_2 \sim \pm 1,\\
&q_1 - 1 = 0, \quad q_2 \sim \pm 1.
\end{aligned} \tag{64}$$

For all of these alternatives each (11) representation contains only one doublet of quarks with charges $\frac{2}{3}, -\frac{1}{3}$. This is the reason it is necessary to consider the "doubled" $\mathrm{SU}(3) \times \mathrm{O}(11)$ model. In the undoubled version the u, d and c, s doublets must be in the same (3,11) block; therefore Eq. (64) cannot be satisfied, and either θ or the ν_e charge is wrong. In the "doubled" model the left-handed u, d and c, s doublets are in separate (3,11) blocks; the corresponding right-handed particles must be in an additional two (3,11) blocks. Only the first choice in Eq. (64) allows the (11) representation to contain a second quark with charge $-\frac{1}{3}$. Since some experiments[26] suggest that such a "bottom" or "b" quark does exist, we only discuss the embedding with $q_2 = \pm\frac{1}{3}$ and $q_1 = \frac{4}{3}$ (or $-\frac{1}{3}$) in this subsection.

We present two ways of embedding the complete low-mass particle spectrum in the above model. The first method depends on the assumption that dynamical symmetry breaking picks out the following low-mass quarks from each of the four (3,11) blocks:

$$\text{each}(3,11) \to 3 \text{ colors} \times \left[\begin{pmatrix} \frac{2}{3} \\ -\frac{1}{3} \end{pmatrix} + (-\tfrac{1}{3})\right]$$
$$+ \text{ states with masses} \sim M.$$

In one (3,11) block the "weak isospin" doublet with charges $\frac{2}{3}, -\frac{1}{3}$ is identified as the left-handed u, d_C quark pair and the singlet with charge $-\frac{1}{3}$ is the left-handed b quark

$$\begin{pmatrix} u \\ d_C \end{pmatrix}_L + (b)_L.$$

The other three (3,11) blocks contain the following low-mass states:

$$\begin{pmatrix} c \\ s_C \end{pmatrix}_L + (a)_L,$$
$$\begin{pmatrix} u \\ b \end{pmatrix}_R + (d)_R,$$
$$\begin{pmatrix} c \\ a \end{pmatrix}_R + (s)_R,$$

where the subscript "C" refers to Cabibbo rotation and "a" is a new quark with charge $-\frac{1}{3}$ which has not yet been observed. There are six (1,55) blocks, of which three are left-handed and three are right-handed. Dynamical symmetry breaking is supposed to pick out the following low-mass leptons from each of the three left-handed (1,55)'s:

$$\text{each } (1,55)_L \to \begin{pmatrix} 0 \\ -1 \end{pmatrix}_L + (0)_L$$
$$+ \text{ states with masses} \sim M.$$

These are interpreted as

$$\begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L + (X^0)_L,$$
$$\begin{pmatrix} \nu_\mu \\ \mu^- \end{pmatrix}_L + (Y^0)_L,$$
$$\begin{pmatrix} \nu_P \\ P^- \end{pmatrix}_L + (U^0)_L,$$

where P^-, ν_P are the new leptons discovered at SPEAR and X^0, Y^0, U^0 are leptons which have not yet been produced. Each of the three right-handed (1,55)'s has the particle content

$$\text{each } (1,55)_R \to \begin{pmatrix} 0 \\ -1 \end{pmatrix}_R + \text{ states with masses} \sim M.$$

These doublets contain the following particles:

$$\begin{pmatrix} X^0 \\ e^- \end{pmatrix}_R, \begin{pmatrix} Y^0 \\ \mu^- \end{pmatrix}_R, \begin{pmatrix} U^0 \\ P^- \end{pmatrix}_R.$$

Equation (45) can now be used to find S_1, S_2, S_3 from the quantum numbers of the low-mass particles

$$S_3 = 2, \quad S_2 = \tfrac{9}{2}, \quad S_1 = 5. \tag{65}$$

Substituting Eqs. (65), (54), (58)–(59), and (47) into Eq. (46) yields

$$\log_{10}(M/\mu) = 3.41 - \frac{0.38}{\alpha_S(\mu)}, \quad \sin^2\theta(\mu) = 0.37 + \frac{0.0022}{\alpha_S(\mu)}, \quad \left(\frac{c^2}{4\pi}\right)_{\mu=M} = \frac{28}{82.1 + 0.17/\alpha_S(\mu)} \tag{66}$$

for low values of μ. Analyses of scaling behavior[24] suggest that $\alpha_S(\mu)$ lies between 0.2 and 0.4 at $\mu^2 = 8\ \text{GeV}^2$. If we take $\alpha_S = 0.4$, then Eq. (66) gives

$$M = 815\ \text{GeV}, \quad \sin^2\theta = 0.38, \quad \left(\frac{c^2}{4\pi}\right)_{\mu=M} = 0.34. \tag{67}$$

Notice that the particle spectrum and Weinberg angle of this model make it identical to one of the theories (the "E" model) which was picked out by Barnett[26] to describe weak and electromagnetic interactions. Therefore the unified theory of this section is compatible with experiments in the low-energy domain. If α_S is as low as 0.2, then we get

$$M = 92\ \text{GeV}, \quad \sin^2\theta = 0.38, \quad \left(\frac{c^2}{4\pi}\right)_{\mu=M} = 0.34. \tag{68}$$

The value of M in Eq. (68) is too low to be consistent with the initial assumption, $M \gg M_W$, which justifies the use of $\mathcal{L}_{\text{eff}}$ instead of the full theory in the low-energy domain. This problem can be avoided by introducing four more quarks into the low-mass particle spectrum. Each of the four (3, 11) blocks is now assumed to have the particle content

$$\text{each } (3,11) \to 3 \text{ colors} \times \left[\begin{pmatrix} \frac{2}{3} \\ -\frac{1}{3} \end{pmatrix} + \begin{pmatrix} \frac{1}{3} \\ -\frac{2}{3} \end{pmatrix} + (-\tfrac{1}{3})\right]$$
$$+ \text{ states with masses } \sim M.$$

The u, d, c, s, b, and a quarks are identified as before; now, however, there are four additional quarks with charges $\frac{1}{3}$, $-\frac{2}{3}$, $\frac{1}{3}$, and $-\frac{2}{3}$. The lepton spectrum is unchanged. Equation (45) shows that

$$S_3 = \tfrac{10}{3}, \quad S_2 = \tfrac{15}{2}, \quad S_1 = \tfrac{17}{3}, \tag{69}$$

and Eq. (46) gives

$$\log_{10}(M/\mu) = 5.12 - \frac{0.57}{\alpha_S(\mu)}, \quad \sin^2\theta(\mu) = 0.39 - \frac{0.0002}{\alpha_S(\mu)}, \quad \left(\frac{c^2}{4\pi}\right)_{\mu=M} = \frac{28}{76.5 + 0.82/\alpha_S(\mu)} \tag{70}$$

at low μ. If $\alpha_S = 0.4$ at $\mu^2 = 8\ \text{GeV}^2$, then

$$M = 1.42 \times 10^4\ \text{GeV}, \quad \sin^2\theta = 0.39, \quad \left(\frac{c^2}{4\pi}\right)_{\mu=M} = 0.36. \tag{71}$$

On the other hand, if $\alpha_S = 0.2$, then

$$M = 526\ \text{GeV}, \quad \sin^2\theta = 0.39, \quad \left(\frac{c^2}{4\pi}\right)_{\mu=M} = 0.35. \tag{72}$$

As long as the "a" quark and the four additional quarks are sufficiently heavy, these models[31] will be in agreement with experimental data on the strong and weak interactions.

Figure 2 gives a physical picture of the models which are associated with $\alpha_S(\sqrt{8}\ \text{GeV}) \simeq 0.4$ [Eqs. (67) and (71)]. The unification energy is $M \sim 10^3$ or 10^4 GeV. For energies below M the S matrix

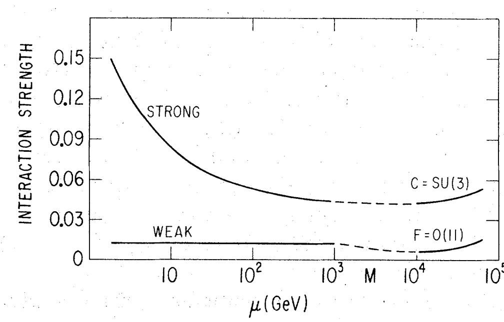

FIG. 2. The strengths of the various interactions in an SU(3) × O(11) model with unification energy $M \sim 10^3$ or 10^4 GeV. As μ approaches M from below the strong and weak interactions blend into the C and F interactions, respectively.

can be calculated approximately by using the "effective" $SU(3)\times SU(2)\times U(1)$ theory. An argument of the type given in Sec. IV shows that the size of the strong interaction is characterized by $\frac{3}{8}\alpha_S(\mu)$. This quantity falls from 0.150 at $\mu=\sqrt{8}$ GeV to 0.045 at $\mu=M$. The strength of the weak interaction is given by $\frac{2}{3}g(\mu)^2/4\pi$, which has a nearly constant value of 0.012 as μ varies between $\sqrt{8}$ GeV and M. At the unification energy many new leptons and F gauge bosons appear suddenly. At an energy above M the S matrix must be calculated from the full $C\times F$ theory. The strength of the C interaction is described by $\frac{1}{8}(c^2/4\pi)_{\mu=M}\simeq 0.044$ while the size of the F interaction is given by $\frac{1}{55}(c^2/4\pi)_{\mu=M}\simeq 0.006$. Thus, even at 10^3 or 10^4 GeV, there is a natural hierarchy of interaction strengths which is scaled by the relative sizes of the C and F groups. Since the full $SU(3)\times O(11)$ model is not asymptotically free, both the C and F interactions grow as the energy becomes substantially greater than M.[32]

VI. CONCLUSIONS

In general, there are two coupling constants in a gauge model for the group $C\times F$. Usually, it is not possible to relate these coupling constants in a renormalizable (i.e., cutoff independent) way. In this work a computer is used to scan millions of semisimple gauge theories for various groups and fermion representations. The computer picks out those groups and representations which are compatible with the existence of any renormalizable coupling-constant relation at the two-loop level. A small number of such models is found. In every case the coupling-constant relation turns out to be linear and fixes the C and F interaction strengths so that they are inversely proportional to the size of the respective groups. Thus a semisimple gauge model with one coupling constant, which is based on a small C group and a large F group, will have a natural hierarchy of interaction strengths. This hierarchy can be exploited to construct unified theories of strong, weak, and electromagnetic interactions, in which the unification energy M is low. At energies above M, the C interactions will be stronger than the F interactions. Therefore at very high energies the strong interactions, which are embedded in C, are larger than the weak interactions, which are contained in F. In a sense the great strength of the strong interactions derives from the fact that color SU(3) is one of the smallest Lie groups. This group-theoretical effect accounts for much of the observed difference between the strong and weak interactions. Since the remaining difference can be explained by a relatively small renormalization effect, the unification energy M can be low. These ideas are illustrated in an $SU(3)\times O(11)$ model in which M is 10^3 or 10^4 GeV. At low energies the model contains the usual quark and lepton spectrum, and it is consistent with phenomenology. As the energy approaches M from below, we expect a large number of leptons and O(11) gauge bosons to appear suddenly.

It is useful to contrast these semisimple models with unified theories based on simple groups. In the latter case the entire difference between strong and weak interactions is explained by a renormalization effect; therefore M is very large, 10^9 to 10^{17} GeV. In the semisimple theories it is not necessary to extrapolate the principles of local quantum field theory up to such high energies. For this reason such models are less speculative than the ones based on simple groups. The small values of M in the semisimple models are also more consistent with Gildener's discovery that spontaneous symmetry breaking cannot produce mass ratios greater than $\alpha^{-1/2}$. Finally, the semisimple approach has the theoretical advantage that renormalizability almost fixes the gauge group and the fermion representation.

There may be many types of microscopic structure for which the semisimple theories of Table III are effective Lagrangians at low energies. In fact, the main virtue of the work in this paper is that it provides a unification of all interactions without making a commitment to a particular picture of physics at extremely short distances. Nevertheless, it is amusing to propose that spontaneously compactified gravity[33] in $4+N$ dimensions is an example of a microscopic structure which generates semisimple unified theories at relatively low energies. Consider the coupling of gravitational and Yang-Mills fields for a simple group H in $4+N$ dimensions. The Lagrangian can be written in terms of one parameter with dimension (the gravitational constant G) and one dimensionless parameter (the gauge coupling constant h). Cremmer and Scherk[33] discovered that the presence of the gauge fields causes N of the spacetime dimensions to compactify into a hypersphere while the other four dimensions remain flat or, at least, open. Since the hypersphere has a tiny radius,

$$R=\left(\frac{8\pi G}{h^2}\right)^{1/2}\sim\left(\frac{8\pi G}{e^2}\right)^{1/2}\sim 10^{-33}\text{ cm},$$

its existence is not manifest until the Planck energy ($G^{-1/2}\sim 10^{19}$ GeV) is reached. The effective four-dimensional Lagrangian at lower energies contains the following fields: the usual gravitational field $g_{\mu\nu}$ ($\mu,\nu=1,\ldots,4$), massless gauge bosons associated with an unbroken subgroup (call it F) of

H, massless gauge bosons described by remnant fields $g_{a\mu}$ ($a=5,\ldots,N$; $\mu=1,\ldots,4$) of the $(4+N)$-dimensional metric, and various scalar fields. The gauge bosons of F will have a coupling f which is proportional to h. The $g_{a\mu}$ fields will be the gauge bosons of some group (call it C) and will have a coupling constant c, which is proportional to $(8\pi G/R^2)^{1/2}=h$. Therefore the $(4+N)$-dimensional theory automatically generates a low-energy effective Lagrangian which is a gauge model for $C\times F$ with a linear coupling-constant relation between c and f. Perhaps this effective Lagrangian (minus gravity) is a renormalizable theory like those in Table III. In that case the color group would have a purely geometric origin while the weak-electromagnetic group would be related to a truly internal symmetry.

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[9]Notation: Conventions are taken from J. D. Bjorken and S. D. Drell, *Relativistic Quantum Fields* (McGraw-Hill, New York, 1965). Greek letters denote Lorentz indices. Latin letters denote internal indices: a, b, c for group C and k, l, m for group F.

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[15]For a tabulation of these and other invariants, see P. Cvitanović, Phys. Rev. D 14, 1536 (1976).

[16]For simplicity μ is taken to be unity.

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[19]This includes the possibility of "shiny" quarks (i.e., not triplets under color). See Ref. 18.

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[28]In the Georgi-Glashow SU(5) model of Ref. 1, $X_2{}^2/X_3{}^2=1$, and $M\sim 10^{17}$ GeV.

[29]Even though the adjoint representation matrices are real, we take the lepton fields to be complex.

[30]These fields cannot represent antiquarks since they transform as a (3) of color SU(3).

[31]The low-mass particle assignments are arranged so that $\mathcal{L}_{\text{eff}}$ has no anomalies.

[32]Because the color SU(3) sector of $\mathcal{L}_{\text{eff}}$ is asymptotically free, strong interactions will appear to scale until the energy approaches the unification mass.

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