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Classification of SU(2) gauge fields

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Can two SU(2) gauge fields at one space-time point share the same quadratic Lorentz invariants, yet be not gauge-Lorentz equivalent to each other? The analysis of this question leads to a classification of SU(2) gauge field at one space-time point.

Recently there has been considerable interest in the classifications of gauge fields.[1] Here we present a classification of SU(2) gauge fields, addressing specifically the following questions: Given the quadratic Lorentz invariants for the field, are they realizable? If they are, how many inequivalent realizations[2] are there? Can one choose some standard forms of realizations?

In Sec. I, we solve this classification problem for the electromagnetic field, to demonstrate our procedure. In Sec. II, the classification of SU(2) gauge fields is solved. The result is summarized in Table II. In the appendixes we give proofs of lemmas which are used in the text.

All discussions in this paper are only concerned with the field strengths at *one* space-time point.

I. ELECTROMAGNETIC FIELDS

First we consider the electromagnetic field. Its classification is very simple. We shall discuss it in order to demonstrate our procedure and develop some tools for the study of SU(2) fields. We represent the electromagnetic field $\vec{E}$ and $\vec{H}$ as two real column vectors E and H, and the combination $E+iH$ as a complex column vector A:

$$E \equiv \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}, \quad H \equiv \begin{bmatrix} H_x \\ H_y \\ H_z \end{bmatrix}, \quad \text{and } A = E + iH. \tag{1.1}$$

Consider the matrix, in this case a number

$$\Delta \equiv \tilde{A}A \equiv K + iJ\,, \tag{1.2}$$

where $K = \vec{E}\cdot\vec{E} - \vec{H}\cdot\vec{H}$, $J = 2\vec{E}\cdot\vec{H}$.

Under a Lorentz transformation,[3] A is transformed by a 3×3 complex orthogonal matrix L with determinant equal to one:

$$A' = LA, \quad \tilde{L}L = 1, \text{ and } \det L = 1. \tag{1.3}$$

To show this we observe that a space rotation is represented by such an orthogonal matrix L, which is in fact real. A boost along the z direction with velocity v is represented by

$$L = \begin{bmatrix} \gamma & i\gamma\beta & 0 \\ -i\gamma\beta & \gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{1.4}$$

where

$$\beta = \frac{v}{c}\,, \quad \gamma = \frac{1}{(1-v^2/c^2)^{1/2}}\,.$$

Such an L clearly satisfies Eq. (1.3). A general Lorentz transformation can be written as a product of space rotations with boosts. Hence it generates a transformation $A' = LA$ with L satisfying (1.3). We leave to Appendix A the demonstration that any 3×3 complex orthogonal matrix with determinant $=+1$ represents a Lorentz transformation.

Thus under a Lorentz transformation

$$\Delta' = \tilde{A}'A' = \Delta\,,$$

i.e., Δ, a complex number, is a Lorentz invariant.

If Δ is given, is it realizable, i.e., does there exist an electromagnetic field which gives this Δ? (The answer is yes.) Furthermore, is there more than one Lorentz-inequivalent realization? To answer these questions it is convenient to consider two cases separately.

Case 1. $\Delta \neq 0$

In this case one standard realization is always possible by taking

$$A = \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix}, \tag{1.5}$$

where a = positive complex[4] $=\sqrt{\Delta}$. This realization is a one-dimensional one, in that both the electric and magnetic fields are along the x axis.

We shall now show that when $\Delta \neq 0$, any realization of Δ is equivalent to the standard one. Consider any realization. The vectors $\vec{E}$ and $\vec{H}$ can be made to lie in the x-y plane by a space rotation. Thus we can write

$$A=\begin{bmatrix} A_x \\ A_y \\ 0 \end{bmatrix} . \qquad (1.6)$$

Apply a z boost described by (1.4) to A. We find $A_z'=0$ and

$$A_x' : A_y' = (A_x + i\beta A_y) : (A_y - i\beta A_x) . \qquad (1.7)$$

It is easy to prove that unless

$$A_y = \pm i A_x , \qquad (1.8)$$

there always exists a real β with $\beta^2<1$ so that the ratio (1.7) is real. But the reality of $A_x' : A_y'$ means that $\vec{E}'$ and $\vec{H}'$ are collinear. By a further coordinate rotation both vectors can be lined up along the x axis. Thus the realization can be transformed into the standard one by a Lorentz transformation, if (1.8) is not satisfied.

The geometrical meaning of (1.8) is that $\vec{E}$ and $\vec{H}$ are perpendicular to each other and of the same length. It is thus equivalent to the condition $\Delta=0$.

Case 2. $\Delta = 0$ (Radiationlike case)

We can find two standard realizations in this case:

$$A=\begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix} , \qquad (1.9)$$

and

$$A=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} , \text{ i.e., vacuum.} \qquad (1.10)$$

These two are obviously inequivalent.

Any realization of $\Delta=0$ has $\vec{E}^2=\vec{H}^2$ and $\vec{E}\cdot\vec{H}=0$. Thus by a space rotation it can be brought to the form

$$A=\begin{bmatrix} E \\ \pm iE \\ 0 \end{bmatrix} , \quad E \geq 0 . \qquad (1.11)$$

The two cases distinguished by the $\pm$ sign are transformable to each other by a rotation around the x axis by π. Thus we can choose

$$A=E\begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix} . \qquad (1.12)$$

A z boost (1.4), leaves the form of (1.12) unchanged but multiplies E by a factor $\gamma(1-\beta)$ which can be made to assume any positive real value. Thus any realization is equivalent to one of the two standard ones, (1.9) or (1.10).

These results are summarized in Table I.

II. SU(2) GAUGE FIELDS

Now we consider the classification of the SU(2) gauge fields. Here the field strengths are E^a and H^a, where $a=1,2,3$ is the isospin index. We represent these nine E^a's and nine H^a's by 3×3 matrices

$$E=\begin{bmatrix} E_x^1 & E_x^2 & E_x^3 \\ E_y^1 & E_y^2 & E_y^3 \\ E_z^1 & E_z^2 & E_z^3 \end{bmatrix} , \quad H=\begin{bmatrix} H_x^1 & H_x^2 & H_x^3 \\ H_y^1 & H_y^2 & H_y^3 \\ H_z^1 & H_z^2 & H_z^3 \end{bmatrix} . \qquad (2.1)$$

Define $A=E+iH$. As in the electromagnetic case, under a Lorentz transformation $A'=LA$, where $\tilde{L}L=1$ and $\det L=1$. In addition, the row vectors of A transform like a vector under a local gauge transformation

$$A'=AG, \text{ where } \tilde{G}G=1, \quad \det G=1, \quad G=\text{real}. \qquad (2.2)$$

Again we define

$$\Delta \equiv \tilde{A}A \equiv K+iJ , \qquad (2.3)$$

where $K^{ab}=\vec{E}^a\cdot\vec{E}^b-\vec{H}^a\cdot\vec{H}^b$, $J^{ab}=\vec{E}^a\cdot\vec{H}^b+\vec{E}^b\cdot\vec{H}^a$. Notice that both K and J are real symmetric matrices. Clearly Δ is a quadratic Lorentz invariant, and transforms under a gauge transformation like

$$\Delta'=\tilde{G}\Delta G . \qquad (2.4)$$

In the case of the electromagnetic field, there are two independent real parameters in Δ, its real and imaginary parts. How many gauge-independent parameters are there in Δ in the present case? Δ is complex symmetrical. So there are to start with 12 real parameters. But the matrix G in (2.4) contains 3 real parameters. So the number of gauge-independent parameters is $12-3=9$.

We now come to the question of the realizability and its uniqueness once Δ is given. We consider separate cases, again according to the rank of Δ.

TABLE I. Classification of electromagnetic fields. In each case, Δ can be realized by one of the standard realizations. Two different standard realizations are Lorentz-gauge-inequivalent. The column labelled "No. of space dimensions spanned" refers to the standard realizations.

Case	Rank (Δ)	Standard realization	No. of space dimensions spanned	No. of inequivalent realizations
1	1	$\begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix}$, a = positive complex	1	1
2	0	$\begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix}$	2	1 } 2
		or $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$	0	1 }

Case 1. Rank $(\Delta)=3$

This is the case $\det\Delta \neq 0$. We shall demonstrate first that in this case there exists a gauge frame such that Δ is realized by

$$A=\begin{bmatrix} a & f & e \\ 0 & b & d \\ 0 & 0 & c \end{bmatrix}, \quad a,\ldots,f=\text{complex} \quad a,b,c=\text{positive}^4\text{ complex} \tag{2.5a}$$

or

$$A=\begin{bmatrix} a & f & e \\ 0 & b & d \\ 0 & 0 & c \end{bmatrix}, \quad a,\ldots,f=\text{complex} \quad a,b,-c=\text{positive}^4\text{ complex.} \tag{2.5b}$$

Before proceeding with the demonstration we notice that if A is given by (2.5), then

$$\Delta^{11}\neq 0 \tag{2.6}$$

(since it is equal to a^2), and

$$\begin{vmatrix} \Delta^{11} & \Delta^{12} \\ \Delta^{21} & \Delta^{22} \end{vmatrix} \neq 0, \tag{2.7}$$

since it is equal to $(ab)^2$. We thus have to demonstrate first that if $\det\Delta\neq 0$, there is always a gauge (to be called a proper gauge), in which (2.6) and (2.7) are valid. This will be done in Appendix B.

In a proper gauge, we substitute (2.5) into $\Delta=\tilde{A}A$ and try to solve for $a, b, \ldots, f$. First, the 11 elements of both sides show that $\Delta^{11}=a^2$. Thus $a=(\Delta^{11})^{1/2}$ and is nonvanishing and uniquely determined, because of (2.6) and the requirement in (2.5) that a = positive complex. Next, equate the 12 and 13 elements of both sides of $\Delta=\tilde{A}A$. We thus uniquely determine f and e. Then equate the 22 elements of both sides of $\Delta=\tilde{A}A$. We obtain

$$b^2=\Delta^{22}-f^2=\Delta^{22}-(\Delta^{12})^2(\Delta^{11})^{-1}, \tag{2.8}$$

which is $\neq 0$ because of (2.6) and (2.7). Proceeding this way we find that in a proper gauge, $\Delta=\tilde{A}A$ uniquely determines an A of the form (2.5a).

Thus in a proper guage, Δ is realizable by a standard realization, (2.5a). In the same proper gauge,

$$A^R=\begin{bmatrix} 1 & & \\ & 1 & \\ & & -1 \end{bmatrix} A \tag{2.9}$$

is clearly also a realization, since $\tilde{A}^R A^R=\tilde{A}A=\Delta$. Now A^R and A are not gauge-Lorentz-equivalent, since their determinants differ by a sign. Thus in the proper gauge we have two inequivalent realizations. They are the standard realizations, (2.5a) and (2.5b).

Consider any realization A_0 of Δ in the same proper gauge. We shall now show that it can be Lorentz transformed into either of the standard realizations. Consider the first column of A_0. It is a column of three complex numbers and can be thought of as an electromagnetic field discussed in Sec. I. $\Delta^{11}\neq 0$ then puts this electromagnetic field into Case 1 of that section. Now perform a Lorentz transformation to bring this electromagnetic field to its standard realization. Thus A_0 becomes realization A_1 whose first column is the same as that of (1.5), i.e.,

$$A_1 = \begin{bmatrix} a' & f' & e' \\ 0 & b' & d' \\ 0 & g' & c' \end{bmatrix}, \quad a' = \text{positive complex.} \tag{2.10}$$

Now $\Delta = \tilde{A}_1 A_1 = \tilde{A}A$, where A is the standard realization (2.5). It follows easily that $a=a'$, $f=f'$, $e=e'$. Furthermore (2.7) implies that

$$b'^2 + g'^2 \neq 0 . \tag{2.11}$$

Consider the electromagnetic field

$$\begin{bmatrix} 0 \\ b' \\ g' \end{bmatrix} .$$

Using (2.11) we find it is of Case 1 of Sec. I. Thus by a boost along the x direction we can make its $\vec{E}$ and $\vec{H}$ collinear in the y-z plane. A rotation of the y-z axis then brings it into the form

$$\begin{bmatrix} 0 \\ b'' \\ 0 \end{bmatrix} .$$

Neither this rotation nor the x boost changes any x components. Thus we have transformed A_1 into realization A_2 by a Lorentz transformation, where

$$A_2 = \begin{bmatrix} a & f & e \\ 0 & b'' & d'' \\ 0 & 0 & c'' \end{bmatrix} . \tag{2.12}$$

Now $\Delta = \tilde{A}_2 A_2 = \tilde{A}A$. Hence $b''=b^2$. If $b''=-b$ we can make a 180° rotation around the x axis to change the sign of b''. Continuing this way we conclude that A_0 can be Lorentz transformed into one of the two standard realizations, (2.5a) or (2.5b).

Thus in this case, by a gauge transformation Δ can be brought into a proper gauge. In a proper gauge, there are exactly two Lorentz-inequivalent realizations, which can be respectively Lorentz transformed into standard realizations 1 and 2. These results are summarized in the first row of Table II.

Case 2. Rank (Δ) = 2

There is only one standard realization in this case:

$$A = \begin{bmatrix} a & f & e \\ 0 & b & d \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{array}{l} a, \ldots, f = \text{complex} \\ a, b = \text{positive}^4 \text{ complex.} \end{array} \tag{2.13}$$

We notice that if A is of this form, (2.6) and (2.7) are satisfied. Thus we have to demonstrate that

TABLE II. Classifications of SU_2 gauge fields. In each case, Δ can be realized by one of the standard realizations. Two different standard realizations are Lorentz-gauge-inequivalent. Two standard realizations with different parameters λ (or μ) are Lorentz-gauge-inequivalent. The column labelled "No. of space dimensions spanned" refers to the standard realizations.

Case	Rank (Δ)	Standard realization	No. of space dimensions spanned	No. of isospin dimensions spanned	No. of inequivalent realizations
1	3	(2.5a)	3	3	1 } 2
		or			
		(2.5b)	3	3	1
2	2	(2.13)	2	3 or 2	1
3a	1	(2.20)	3	3	∞^2 } ∞^2+2
		or			
		(2.21)	3	2	1
		or			
		(2.22)	1	2	1
3b	1	(2.25)	3	3 or 2	∞ } $\infty+1$
		or			
		(2.26)	1	1	1
4	0	(2.31)	2	2 or 1	∞ } $\infty+1$
		or			
		(2.32)	0	0	1

there is a gauge, to be called a proper guage, in which (2.6) and (2.7) are valid. This will be done in Appendix C.

In the proper gauge, following the same argument as for the case rank$(\Delta)=3$, we find that (2.13) is always a realization. Since the bottom element of the diagonal is now 0, there is only one standard realization in the present case.

Again using the same argument as for the case rank$(\Delta)=3$, we find that for the present case any realization in the proper gauge is Lorentz transformable to the standard one (2.13).

Case 3. Rank $(\Delta)=1$

Any symmetrical complex matrix Δ of rank 1 can be written in the form

$$\Delta=\begin{bmatrix} a \\ f \\ e \end{bmatrix}[a\ f\ e], \quad a,f,e=\text{complex}. \qquad (2.14)$$

The real and imaginary parts of the isovector $[a,f,e]$ represent two real vectors in isospin space. We can always choose a gauge so that they both have only 1 and 2 components, i.e., there always exists a gauge in which

$$\Delta=\begin{bmatrix} a \\ f \\ 0 \end{bmatrix}[a\ f\ 0], \quad a=\text{positive complex}. \qquad (2.15)$$

We shall call such a gauge a proper gauge. In a proper gauge Δ is realizable by

$$A=\begin{bmatrix} a & f & 0 \\ 0 & \lambda & \mu \\ 0 & \lambda i & \mu i \end{bmatrix}, \qquad (2.16)$$

where λ and μ are two arbitrary complex numbers. This statement is true because (2.16) implies that $\Delta=\tilde{A}A$ is of the form (2.15). The realization

$$A'=\begin{bmatrix} a & f & 0 \\ 0 & \lambda' & \mu' \\ 0 & -\lambda' i & -\mu' i \end{bmatrix} \qquad (2.17)$$

is Lorentz-gauge-transformable to (2.16) because

$$A'=\begin{bmatrix} -1 & & \\ & -1 & \\ & & 1 \end{bmatrix} A \begin{bmatrix} -1 & & \\ & -1 & \\ & & 1 \end{bmatrix}$$

if we put $\lambda'=\lambda$, $\mu'=-\mu$.

Any realization A_1 of Δ in the proper gauge (2.15) has its first column describing an electromagnetic field of class 1 or Sec. I. Thus there is a Lorentz frame in which this electromagnetic field is brought to its standard realization. In such a Lorentz frame, A_1 becomes

$$A_2=\begin{bmatrix} a & f' & e' \\ 0 & b' & d' \\ 0 & g' & c' \end{bmatrix}. \qquad (2.18)$$

Now $a\neq 0$. The equation $\tilde{A}_2A_2=\Delta$ of (2.15) then implies that A_2 is of the form of the A of (2.16), or A' of (2.17).

We have thus proved that if rank$(\Delta)=1$, in a proper gauge (2.15), all realizations are Lorentz equivalent to the realizations (2.16) for some values of λ and μ.

It remains to investigate the following question: Given two sets of (λ,μ)'s,

$$(\lambda_1,\mu_1) \text{ and } (\lambda_2,\mu_2),$$

with the corresponding realization (2.16), to be denoted by A_1 and A_2, what is the conditon that they are Lorentz-gauge-equivalent? It is easy to verify that a rotation in the y-z plane multiplies λ and μ simultaneously by one and the same phase factor $e^{i\phi}$. It is also easy to verify that an x boost on (2.16) multiplies λ and μ simultaneously by one and the same factor $\gamma(1-\beta)$ which can assume any positive value between 0 and ∞. Thus if there exists a complex number α so that

$$\alpha\neq 0, \quad \alpha\lambda_1=\lambda_2, \quad \alpha\mu_1=\mu_2, \qquad (2.19)$$

then A_1 and A_2 are Lorentz-gauge-equivalent.

Condition (2.19) is sufficient for the Lorentz-gauge equivalence of A_1 and A_2. Is it necessary? To analyze this question we need to discuss two subcases.

Subcase 3a. Rank $(\Delta)=1$, $f/a\neq$ *real*

In this subcase, one can prove that (2.19) is also necessary for the Lorentz-gauge equivalence of A_1 and A_2 (proof omitted). Thus there are many Lorentz-gauge-inequivalent realizations of Δ, one for each value of the ratio $\lambda:\mu$. In other words, in this subcase, it is always possible to realize Δ with one of the three standard realizations

$$A=\begin{bmatrix} a & f & 0 \\ 0 & \lambda & 1 \\ 0 & \lambda i & i \end{bmatrix}, \quad \lambda=\text{complex}, \qquad (2.20)$$

$$A=\begin{bmatrix} a & f & 0 \\ 0 & 1 & 0 \\ 0 & i & 0 \end{bmatrix}, \qquad (2.21)$$

or

$$A=\begin{bmatrix} a & f & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{2.22}$$

In all these three realizations a = positive complex. The three are not Lorentz-gauge-equivalent. Furthermore, the realizations (2.20) with different λ's are not Lorentz-gauge-equivalent.

Subcase 3b. Rank (Δ) = 1, f/a = real

In this case one can always make a gauge transformation to make $f=0$. Thus Eq. (2.16) becomes

$$A_{\lambda\mu}=\begin{bmatrix} a & 0 & 0 \\ 0 & \lambda & \mu \\ 0 & \lambda i & \mu i \end{bmatrix}, \tag{2.23}$$

where a = positive complex $\neq 0$. Now we can make a further gauge transformation, mixing the second and third columns of $A_{\lambda\mu}$. Combining such a gauge transformation with the Lorentz transformation (2.19), we can always have a realization $A_{i\mu_1}$ where μ_1 is real, unless $\lambda=\mu=0$. Now

$$A_{i\mu_1}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}=\begin{bmatrix} a & 0 & 0 \\ 0 & -\mu_1 & i \\ 0 & -\mu_1 i & -1 \end{bmatrix}.$$

Thus by a further Lorentz transformation we see that if $\mu_1 \neq 0$,

$$A_{i\mu_1} \text{ is gauge-Lorentz equivalent to } A_{i(\mu_1{}^{-1})}. \tag{2.24}$$

Thus in this subcase 3b, there are two standard realizations,

$$A_{i\mu}=\begin{bmatrix} a & 0 & 0 \\ 0 & i & \mu \\ 0 & -1 & \mu i \end{bmatrix}, \quad -1 \leq \mu \leq 1 \tag{2.25}$$

or

$$A_{00}=\begin{bmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{2.26}$$

where a = positive complex $\neq 0$. These are not Lorentz-gauge-equivalent. Furthermore for two different μ's the realizations (2.25) are Lorentz-gauge-inequivalent (proof omitted).

Case 4. Rank (Δ) = 0

In this case $\Delta=0$, if a realization is not the vacuum (i.e., if $A\neq 0$), we can always choose a gauge where the last column of A is not zero. $\Delta^{33}=0$ implies that $\vec{E}^3$ and $\vec{H}^3$ are perpendicular and of equal length $\neq 0$. Rotate coordinates so that $\vec{E}^3$ is along the y axis and $\vec{H}^3$ along the z axis. By an x boost we can always bring them to lengths 1. Thus any realization $A\neq 0$ is gauge-Lorentz-equivalent to

$$A=\begin{bmatrix} a & d & 0 \\ b & e & 1 \\ c & f & i \end{bmatrix}. \tag{2.27}$$

$\Delta^{11}=\Delta^{13}=0$ imply $a^2+b^2+c^2=0$, $b+ci=0$. Hence $a=0$, $c=bi$. Similarly $d=0$, $f=ei$, and

$$A=\begin{bmatrix} 0 & 0 & 0 \\ b & e & 1 \\ bi & ei & i \end{bmatrix}. \tag{2.28}$$

A gauge transformation on (2.28) is an orthogonal transformation on the row matrix $[b, e, 1]$. It is always possible to find a real vector perpendicular to both the real and imaginary parts of $[b, e, 1]$. Thus A is gauge-equivalent to

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & \lambda & \mu \\ 0 & \lambda i & \mu i \end{bmatrix}. \tag{2.29}$$

We can now proceed exactly as we did after Eq. (2.23) and conclude that any realization $A\neq 0$ of $\Delta=0$, is gauge-Lorentz-equivalent to

$$A_{i\mu}=\begin{bmatrix} 0 & 0 & 0 \\ 0 & i & \mu \\ 0 & -1 & \mu i \end{bmatrix}, \quad -1 \leq \mu \leq 1. \tag{2.30}$$

Now

$$A_{i\mu}\begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}=\begin{bmatrix} 0 & 0 & 0 \\ 0 & \mu & i \\ 0 & \mu i & -1 \end{bmatrix},$$

which can be transformed by a Lorentz transformation (2.19) to $A_{i(-\mu^{-1})}$ if $\mu\neq 0$. Thus any realization can be gauge Lorentz transformed to one of the two standard realizations

$$A=\begin{bmatrix}0 & 0 & 0\\ 0 & i & \mu\\ 0 & -1 & \mu i\end{bmatrix}, \quad 0\le\mu\le 1 \tag{2.31}$$

or

$$A=0 \tag{2.32}$$

Two standard realizations with different μ are not gauge-Lorentz-equivalent.

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APPENDIX A

Lemma. Any 3×3 orthogonal matrices L with determinant equal to +1 represents a Lorentz transformation.

Proof. We prove the lemma by showing that any such matrix can be reduced to the identity matrix by a series of rotations and boosts. Actually, many of the reasonings have been used through the text. The 3×3 matrix

$$L=\begin{bmatrix}L_{xx} & L_{xy} & L_{xz}\\ L_{yx} & L_{yy} & L_{yz}\\ L_{zx} & L_{zy} & L_{zz}\end{bmatrix}\equiv(L^1,L^2,L^3) \tag{A1}$$

can be viewed as made up of three complex column vectors, L^1, L^2, L^3. Each column vector is just like the column vector of an electromagnetic field. The fact that $\tilde{L}L=1$ means that none of the three column vectors is radiation-field-like. As shown in the discussion of *Case* 1 in Sec. I, we can always apply a rotation and then a boost in the z direction and another rotation so that after these operations L^1 has only the first element, i.e.,

$$L'\equiv ML=\begin{bmatrix}L'_{xx} & L'_{xy} & L'_{xz}\\ 0 & L'_{yy} & L'_{yz}\\ 0 & L'_{zy} & L'_{zz}\end{bmatrix}, \tag{A2}$$

where L' is still orthogonal and has determinant equal to one. Hence, $L'_{xx}=\pm 1$, $L'_{xy}=L'_{xz}=0$. If $L'_{xx}=-1$, we can rotate around the z axis by 180°, thus changing it to +1. Hence we can always take

$$L'=ML=\begin{bmatrix}1 & 0 & 0\\ 0 & L'_{yy} & L'_{yz}\\ 0 & L'_{zy} & L'_{zz}\end{bmatrix}.$$

The condition that L' is orthogonal and has determinant 1 can be easily shown to imply that L' is a rotation in the y-z plane multiplied by an x boost. Thus $L=M^{-1}L'$ is a Lorentz transformation.

APPENDIX B

Lemma. If $\det\Delta\neq 0$, there always exists a gauge in which (2.6) and (2.7) are satisfied.

Proof. Δ^{-1} exists and is symmetrical. Separate it into real and imaginary parts:

$$\Delta^{-1}=R+iI, \tag{B1}$$

where R and I are both real symmetrical. The 6 eigenvalues of R and I cannot all be zero, for if so $R=I=0$, which is not possible. Let ψ be an eigenvector of R or I with a nonvanishing eigenvalue. Then $\tilde{\psi}\Delta^{-1}\psi\neq 0$. Make a gauge transformation so that ψ becomes the third isospin direction. After the transformation $(\Delta^{-1})^{33}\neq 0$. Since

$$(\Delta^{-1})^{33}=(\det\Delta)^{-1}\begin{vmatrix}\Delta^{11} & \Delta^{12}\\ \Delta^{21} & \Delta^{22}\end{vmatrix}.$$

we see that in the new gauge (2.7) is satisfied.

Now consider the 2×2 matrix in (2.7) and separate it into real and imaginary parts. By a reasoning identical to the one following (B1) we finish the proof of the lemma.

APPENDIX C

Lemma. If $\det\Delta=0$ and all 2×2 diagonal minors of Δ are $=0$, then Δ has Rank 1.

Proof. Since all 3 diagonal minors of Δ are zero we can always choose signs in $a=\pm(\Delta^{11})^{1/2}$, $b=\pm(\Delta^{22})^{1/2}$, $c=\pm(\Delta^{33})^{1/2}$, so that

$$\Delta=\begin{bmatrix}a^2 & ab & ac\\ ab & b^2 & \pm bc\\ ac & \pm bc & c^2\end{bmatrix}. \tag{C1}$$

$\det\Delta=0$ then implies that the sign is + in (C1), or else $abc=0$. In both cases the lemma follows.

From this lemma it follows that if $\operatorname{rank}(\Delta)=2$, there always is a permutation of the isospin axis so that (2.7) is valid. It then follows easily that there is a gauge in which (2.6) is also valid.

[1]R. Roskies, Phys. Rev. D 15, 1722 (1977); M. Carmeli, Phys. Rev. Lett. 39, 523 (1977).

[2]Two realizations are inequivalent if they are not related by Lorentz and gauge transformations. We call a realization unique if there is only one inequivalent realization.

[3]Throughout this paper we mean by a Lorentz transformation an orthochronous proper one, i.e., one that does not involve a time reversal and is represented by a 4×4 matrix whose determinant is +1.

[4]A positive complex number a is one that is nonvanishing and either (i) has a positive real part or (ii) is equal to yi where $y=$ real and is >0.