Order and disorder in gauge systems and magnets

Eduardo Fradkin
Stanford Linear Accelerator Center, Stanford University, Stanford, California 94305

Leonard Susskind
Stanford Linear Accelerator Center, Stanford University, Stanford, California 94305
and Tel-Aviv University, Tel-Aviv, Israel

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We show how phase transitions in Abelian two-dimensional spin and four-dimensional gauge systems can be understood in terms of condensation of topological objects. In the spin systems these objects are kinks and in the gauge systems either magnetic monopoles or fluxoids (quantized lines of magnetic flux). Four models are studied: two-dimensional Ising and XY models and four-dimensional Z_2 and U(1) gauge systems.

I. INTRODUCTION

In the present paper we discuss several underlying analogies between four-dimensional gauge systems and two-dimensional magnets. Our main purpose is to clarify the meaning of concepts such as confinement and its relationship with order and disorder in the system. Most generally we shall be interested in studying the long-distance behavior, i.e., long-distance correlations, of these systems in order to understand their phase diagrams. It is well known that most four-dimensional gauge theories are similar to their two-dimensional spin system counterparts in the sense that the renormalization-group equations have the same structure in both systems[1]; they exhibit the same kind of instantons,[2] etc. Explicitly we show a remarkable analogy between the 3+1 Abelian gauge theory and the 1+1 XY ferromagnet. Roughly speaking, the long-distance behavior of both systems is similar when proper analogous quantities are discussed. For instance, the behavior of Wilson's loop integral[3] (for the gauge theory) is similar to the behavior of the two-point correlation function in the XY model[4] once we recognize that a decay of the loop integral as the area of the loop means disorder in a gauge theory. On the other hand, the XY model has a phase in which the correlation function falls off at large distances with a power-law behavior,[4] which is also true for the Abelian gauge theory.

In general the systems shown in the following table exhibit analogous behaviors:

Two-dimensional magnets	Four-dimensional gauge theories
Ising model	Z_2
XY model	Abelian gauge
Heisenberg	Non-Abelian (Yang-Mills)

We shall restrict our discussion in this paper to the first two analogies. These analogies were first pointed out by Migdal, who discussed them in the framework of his recursion relations.[1]

The Z_2 gauge theory, which is a guage theory in which the degrees of freedom are elements of the permutations group of two elements, was first discussed and solved by Wegner.[5]

All our discussions will be done by putting the fields on a lattice with the time direction continuous and the space directions discrete.[6] In Sec. II we discuss the transfer matrix,[7] a formulation which we shall use as a tool to build up the Hamiltonian form of all the models. Later on in Sec. II, we discuss the one-dimensional quantum Ising model in a transverse field[8] and we introduce duality and dual order parameters.[9] Dual order parameters will be related to the existence of condensates of kinks (in magnetic systems) and magnetic monopoles (in gauge theories) which randomize the system. In Sec. III we discuss the Z_2 gauge theory, in Sec. IV the XY model and, finally, Sec. V is devoted to Abelian gauge theory.

II. HAMILTONIAN THEORY OF THE ISING MODEL

A. One-dimensional case

We shall begin our discussion with the Ising model (IM). Identify one of the lattice directions as the (Euclidean) time axis. We will look for a limit in which this direction can be considered continuous. The Ising model in this limit becomes formally equivalent to a quantum-mechanical system with a well-defined Hamiltonian describing a continuous development in time. The method is most easily illustrated using the transfer matrix formalism.[7]

Let us construct the transfer matrix formalism for the one-dimansional[10] Ising model. The action is

$$\mathcal{A} = -\beta \sum_{\text{sites}} [\sigma_3(i)\sigma_3(i+1) + h\sigma_3(i)], \tag{2.1}$$

where $\sigma_3 = \pm 1$ and i runs over all the sites. The parameter h represents an external magnetic field. It is convenient to add a constant to the action to normalize the ground-state energy to zero when $h = 0$. We also rewrite the term proportional to h in a form which will prove more convenient. Thus

$$\mathcal{A} = \frac{\beta}{2} \sum_{\text{sites}} \{[\sigma_3(i) - \sigma_3(i+1)]^2 - h[\sigma_3(i) + \sigma_3(i+1)]\}. \tag{2.2}$$

Define

$$\mathcal{L}(i, i+1) = \tfrac{1}{2}\beta\{[\sigma_3(i) - \sigma_3(i+1)]^2 - h[\sigma_3(i) + \sigma_3(i+1)]\}, \tag{2.3}$$

so that

$$\mathcal{A} = \sum_i \mathcal{L}(i, i+1). \tag{2.4}$$

The partition function is

$$Z = \sum_{\text{configurations}} \exp(-\mathcal{A}).$$

$$= \sum_{\text{configurations}} \prod_i \exp[-\mathcal{L}(i, i+1)]. \tag{2.5}$$

It is easy to see that this is the trace[11] of the Nth power of the transfer matrix, T, where the rows (columns) of T are labeled by the possible configurations of the initial (final) member of a neighboring pair of spins:

$$T = \exp[-\mathcal{L}(i, i+1)] = \begin{pmatrix} e^{\beta h} & e^{-2\beta} \\ e^{-2\beta} & e^{-\beta h} \end{pmatrix}, \tag{2.6}$$

$$Z = \mathrm{Tr} T^N, \tag{2.7}$$

where N is the total number of sites of the lattice.

Now we shall imagine that the axis of the lattice is the time axis of quantum mechanics. Thus T carries information from one time to a neighboring time. We will in fact identify it with the time evolution operator for a quantum system of a single spin.

We want to take a limit in which neighboring lattice sites are treated as infinitesimal transformations of the form

$$T = 1 - \tau H, \tag{2.8}$$

where τ is infinitesimal and H is the Hamiltonian. Of course this is not true in general [see Eq. (2.6)], but there exists a limit in which Eq. (2.6) has the form (2.8). The limit is

$$\beta \to \infty \quad \text{(low temperature)}, \qquad \beta h \to \lambda e^{-2\beta} \quad (\lambda \text{ is any constant}), \tag{2.9}$$

and

$$\tau = e^{-2\beta}. \tag{2.10}$$

Then

$$T = \begin{pmatrix} 1 + \lambda\tau & \tau \\ \tau & 1 - \lambda\tau \end{pmatrix}. \tag{2.11}$$

We can write T in terms of Pauli matrices along on the Hilbert space of the quantum spin

$$T = 1 + (\lambda\sigma_3 + \sigma_1)\tau$$

or (2.12)

$$H = -\sigma_1 - \lambda\sigma_3.$$

In taking the continuum limit we must imagine that the number of sites separating any two times increases as $e^{2\beta}$. This dependence of coupling constant (β) on lattice spacing is a simple example of the renormalization group.

The correspondences between classical statistical mechanics and the equivalent Euclidean quantum system are summarized in the following scheme:

Quantum system	Statistical system
(1) Ground state	Equilibrium state
(2) Ground-state expectation values of time-ordered operators	Averages on the ensemble
(3) Ground-state energy	Free energy

B. The two-dimensional case

The two-dimensional Ising model will be more interesting than the one-dimensional case. The action of the anisotropic IM is

$$\mathcal{A} = \sum_{\vec{r}} \{\tfrac{1}{2}\beta_t[\sigma_3(\vec{r}) - \sigma_3(\vec{r} + \hat{n}_t)]^2 - \beta_z[\sigma_3(\vec{r})\sigma_3(\vec{r} + \hat{n}_z)]\}, \tag{2.13}$$

where $\vec{r}$ runs over all the sites of a two-dimension-

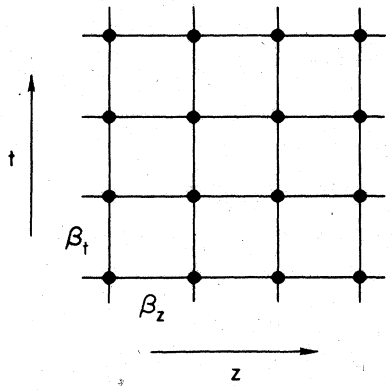

FIG. 1. The space-time lattice.

al rectangular lattice and $\hat{n}_t$ ($\hat{n}_z$) is the unit vector in the t (z) direction (see Fig. 1). The coupling constants in the directions t, z are β_t, β_z which are not necessarily equal. We will not bother with an external field in this case.

Before dealing with the technical details of the transfer matrix we will qualitatively describe a limit in which the t direction becomes continuous leaving discrete the z axis. In this limit the IM becomes equivalent to a Hamiltonian quantum system consisting of a one-dimensional (z) discrete system of interacting spins.

The two-dimensional IM has a phase transition. In the space of the parameters β_z, β_t there is a critical curve which separates the ordered (ferromagnetic) and disordered (paramagnetic) phases. This is shown in Fig. 2. The critical curve is given by[7]

$$(\sinh 2\beta_z)(\sinh 2\beta_t) = 1. \tag{2.14}$$

We will illustrate the main ideas in terms of the two-point correlation function $\langle \sigma_3(0)\sigma_3(r)\rangle = C(r)$. For $\beta_z = \beta_t$, $C(r)$ has cubic symmetry (symmetry under rotations by $\pi/2$). To illustrate this we draw

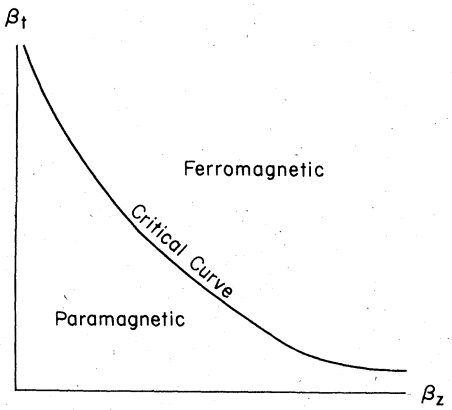

FIG. 2. Phase boundary for the anisotropic two-dimensional Ising model.

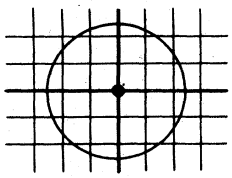

FIG. 3. The equal correlation contours for the symmetrical lattice.

the contours of the curves $C(\vec{r}) = \text{const}$ as circles in the case $\beta_z = \beta_t$ (see Fig. 3). However, in the anisotropic case ($\beta_t > \beta_z$) the contours at large r are deformed into ellipses with major axis along t (see Fig. 4).

Now imagine rescaling the t axis in such a way that the ellipses are transformed back into circles. This is shown in Fig. 5. In this way we can approximately compensate the effects of the anisotropy by a rescaling of t relative to z. The form of the correlation function in the new model is similar to the symmetric case.

We can repeat this process until we reach a limit in which the lattice in the time direction becomes dense. This is the time-continuum limit.

C. Transfer matrix for the two-dimensional case

We will now construct the time-continuum limit in a precise way using the transfer matrix method. Consider two neighboring rows of spins as in Fig. 6. The spins on the "earlier" ("later") row are denoted by $s(n)$ [$\sigma(n)$] where n labels discrete location along the space z axis. The Lagrangian for this pair of rows is

$$\mathcal{L} = \frac{\beta_t}{2} \sum_n [s_3(n) - \sigma_3(n)]^2 - \frac{\beta_z}{2} \sum_n [s_3(n)s_3(n+1) + \sigma_3(n)\sigma_3(n+1)]. \tag{2.15}$$

The rows and columns of the transfer matrix are labeled by the spin configurations of both layers. Since for N spins on a layer there are 2^N configurations, the T matrix is $2^N \times 2^N$.

The diagonal elements of T are given by setting $s_3(n) = \sigma_3(n)$ for all n. Thus

$$T_{\text{diagonal}} = \exp\left[\beta_z \sum_n \sigma_3(n)\sigma_3(n+1)\right]. \tag{2.16}$$

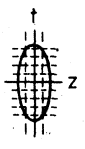

FIG. 4. The equal correlation contours become ellipses for the anisotropical lattice.

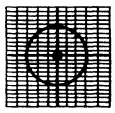

FIG. 5. The ellipses are deformed back into circles by squeezing the lattice in the time direction.

The off-diagonal elements can be classified by the number of spin flips [the number of sites for which $\sigma_3(n) = -s_3(n)$]. The single-flip elements are

$$T_{1\ \text{flip}} = \exp(-2\beta_t)\exp[-E(\sigma,s)\beta_z], \tag{2.17}$$

where $E(\sigma,s)$ is the sum of the energy of the two independent rows.

Similarly the n-flip elements are

$$T_n = \exp(-2n\beta_t)\exp[-E(\sigma,s)\beta_z]. \tag{2.18}$$

Now consider the limit

$$\begin{aligned} &\beta_t \to \infty, \\ &\beta_z \to \lambda e^{-2\beta_t}. \end{aligned} \tag{2.19}$$

This limit is similar to that used in the one-dimensional case. In fact the replacement $\beta_z \to \beta h$ is natural since the interaction with neighboring columns exerts a field on each spin.

The limiting form of the matrix elements of T are

$$\begin{aligned} &T_{\text{diag}} \to 1 + e^{-2\beta_t}\sum_n \sigma_3(n)\sigma_3(n+1), \\ &T_1 \to e^{-2\beta_t} + O(e^{-4\beta_t}), \\ &T_2 \to e^{-4\beta_t} + O(e^{-6\beta_t}), \\ &\cdots . \end{aligned} \tag{2.20}$$

It is now possible to put T into the infinitesimal form

$$T = 1 - \tau H.$$

We again identify the quantity $e^{-2\beta_t}$ as τ—the infinitesimal spacing along the t axis:

$$\begin{aligned} T = 1 + \tau\Big[\lambda\sum_n \sigma_3(n)\sigma_3(n+1) + \sum_n \sigma_1(n)\Big] \\ + \tau^2 \sum_{n\neq m} \sigma_1(n)\sigma_1(m) + \cdots . \end{aligned} \tag{2.21}$$

The Pauli matrices σ_1 have been used to flip the spin so that the n-flip terms of T contain n factors of σ_1. However, since τ is infinitesimal we may ignore all terms of order $\tau^2, \tau^4, \ldots$ by comparison with the order-τ term. The result is that the Hamiltonian H contains only no-flip and single-flip terms:

$$H = -\sum_n \sigma_1(n) - \lambda \sum \sigma_3(n)\sigma_3(n+1). \tag{2.22}$$

The connection between the spacing τ and $e^{-2\beta_t}$ provides a quantitative estimate of the amount of rescaling of t which is required to compensate the anisotropy when β_t becomes large.

The correlation functions will approach the limiting forms of the equivalent quantum system as $\beta_t \to \infty$, $\beta_z \to \lambda e^{-2\beta_t}$. Suppose for example

$$\langle 0 \,|\, T\sigma_3(0,0)\sigma_3(n,t) \,|\, 0\rangle = \Gamma(n,t) \tag{2.23}$$

for the quantum system. Then in original discrete integer-valued coordinates of the lattice the correlation function behaves like

$$C(n,m;\beta_z,\beta_t) \to \Gamma(n,m\tau,\lambda), \tag{2.24}$$

where

$$\begin{aligned} &\tau = e^{-2\beta_t}, \\ &\lambda = \beta_z e^{2\beta_t}. \end{aligned}$$

In particular as $\beta_t \to \infty$ the spatial correlation length [decay length of the function $C(n,0)$] tends to a limiting function of λ.

In the space β_z, β_t there exists a set of curves along which the spatial correlation length is constant. In particular, the critical curve is the curve where the correlation length is infinite. The above discussion shows us that for $\beta_t \to \infty$ these curves have the form (see Fig. 7)

$$\beta_z = \lambda e^{-2\beta_t}. \tag{2.25}$$

The parameter λ can be used to label the curves. We can relate any point on the symmetric line $\beta_z = \beta_t$ with a limiting theory by extrapolating along

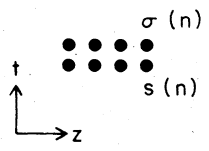

FIG. 6. Two neighboring rows of spins.

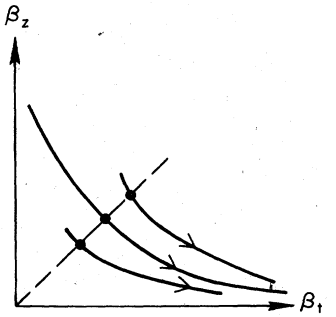

FIG. 7. Asymptotic behavior of the equal correlation length curves in the $\beta_t\,\beta_z$ parameter space. Each curve is labeled by a single value of λ.

these curves. The qualitative long-range behavior is unchanged along a line of fixed λ. The points on the symmetrical line correspond to different temperatures of the classical square Ising model. Thus we can relate each temperature of the isotropic model to a unique quantum model with a corresponding value of λ. Generally large λ (small λ) corresponds to large β (small β). Thus we will refer to the large (small) λ as low (high) temperature.

D. Lattice duality

The two-dimensional IM and the equivalent one-dimensional quantum problem have the remarkable property of *self-duality*.[9] The dual of a cubic lattice is a new lattice whose sites are located at the centers of the old cubes. In particular, for a one-dimensional lattice the sites of the dual lattice correspond to the links of the original lattice (see Fig. 8).

The original system can be redescribed by a new system with degrees of freedom attached to the dual lattice. For the one one-dimensional quantum Ising model with transverse field [Eq. (2.22)] the dual lattice operators are called μ. They can be written in terms of the original σ's:

$$\mu_1(n) = \sigma_3(n+1)\sigma_3(n), \qquad \mu_3(n) = \sum_{m<n} \sigma_1(m). \tag{2.26}$$

The operators $\mu_1(n)$ describe the mutual state of two neighboring σ's. μ_3 flips all the spins to the left of the site n.

The σ operators satisfy the following relations which specify their algebra completely:

$$[\sigma_i(m), \sigma_j(n)] = 0 \text{ for } n \neq m, \tag{2.27}$$

$$\sigma_1^2(n) = 1, \qquad \sigma_3^2(n) = 1, \qquad \sigma_3(n)\sigma_1(n)\sigma_3(n) = -\sigma_1(n). \tag{2.28}$$

From Eqs. (2.26), (2.27), and (2.28) it is easily seen that the μ's satisfy the same relations. Thus the variables on the dual sites are isomorphic to the original variables.

The Hamiltonian in Eq. (2.22) can be expressed in terms of the μ's as

$$H = -\sum_n \mu_3(n)\mu_3(n+1) - \lambda \sum_n \mu_1(n) = \lambda\left[-\sum_n \mu_1(n) - \frac{1}{\lambda}\sum_n \mu_3(n)\mu_3(n+1)\right]. \tag{2.29}$$

The remarkable thing about the Hamiltonian is that it has the same form in terms of the μ's and the σ's. The only differences are the overall factor of λ in (2.29) and the replacement $\lambda \rightarrow 1/\lambda$ inside the brackets. We may summarize this property by the formula

$$H(\sigma;\lambda) = \lambda H(\mu, \lambda^{-1}). \tag{2.30}$$

The self-duality of H is a very powerful result. It shows that the high-temperature behavior ($\lambda < 1$) and low-temperature behavior ($\lambda > 1$) are in a sense equivalent. For example, we can map any eigenstate of $H(\lambda)$ to a unique eigenstate of $H(1/\lambda)$. The energy spectrum has the property that if $E(\lambda)$ is the energy of some state then $E(\lambda)/\lambda$ is the energy of a related state of $H(1/\lambda)$.

For example, the energy gap between the ground and first excited states satisfies

$$G(\lambda) = \lambda G(1/\lambda). \tag{2.31}$$

The one-dimensional Ising model with transverse field is exactly soluble for the spectrum.[8] The gap $G(\lambda)$ is given by

$$G(\lambda) = 2|1-\lambda|, \tag{2.32}$$

which is easily seen to satisfy (2.31).

The symmetry point of the duality transformation is $\lambda = 1$. From (2.32) we see that the gap vanishes at this point signaling the presence of massless excitations and a divergent correlation length. In other words, the point $\lambda = 1$ separates both the ordered and disordered phases.

E. Order and disorder parameters

The duality transformation relates the high- and low-temperature behaviors of the system. We will discuss the properties of these phases now.

(1) *Large* λ (*small temperature*). For $\lambda \gg 1$ the term $\sum[-\lambda\sigma_3(n)\sigma_3(n+1)]$ dominates the Hamiltonian. The ground state for $\lambda = \infty$ is doubly degenerate with all spins parallel either up or down (see Fig.

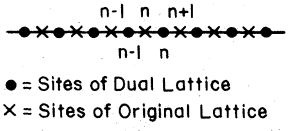

FIG. 8. Dual lattices in one dimension.

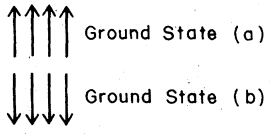

FIG. 9. The ground state of the quantum-mechanical Ising model in a transverse field at large values of λ is doubly degenerate.

9). By picking boundary conditions at ∞ we may choose the ground state (a). Then the expectation value of σ_3 is 1. Defining $|0\rangle_\lambda$ to be the ground state for given λ we have

$$ {}_\infty\langle 0|\sigma_3|0\rangle_\infty = 1. \tag{2.33}$$

More generally

$$ \langle\sigma_3\rangle = {}_\lambda\langle 0|\sigma_3|0\rangle_\lambda \neq 0 \text{ (for } \lambda > 1). \tag{2.34}$$

The quantity $\langle\sigma_3\rangle$ is known as the order parameter or magnetization.

That the magnetization persists for noninfinite λ is not completely trivial. For example, the ordinary one-dimensional Ising model is ordered for zero temperature but not for finite temperature.

To see that $\langle\sigma_3\rangle \neq 0$ for large but finite λ we may apply perturbation theory to see how $\langle\sigma_3\rangle$ changes with $1/\lambda$. For large λ we write

$$ \frac{H}{\lambda} = -\sum \sigma_3(n)\sigma_3(n+1) - \frac{1}{\lambda}\sum_n \sigma_1(n) $$

$$ = H_0 + \frac{1}{\lambda} H_1. \tag{2.35}$$

Applying standard perturbation theory we get

$$ |0\rangle_\lambda = |0\rangle_\infty + \frac{\lambda^{-1}}{E_0 - H_0} H_1 |0\rangle_\infty + \cdots . \tag{2.36}$$

H_1 flips one spin at a time. The state with the nth spin flipped is called $|n\rangle$:

$$ |0\rangle_\lambda = |0\rangle_\infty - \frac{\lambda^{-1}}{4}\sum_n |n\rangle + \cdots . \tag{2.37}$$

To order λ^{-2} we find

$$ \frac{{}_\lambda\langle 0|\sigma_3(m)|0\rangle_\lambda}{{}_\lambda\langle 0|0\rangle_\lambda} = {}_\infty\langle 0|\sigma_3(m)|0\rangle_\infty \left(1 - \frac{N}{16\lambda^2}\right) $$

$$ + \frac{\lambda^{-2}}{16}\sum_n \langle n|\sigma_3(m)|n\rangle. \tag{2.38}$$

The factor $(1 - N/16\lambda^2)$ is the normalization factor ${}_\lambda\langle 0|0\rangle_\lambda^{-1}$ to order λ^{-2} and N is the total number of sites. For $m \neq n$, $\langle n|\sigma_3(m)|n\rangle = +1$, while for $m = n$, $\langle n|\sigma_3(m)|n\rangle = -1$. Thus

$$ \langle\sigma_3\rangle = \left(1 - \frac{N}{16\lambda^2}\right) + (N-1)\frac{1}{16\lambda^2} - \frac{1}{16\lambda^2} $$

$$ = 1 - \frac{1}{8\lambda^2} . \tag{2.39}$$

The important feature of this result is that the N dependence of the order λ^{-2} correction cancels leaving a finite coefficient. This is true to all orders and therefore we expect a finite region of λ to have a nonvanishing magnetization. This result may be contrasted with a calculation of the derivative of the magnetization with respect to temperature for the ordinary one-dimensional Ising model. There the N dependence does not disappear and the magnetization is not a smooth function of T for $N \to \infty$.

It will prove to be interesting to define a dual order parameter or "disorder parameter" which actually measures the degree of disorder of the σ_3 variables. To do this we perform the duality transformation on the order parameter, $\langle\sigma_3\rangle$.

From (2.26) we define the disorder parameter to be

$$ \langle\mu_3(n)\rangle = \left\langle \sum_{m<n} \sigma_1(m) \right\rangle . \tag{2.40}$$

This object generally vanishes in the ordered phase and has a nonvanishing expectation value in the disordered phase. To see intuitively why this is so, we consider the action of the operator $\mu_3(n)$ when applied to a basis state in the σ_3 representation. The result is to flip all the spins up to the site n (Fig. 10). Therefore when applied to a magnetized state $\mu_3(n)$ reverses the sign of the magnetization at an infinite number of sites. The resulting state is obviously orthogonal to the original.

Accordingly for any magntized state

$$ \langle\mu_3\rangle = 0. \tag{2.41}$$

On the other hand, if the state is sufficiently disordered it may be possible for the state resulting from an infinite number of spin flips to have a projection onto the original state.

(2) *Small* λ *(high temperature)*. The ground state is that of $H_0 = \sum_n \sigma_1(n)$ for $\lambda \ll 1$. The σ_1's are all aligned with positive value 1. Then we define the ground state for $\lambda = 0$ as a state $|0\rangle_0$ such that

$$ \sigma_1(n)|0\rangle_0 = |0\rangle_0 \quad \text{(all } n\text{)}. \tag{2.42}$$

Evidently the average of σ_3 (magnetization) satisfies

$$ {}_0\langle 0|\sigma_3(n)|0\rangle_0 = 0. \tag{2.43}$$

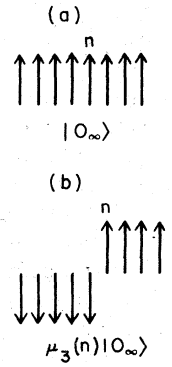

FIG. 10. A kink applied at site n flips all the spins from $-\infty$ up to site n.

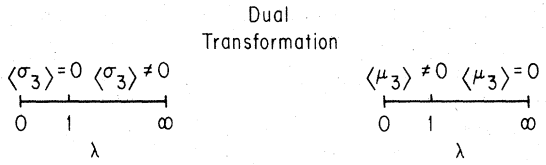

Fig. 11. The effect of the dual transformation.

In fact Eq. (2.33) is true for all $\lambda<1$. This is true because the transformation

$$\sigma_3 \to -\sigma_3, \qquad \sigma_1 \to \sigma_1 \tag{2.44}$$

is a symmetry of H. Unless this symmetry is spontaneously broken ${}_\lambda\langle 0|\sigma_3|0\rangle_\lambda$ must vanish. Now consider the disorder parameter for $\lambda=0$. Since $|0\rangle_0$ is an eigenvector of σ_1 [see Eq. (2.42)] it follows that

$${}_0\langle 0|\mu_3(n)|0\rangle_0 = 1. \tag{2.45}$$

By the same arguments as in Eqs. (2.36)–(2.39) we can prove that the disorder parameter is not vanishing for a finite range of λ:

$${}_\lambda\langle 0|\mu_3(n)|0\rangle_\lambda \neq 0; \tag{2.46}$$

we summarize these results in Fig. 11.

F. Kink condensates and disorder

In the preceding section we showed that $\langle\mu_3(n)\rangle$ measures the amount of disorder in the system. However, we can reinterpret all these results in an interesting way.

The operator $\mu_3(n)$ acting on an ordered state creates a spin configuration which we shall call a kink. This object has finite energy and the number of such an object is a conserved quantity. Thus we can regard these configurations as massive particles.

Since a kink configuration is orthogonal to the ground state in an ordered phase there will be no kinks present in this phase. However, we know that if λ is very large but finite there will be a finite (and small) number of spins flipped. As we see from Fig. 12 a single spin flip is equivalent as a pair kink and antikink at two neighboring dual sites. At lower values of λ there will be blocks of spins flipped, which are clearly equivalent to the kink-antikink pairs with some size. If λ is large the distance between pairs will be much larger than the size of the pair. However, as λ approaches its critical value 1 the interpair distance becomes comparable to the pair size. Thus the phase transition is a kink condensation phenomenon (Fig. 13). Moreover, the kink-antikink pairs become "ionized" at $\lambda=1$ without any cost of energy.

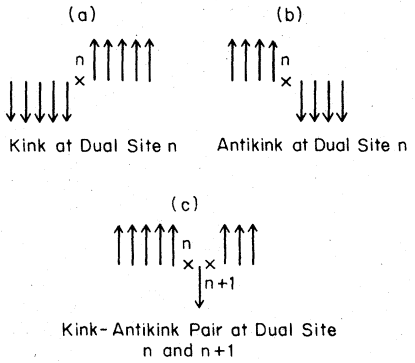

FIG. 12. Kinks in action.

Kinks have very important features. Unpaired kinks cannot be present in any ordered phase of the system since they violate the imposed boundary conditions. They only can exist in the system paired with antikinks (in the ordered phase) or as a condensate in the disordered phase where the system ignores boundary conditions. Moreover, they are topological objects because they are large perturbations of the system which change the boundary conditions. Finally they disorder the system, and above the critical temperature their presence as a condensate is responsible for the short range of the two-point correlation function. Thus, if we are considering $\langle\sigma_3(0)\sigma_3(n)\rangle$, an indefinite or random number of kinks occurring between the two points will destroy the correlation between the two spins.

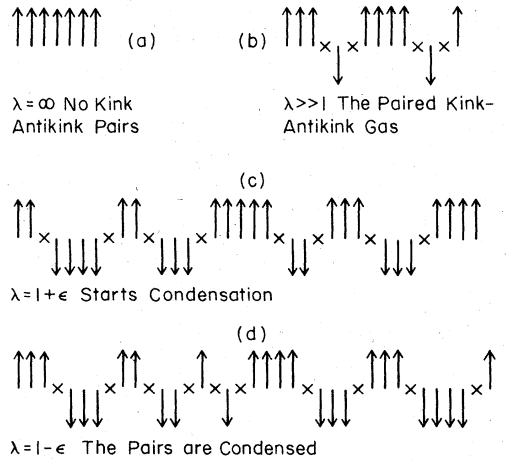

FIG. 13. The sequence shows how kinks disorder the system. The disordered state is the kink condensate.

III. GAUGE SYSTEMS IN 3+1 DIMENSIONS

A. Gauge invariance

The Ising model studied in the previous sections has a *global* symmetry consisting of flipping all the $\sigma_3(n)$ *simultaneously*. We call this a global symmetry because the symmetry operation involves all the spins.

Gauge symmetries[3,5,6] are *local* symmetries in which the operation only involves degrees of freedom localized near some point. In this section we will consider the simplest example of a guage system in (3+1)-dimensional space-time. We will call it the Z_2 gauge system.

Let us imagine a simple cubic lattice in $(d=4)$-dimensional space. The elements of the lattice are sites labeled by four integers $X=(x_1,x_2,x_3,x_4)$ and links labeled by a site X and a unit vector $\hat{n}_i$ pointing in one of eight lattice directions. Alternatively the links can be labeled by a pair of nearest-neighbor sites (X_1,X_2). The spin degrees of freedom for the gauge system are defined on the links (see Fig. 14). Each site of the lattice is connected to eight links (Fig. 15) and therefore to eight spins. A local gauge transformation at the site X flip all eight spins leaving the remaining spins unchanged.

Let us now build an action which is invariant under such gauge transformations. The terms of the action are identified with the *faces* or elementary boxes of the lattice (see Fig. 16).

For each box, define an action

$$\mathcal{L}_{\text{box}} = -\beta\sigma_3(1)\sigma_3(2)\sigma_3(3)\sigma_3(4),$$
$$\mathcal{L} = -\beta \sum_{\text{boxes}} \sigma_3\sigma_3\sigma_3\sigma_3, \tag{3.1}$$

where $\sigma_3\sigma_3\sigma_3\sigma_3$ represents the product of spins on the edges of the box.

Now consider the behavior of $\mathcal{L}_{\text{box}}$ under a local guage transformation at X. If X is not a corner of "box" then none of the spins in $\mathcal{L}_{\text{box}}$ are flipped and $\mathcal{L}_{\text{box}}$ is unchanged. If X is a corner of box then two

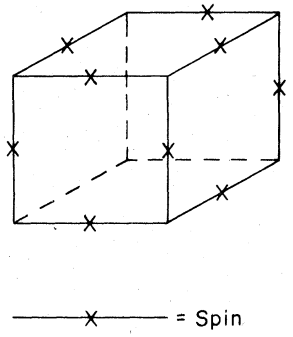

FIG. 14. The degrees of freedom of the gauge theory are defined on the links of the lattice.

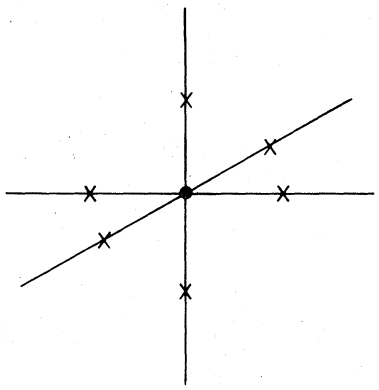

FIG. 15. In 3 + 1 dimensions, each lattice site is connected with eight links.

spins are flipped and $\mathcal{L}_{\text{box}}$ is again unchanged. Therefore $\mathcal{L}_{\text{box}}$ and $\mathcal{L}$ are invariant under local gauge transformations. A more general class of gauge-invariant objects can be formed by considering arbitrary closed paths of links as in Fig. 17. The products of σ_3's on the links forming such paths are gauge invariant.

Consider the expectation value of any gauge-invariant object Γ

$$\langle\Gamma\rangle = \sum_{\{\sigma\}} \Gamma(\sigma)\exp(-\mathcal{L}) \Big/ \sum_{\{\sigma\}} \exp(-\mathcal{L}). \tag{3.2}$$

The sum $\sum_{\{\sigma\}}$ is over all configurations of the σ's. This means that we will add contributions corresponding to configurations which are identical modulo a gauge transformation. Since both $\mathcal{L}$ and Γ are gauge invariant we are counting the same contributions many times. One way to avoid that is to introduce a gauge-fixing condition or constraint which selects out from each gauge equivalence class a single configuration $\bar{\sigma}$. The sum $\sum_{\{\sigma\}}$ can be replaced by

$$\sum_{\{\bar{\sigma}\}} N(\bar{\sigma})$$

where $\sum_{\{\bar{\sigma}\}}$ means a sum over the unique representative of each class and $N(\bar{\sigma})$ is the number of equivalent configurations to $\bar{\sigma}$. For an infinite lattice $N(\bar{\sigma})$ is infinite but for any finite lattice $N(\bar{\sigma})$ is in fact independent of $\bar{\sigma}$ so that restricting the sum to $\bar{\sigma}$ merely introduces an irrelevant mul-

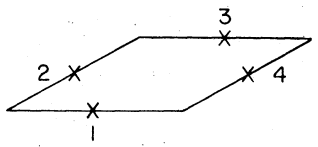

FIG. 16. The terms of the action are identified with the faces of the four-dimensional cubic lattice.

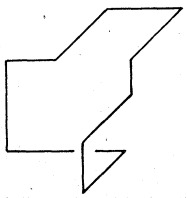

FIG. 17. A closed path of links.

tiplicative factor.

In what follows we will impose such a restriction on the configuration space. It can be shown that any configuration is gauge equivalent to a configuration in which the spins on timelike links are fixed to be equal to 1. However, this condition does not determine a unique configuration. Consider an arbitrary configuration of σ_3's on spacelike links and $\sigma_3 = 1$ on time links. Now consider a transformation which is composed of an infinite product of local gauge transformation which is composed of a an infinite product of local gauge transformations. The product is over all the lattice sites which have given spatial location (x_1, x_2, x_3) and all values of Euclidean time x_4. The relevant sites are shown in Fig. 18. The effect is to reverse only those spins on the six spatial links connected to (x_1, x_2, x_3). In particular, no time link is flipped. Thus the gauge-fixing condition

$$\sigma_3 = 1 \quad \text{(time links)} \tag{3.3}$$

does not uniquely define a configuration within each gauge equivalence class. However, it can be seen that the number of configurations satisfying (3.3) is the same for each equivalence class. Thus imposing (3.3) on the configuration sum introduces a mere numerical factor. Henceforth Eq. (3.3) will be assumed.

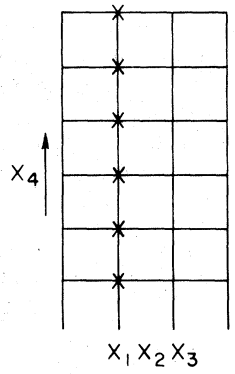

FIG. 18. A time-independent gauge transformation at spatial site (x_1, x_2, x_3).

B. Hamiltonian form

As in the Ising case, we will introduce two coupling constants, one for space-time boxes and one for space-space boxes. The action for a given space-time box (see Fig. 19) is

$$\begin{aligned} \mathcal{L} &= -\beta_t[\sigma_3(1)\sigma_3(2)\sigma_3(3)\sigma_3(4)] \\ &= -\beta_t[\sigma_3(1)\sigma_3(3)] \\ &= +\tfrac{1}{2}\beta_t[\sigma_3(1) - \sigma_3(3)]^2 - \text{const.} \end{aligned} \tag{3.4}$$

Thus for each spatial link the sum over x_4 is an Ising-type action. We denote the space-time term of the action by

$$\sum_{(l,x_4)} \tfrac{1}{2}\beta_t[\sigma_3(l,x_4) - \sigma_3(l,x_4+1)]^2, \tag{3.5}$$

where l labels spatial links.

The space-space boxes contribute with a term

$$-\sum_{ss} \beta_s \sigma_3\sigma_3\sigma_3\sigma_3, \tag{3.6}$$

where the sum is over all spatially oriented boxes. Thus

$$\begin{aligned} \mathcal{L} = &\sum_{(l,x_4)} \tfrac{1}{2}\beta_t[\sigma_3(l,x_4) - \sigma_3(l,x_4+1)]^2 \\ &- \sum_{ss} \beta_s \sigma_3\sigma_3\sigma_3\sigma_3. \end{aligned} \tag{3.7}$$

The passage to a Hamiltonian formulation is performed by the same limiting procedure as for the Ising case, namely

$$\begin{aligned} &\beta_t \to \infty, \\ &\beta_s \to \lambda \exp(-2\beta_t). \end{aligned} \tag{3.8}$$

Thus, we find

$$H = -\sum_{\text{links}} \sigma_1(l) - \lambda \sum_{\text{boxes}} \sigma_3\sigma_3\sigma_3\sigma_3, \tag{3.9}$$

where the sums are over spatial positions only. As in the previous case of the Ising model the σ's are Pauli spin operators acting in a Hilbert space.

The Hamiltonian (3.9) has a local gauge invariance as a consequence of the original gauge invariance of the Lagrangian. Consider the spatial

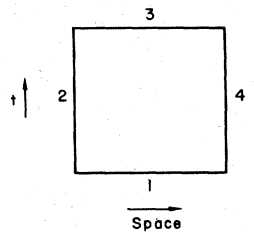

FIG. 19. A space-time box.

site $\vec{r}=(x_1,x_2,x_3)$ and define the operator

$$G_{\vec{r}}=\prod_{1\le i\le 6}\sigma_1(l_i), \tag{3.10}$$

where l_i are the six links attached to r. G_r is a unitary operator which has the following action on the σ's:

$$\begin{aligned} &G_r^{-1}\sigma_1(l)G_r=\sigma_1(l) \quad \text{all } l,\\ &G_r^{-1}\sigma_3(l_i)G_r=-\sigma_3(l_i) \quad (l_i \text{ attached to } r),\\ &G_r^{-1}\sigma_3(l)G_r=\sigma_3(l) \quad (l \text{ not attached to } r). \end{aligned} \tag{3.11}$$

Thus, the action of G_r is to flip the σ_3's linked to site r and leave unchanged all σ_1's. Evidently the Hamiltonian (3.9) is invariant under G_r.

It may also be proved that the ground state of H is invariant under gauge transformations.[10] Calling the ground state $|0\rangle_\lambda$,

$$G_r|0\rangle_\lambda=|0\rangle_\lambda \quad (\text{all } r). \tag{3.12}$$

This is very different from the global invariance of the Ising model. In that case the ground state for $\lambda\gg 1$ is doubly degenerate and the symmetry transformation takes one vacuum to the other. The stability of the spontaneously broken symmetry lies in the fact that it takes an infinite number of steps in perturbation theory (powers of $H_0=-\sum\sigma_1$) to mix the degenerate states. This is not the case here. For example, suppose the vacuum for $\lambda\gg 1$ was all $\sigma_3=1$. The perturbation $-\sum_{\text{links}}\sigma_1$ can act six times to flip the σ spins linked to (x_1, x_2, x_3) thus mixing the ground state with another in the same class. Thus even for $\lambda\gg 1$ the spontaneously broken ground state is unstable.

Since we are interested only in gauge-invariant operators acting on $|0\rangle$ the only states of interest will also be gauge invariant. Accordingly we consider as physically interesting only those states satisfying

$$\begin{aligned} &G(r)|\psi\rangle=|\psi\rangle \quad (\text{all } r)\\ &\text{or}\\ &\prod_{1\le i\le 6}\sigma_1(l_i)|\psi\rangle=|\psi\rangle . \end{aligned} \tag{3.13}$$

Note that since H is gauge invariant, condition (3.13) is consistent with the dynamics.

Equation (3.13) has as a consequence the vanishing of expectation values of all $\sigma_3(l)$. Thus consider

$$\langle\psi|\sigma_3(l)|\psi\rangle .$$

From (3.13) we write

$$\langle\psi|\sigma_3(l)|\psi\rangle=\langle\psi|G^{-1}(r)\sigma_3(l)G(r)|\psi\rangle , \tag{3.14}$$

where r is one of the end points of the link l. But

$$G^{-1}(r)\sigma_3(l)G(r)=-\sigma_3(l) \tag{3.15}$$

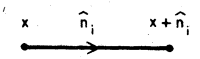

FIG. 20. How to label a link.

so $\langle\psi|\sigma_3|\psi\rangle=0$. Therefore there can be no magnetization in any state satisfying gauge invariance. In particular, no phase transition can lead to a magnetized phase. Nevertheless we shall see that a phase transition exists.

C. Lattice duality

In the first part of this paper we demonstrated the self-duality of the Ising model in the Hamiltonian version. We shall now prove that the Z_2 gauge system is also self-dual in the Hamiltonian form.[5] We shall explicitly construct the variables on the dual lattice. We show that the Hamiltonian takes the same form in terms of the original and dual variables. To carry out this discussion we will need a compact notation to label spatial links. A link may be labeled by a site and a unit vector. The link $(x,\hat{n}_i)$ originates at x and ends at $x+\hat{n}_i$ where i may be any of six unit vectors. The link $(x,\hat{n}_i)$ is evidently equivalent to $(x+\hat{n}_i,-\hat{n}_i)$ (see Fig. 20). The link variables will be denoted by $\sigma(x,\hat{n})$.

The duality transformation turns out to be simplest in a different gauge the gauge we have used up to now. We define the "axial" gauge by

$$\sigma_3=1 \tag{3.16}$$

on those links oriented along the spatial x_3 axis.

The independent variables in the axial gauge are the σ_3's and σ_1's on the x_1,x_2 (transverse) links. The σ_1's on the x_3 links are defined in terms of the independent variables by requiring Eq. (3.13) to be

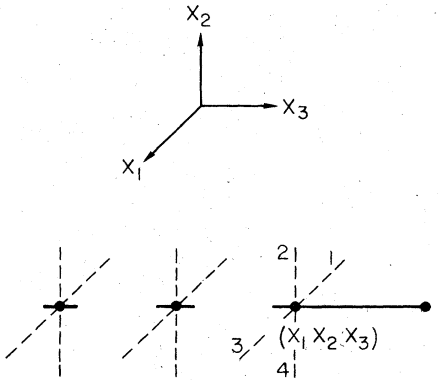

FIG. 21. The operator $\sigma_1(x_1,x_2,x_3;\hat{n}_3)$ is defined in terms of the σ_1's on the $\hat{n}_1$ and $\hat{n}_2$ directions (broken lines).

true. This can be done by defining

$$\sigma_1(x_1,x_2,x_3,\hat{n}_3)=\prod_{x_3'\le x_3}\prod_{i=1}^{4}\sigma_1(x_1,x_2,x_3',\hat{n}_i), \quad (3.17)$$

where $\sigma_1(x_1,x_2,x_3,\hat{n}_3)$ is the (dependent) variable for the link $(x_1,x_2,x_3,\hat{n}_3)$ shown as solid in Fig. 21. The product is over all transverse links shown as broken lines in Fig. 21. $\sigma_1(x,n_3)$ satisfies the identity

$$\sigma_1(x_1,x_2,x_3,n_3)=\left[\prod_{i=1}^{4}\sigma_1(x_1,x_2,x_3,\hat{n}_i)\right]\times\sigma_1(x_1,x_2,x_3-1,\hat{n}_3). \quad (3.18)$$

The reader can now easily prove that

$$\prod_{i=1}^{6}\sigma_1(x,\hat{n}_i)=1 \quad (3.19)$$

is an identity. The Hamiltonian in the axial gauge takes exactly the same form as Eq. (3.9). The only modification is that the $\sigma_3(x,\hat{n}_3)$ are set equal to 1 and $\sigma_1(x,\hat{n}_3)$ is defined by (3.17).

Now we define the dual lattice. The sites of the dual lattice are placed at the body centers of the original lattice (centers of cubes). The dual links pierce the original boxes at their centers. The dual boxes correspond to the original links (see Fig. 22).

Next we must define dual lattice variables μ_1 and μ_3 on the dual links. Each dual link corresponds uniquely to an original box. The variables μ_1 are defined by

$$\mu_1=\prod\sigma_3\sigma_3\sigma_3\sigma_3, \quad (3.20)$$

where the four σ_3's belong to the edges of the box.

For the μ_3 variables we distinguish the x_3 and transverse links of the dual lattice. For the x_3 links the definition of μ_3 is 1 since we are working in the axial gauge. For the transverse link the definition of μ_3 is analogous to the Ising case.

In the Ising model the dual variables μ_3 were

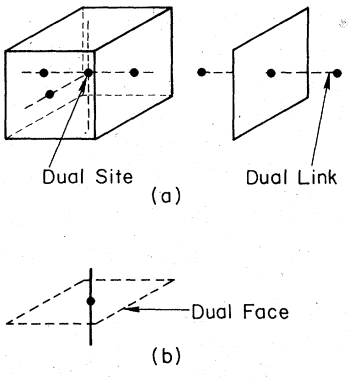

FIG. 22. The dual lattice.

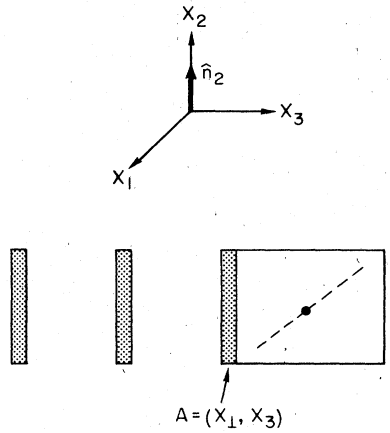

FIG. 23. σ_3 on a dual link is defined as a product of σ_1's on the solid links.

defined by infinite products of σ_1's from $z=-\infty$ to the preceding site. The μ_3 on transverse links are again infinite products of σ_1's. To define this product we note that each transverse link (say in the x_1 direction) may be identified with a box of the original lattice lying in the x_2x_3 plane (see Fig. 23).

Now consider the product

$$\mu_3(\hat{n}_1)=\prod_{x_3'\le x_3}\sigma_1(x_1,x_3,\hat{n}_2). \quad (3.21)$$

The links included in the product are indicated in Fig. 23 by heavy lines. An identical procedure is used for $\mu_3(\hat{n}_2)$.

The following points can be proved very easily:

(i) On transverse dual links the μ_3 and μ_1 satisfy a Pauli algebra.

(ii) Consider the six dual links originating at a dual site. Then

$$\prod_{i}^{6}\mu_1=1 \quad \text{all dual sites.} \quad (3.22)$$

(iii) For each dual box the product $\mu_3\mu_3\mu_3\mu_3$ on the edges of the box is equal to the σ_1 on the corresponding original link. This, however, is only true if we impose Eq. (3.13).

(iv) By definition μ_1 on a dual link equals $\sigma_3\sigma_3\sigma_3\sigma_3$ for the corresponding box.

Thus, it follows that the original Hamiltonian may be reexpressed in terms of the dual variables as

$$H=-\sum_{\text{dual boxes}}\mu_3\mu_3\mu_3\mu_3-\lambda\sum_{\text{dual link}}\mu_1$$

$$=\lambda\left(-\sum_{\text{dual links}}\mu_1-\frac{1}{\lambda}\sum_{\text{dual boxes}}\mu_3\mu_3\mu_3\mu_3\right). \quad (3.23)$$

Thus H is self-dual. As for the Ising case the self-duality relates the physics of $\lambda > 1$ with $\lambda < 1$.

D. Small λ phase

For $\lambda = 0$ the Hamiltonian is

$$H_0 = -\sum_{\text{links}} \sigma_1. \tag{3.24}$$

H_0 has a well-defined nondegenerate gauge-invariant ground state $|0\rangle_{\lambda=0}$ such that

$$\sigma_1(r,n)|0\rangle_{\lambda=0} = |0\rangle_{\lambda=0} \tag{3.25}$$

for all $(\vec{r},\hat{n})$.

The spectrum of excitations includes both gauge-invariant and gauge-noninvariant states. These states are created by flipping the value of σ_1 on any combination of links. However, the gauge-invariant subspace satisfying (3.13) corresponds to special configurations. To construct these states we begin with an arbitrary *closed* path of links. The path may intersect itself and may consist of several disconnected parts but it should have no ends. Now consider the state obtained by slipping the σ_1's on these links. The result is a closed path of links with $\sigma_1 = -1$. It is evident that as long as no end points occur then (3.13) is satisfied at every vertex. These then are a complete set of gauge-invariant excitations. The energy of an excitation for $\lambda = 0$ is simply

$$E = 2n, \tag{3.26}$$

where n is the total number of flipped σ_1's.

In addition to the finite-energy excitations there are a class of interesting excitations whose energy diverges linearly with the radius of the lattice. These consist of infinite lines of inverted σ_1's called strings (see Fig. 24). The simplest such object is a straight line of flipped spins along one of the lattice axes.

The energy of such a configuration is proportional to $2n$ where n is the linear dimension of the lattice. The energy per unit length of such a line is called the string tension. For $\lambda = 0$ the tension is 2.

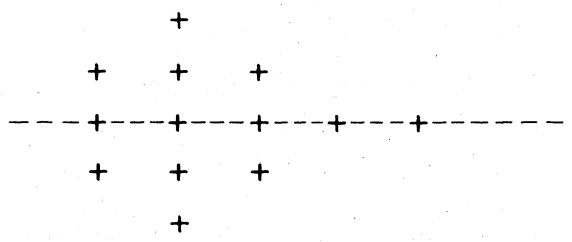

FIG. 24. A stringlike excitation.

FIG. 25. A disconnected graph.

Now suppose λ is small but finite. The term

$$H_1 = -\lambda \sum_{\text{boxes}} \sigma_3\sigma_3\sigma_3\sigma_3 \tag{3.27}$$

will cause modifications of the ground state and excitations. Evidently the action of H_1 on the ground state is to create closed boxes of flipped spins (elementary gauge-invariant excitations). The density of elementary gauge-invariant excitations in the perturbed ground state is $\sim\lambda^2$. Furthermore, the ground-state energy density is lowered. The ground-state energy per site is

$$E = -3 - \tfrac{3}{8}\lambda^2 - \tfrac{9}{512}\lambda^4 + \cdots . \tag{3.28}$$

More interesting is the effect of the perturbation on the strings. The perturbation in this case can act in two different ways. First it can excite an elementary gauge-invariant excitation on a box which is disconnected from the string (see Fig. 25).

These contributions are just renormalizing the vacuum. The other action is to deform the string by putting in a kink. This happens when the perturbation acts on a box containing a side on the string (see Fig. 26). Higher orders in λ cause the string to fluctuate out of the straight line (see Fig. 27). Thus as we let the perturbation act on the string a large number of times, the string will start to percolate. This effect will be more important closer to the critical point.

The fact that strings at λ finite are not straight lines makes ambiguous our definition of the string tension. We can define the string tension, for finite values of λ, as the energy of the string divided by N, the linear dimension of the lattice, i.e., the original string length.

Now we can see that the whole effect of the elementary gauge-invariant excitations acting on the string is just to deform the string as well as to lower its tension. We find

$$T = 2 - \tfrac{1}{2}\lambda^2 - \tfrac{325}{1536}\lambda^4 + \cdots . \tag{3.29}$$

From (3.29) it appears that T might vanish for some finite λ. Suppose this occurs. We argue that this signals a phase transition. The reason is that the tension cannot become negative. If it did the

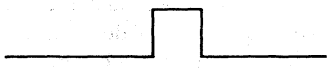

FIG. 26. The string is deformed by the perturbation.

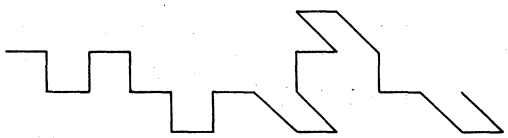

FIG. 27. The string starts to percolate.

string would lower its energy by growing longer. The ground state would be unstable with respect to the creation of infinitely long strings which fill space. Thus at the point where T vanishes a global change in the behavior of the ground state must occur.

If we ignore higher orders in λ then T vanishes at about $\lambda^2 = 2.1$. However, for $\lambda^2 = 2.1$ the series is not yet converging. We can improve the situation by using Padé approximants to extrapolate (3.29). This gives

$$T = 2 \times \frac{1 - 0.67\lambda^2}{1 - 0.42\lambda^2}, \tag{3.30}$$

which vanishes at $\lambda \sim 1.22$. A more refined method is to compute the logarithmic derivative of T and use Padé approximants[13] to determine the pole of

$$\frac{1}{T}\frac{dT}{d\lambda^2} = -\tfrac{1}{4}(1 + 1.1\lambda^2 + \cdots) \xrightarrow[\text{Padé}]{} -\frac{1}{4}\frac{1}{1 - 1.1\lambda^2}. \tag{3.31}$$

The pole (zero of T) occurs at

$$\lambda = 0.912. \tag{3.32}$$

The exact position of the phase transition (assuming one occurs) must be at $\lambda = 1$. This is because of the self-duality relating $\lambda > 1$ and $\lambda < 1$. However, the self-duality does not tell us whether the transition is first order or second order. The apparent vanishing of T for $\lambda \approx 1$ strongly suggests a second-order transition.

E. Large λ phase

Now consider the limit $\lambda \gg 1$. We write

$$\frac{H}{\lambda} = -\sum_{\text{box}} \sigma_3(1)\sigma_3(2)\sigma_3(3)\sigma_3(4) - \frac{1}{\lambda}\sum_{\text{links}} \sigma_1. \tag{3.33}$$

For $\lambda = \infty$ the second term of (3.33) may be ignored. In this case the ground state is determined by the term

$$-\sum_{\text{box}} \sigma_3\sigma_3\sigma_3\sigma_3. \tag{3.34}$$

The lowest eigenvalue of (3.34) occurs when all $\sigma_3 = 1$. However, the ground state is infinitely degenerate. To see this consider a gauge transformation on the state with $\sigma_3(r) = 1$. Such a transformation flips various σ_3's but always leaving the value of (3.34) unchanged. Thus, there is a degeneracy due to the non-gauge-invariance of the state $\sigma_3 = 1$ (all r).

In the Ising model an analogous condition occurs for $\lambda = \infty$. Here the ground state is twofold degenerate. This is connected with the global Z_2 invariance. There is, however, an important difference between the models In the Ising case, the two vacuums cannot mix in any infinite order in perturbation theory in λ^{-1}. In other words, it requires an infinite number of spin flips to go from one to the other.

In the Z_2 gauge case, a gauge transformation flips only six spins. Therefore, to go from one degenerate vacuum to another requires only six orders in λ^{-1}. Accordingly sixth-order perturbation theory lifts the degeneracy.

The correct vacuum for $\lambda = \infty$ is a gauge singlet. It is formed by superposing symmetrically the state $\sigma_3(r) = 1$ with all its gauge-related counterparts. Of course for all gauge-invariant quantities we can ignore this subtlety and use the state $\sigma_3(r) = 1$.

The lightest excitations of the ground state for $\lambda = \infty$ are given by applying σ_1(link) on some link. This flips the corresponding σ_3.

To give a gauge-invariant description of these excitations we must specify the values of some complete set of gauge-invariant functions of the σ_3's. Most simply we can give the value of every box variable $\sigma_3(1)\sigma_3(2)\sigma_3(3)\sigma_3(4)$ or equivalently μ. For example, suppose we apply σ_1(link) on the link shown as the heavy line in Fig. 28. This operation evidently inverts the four box variables on the four boxes containing the link. These are shown in the figure by unbroken light lines. The four boxes can be identified with four links of the dual lattice shown as dashed lines. These four dual links form a closed loop. Thus the resulting excitation is a closed ring of flipped μ_1's dual to the excitations of the $\lambda = \infty$ ground state.

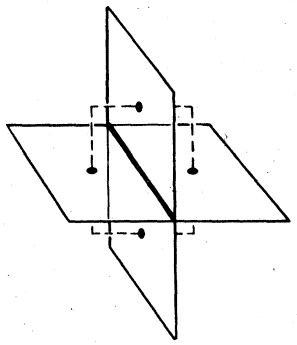

FIG. 28. An excited link (in dark) is dual to a box with inverted flux (dotted line).

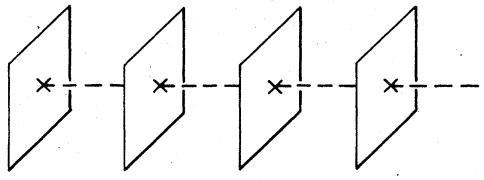

FIG. 29. The string is dual to a tube of boxes with inverted flux.

Next consider the dual of the infinite line of inverted σ_1's. These excitations should be lines of boxes with the box product $\sigma_3\sigma_3\sigma_3\sigma_3=-1$, namely lines of inverted μ_1's (see Fig. 29). The small λ excitations are lines of inverted σ_1's. A line of inverted σ_1's is dual to a half plane of inverted μ_3, as is shown in Fig. 30. However, this state is not gauge invariant. We can get a gauge-invariant state dual to the line of inverted σ_1's if we notice that σ_1 is a gauge-invariant operator dual to the box product $\mu_3\mu_3\mu_3\mu_3$ which is a gauge-invariant operator too. Thus the dual statement to $\sigma_1=-1$ is $\mu_3\mu_3\mu_3\mu_3=-1$ on the boxes which is dual to the link with an inverted σ_1. Therefore a line of boxes with the box product $\sigma_3\sigma_3\sigma_3\sigma_3=-1$ is a gauge-invariant state which is dual to the line with $\mu_1=-1$.

F. Correlation functions

In the preceding sections we stated that only gauge-invariant operators will have a nonvanishing expectation value. Thus it is clear that the two-point correlation function vanishes identically since $\sigma_3(\vec{r},\hat{n})\sigma_3(\vec{r}',\hat{n}')$ is not a gauge:

$$\langle\sigma_3(\vec{r},\hat{n})\sigma_3(\vec{r}',\hat{n}')\rangle\equiv 0 \quad (\text{all } \lambda). \tag{3.35}$$

What we need is a suitable definition of the correlation functions.

The correct object to study is the ground-state expectation value of the product of σ_3's along a closed loop[5,3] on the lattice (see Fig. 31). Let us call $C_\Gamma(R)$ such a magnitude for a loop Γ with typical size R. Since $C_\Gamma(R)$ is a gauge-invariant quantity it may have a nonvanishing value. However, $C_\Gamma(R)$ will depend in general on the details of the loop Γ. But we shall be interested only in its asymptotic behavior as $R\to\infty$. In this limit we shall want to know if there is a phase in which $C_\Gamma(R)$ is asymptotically constant (this should be an ordered phase) or what kind of decay it exhibits as $R\to\infty$ otherwise.

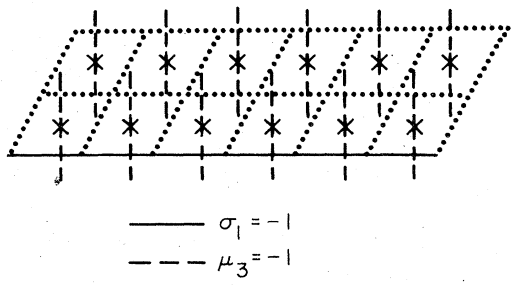

FIG. 30. A half plane of inverted μ_3 is dual to the string, although is not a gauge-invariant state.

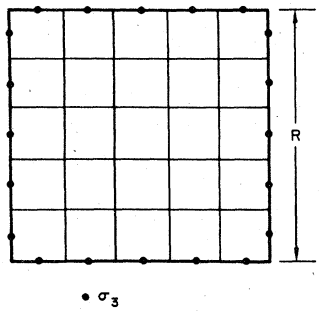

FIG. 31. A closed loop on the lattice.

We shall show that $C_\Gamma(R)$ exhibits two different behaviors for λ large and small although there is nothing like an ordered phase behavior in both cases.

The asymptotic behavior is

$$C_\Gamma(R)\xrightarrow[R\to\infty]{}\begin{cases}\exp(-A), & \lambda<1\\ \exp(-P), & \lambda>1\end{cases} \tag{3.36}$$

where A and P are the area and the perimeter of the loop. Clearly e^{-A} is analogous to the exponential decay in Ising-type systems and therefore we regard the $\lambda<1$ phase as a disordered phase. However, the phase $\lambda>1$ is not ordered since $C_\Gamma(R)$ vanishes as $R\to\infty$, but it decays much more slowly than in other phase. It is clear that the phase transition is not order-disorder, but it is a change in the behavior of the correlation function.

It is easy to understand the behavior of the $\lambda<1$ phase. Since the unperturbed ground state $|0\rangle$ is orthogonal to the state $\prod_{\text{loop}}\sigma_3|0\rangle$, then at zeroth order in perturbation theory

$$\left\langle 0\left|\prod_\Gamma\sigma_3\right|0\right\rangle=0, \quad \lambda=0. \tag{3.37}$$

However, we may get a nonvanishing result if we go to some higher order in perturbation theory. The lowest order needed to get a nonvanishing result is equal to the least number of elementary loops enclosed by Γ, which is exactly the area of Γ. Thus

$$C_\Gamma(R)\sim\lambda^n=e^{-n|\ln\lambda|} \tag{3.38}$$

for the lowest order in λ. Here n is the area of Γ and $\lambda>1$.

Let us consider the other behavior, $\lambda>1$. Now

the operator $\prod_\Gamma \sigma_3$ just counts the number of inverted spins along the loop Γ. More precisely, it counts if there is an odd or even number of inverted σ_3's.

The ground state at first order in λ^{-1} is

$$|0\rangle_{\lambda^{-1}} = \frac{1}{(1+\frac{1}{4}N\lambda^{-2})^{1/2}} \times \left(|0\rangle_{\lambda^{-1}=0} + \tfrac{1}{2}\lambda^{-1} \sum_{\text{boxes}} |n\rangle\right), \tag{3.39}$$

where $|n\rangle$ is the state with all the boxes with flux $(\sigma_3\sigma_3\sigma_3\sigma_3)$ equal to 1 but the nth box with flux -1. Then

$$\prod_\Gamma \sigma_3 |0\rangle_{\lambda^{-1}} = \left(|0\rangle_{\lambda^{-1}=0} + \tfrac{1}{2}\lambda^{-1} \sum_{\substack{\text{boxes} \\ \text{unlinked} \\ \text{to } \Gamma}} |n\rangle - \tfrac{1}{2}\lambda^{-1} \sum_{\substack{\text{boxes} \\ \text{linked} \\ \text{to } \Gamma}} |n\rangle\right) \frac{1}{(1+\frac{1}{4}N\lambda^{-2})^{1/2}} . \tag{3.40}$$

A box is considered to be linked to the loop Γ if its inverted spin lies on Γ. There are four linked boxes per spin flipped lying on Γ.

Thus if $4n$ is the number of boxes linked to Γ, we can write

$$C_\Gamma(R) = {}_{\lambda^{-1}}\left\langle 0 \left| \prod_\Gamma \sigma_3 \right| 0 \right\rangle_{\lambda^{-1}} \cong [1 + \tfrac{1}{4}\lambda^{-2}(N-4n) - \tfrac{1}{4}\lambda^{-2}4n] \times \frac{1}{(1+\frac{1}{4}N\lambda^{-2})} , \tag{3.41}$$

$$C_\Gamma(R) \cong 1 - 2n\lambda^{-2} \cong e^{-2n\lambda^{-2}}.$$

But if P is the perimeter of Γ, we have $n = P$. Then

$$C_L(R) \cong \exp\left(-\frac{2P}{\lambda^2}\right), \tag{3.42}$$

which is the "perimeter decay" behavior. Therefore we can regard the phase transition as a change in the behavior of the correlation function since it decays exponentially as R for $\lambda > 1$ while as R^2 for $\lambda < 1$.

We can understand the perimeter behavior as a boundary effect. However, as λ is decreased more and more box fluxes may be inverted, and when λ is close enough to the critical value 1 the regions with inverted flux become of the same size of the area of Γ. Thus a boundary effect is turned into an area effect. These large regions of inverted flux have boundaries which are lines of flipped spins. Near the critical point long lines of inverted spins go through the loop (see Fig. 32). These lines are closed at infinity and they are topological objects which cannot be removed from the system. These *kinks* change the behavior of the correlation function when they condense. They play here exactly the same role as in the Ising model. In the strongly coupled phase ($\lambda > 1$) they are massive. However, at the critical point they become massless and therefore they condensate randomizing the system. It is clearly seen from their definition that they are large topological objects which change the boundary conditions and cannot be removed by any finite number of spin flippings.

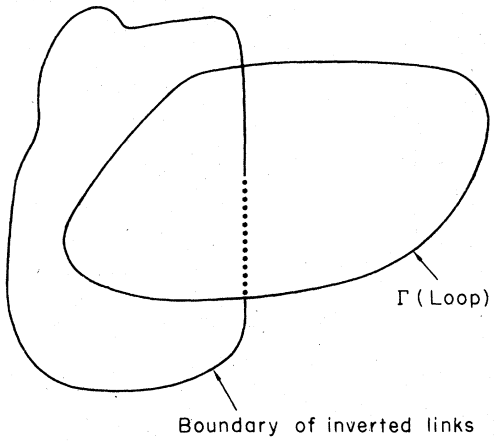

FIG. 32. The kinks of the Z_2 gauge theory in 3 + 1 dimensions are lines of inverted links which are closed at infinity.

IV. THE *XY* MODEL

A. Construction of the Hamiltonian

In the next systems we shall study, the symmetry operations, both global and gauge, are continuous. The group Z_2 is replaced by the continuous rotation group O_2. The first example with O_2 symmetry is the two-dimensional XY model of a magnet.[4] At each site of a two-dimensional square lattice there is a unit two-vector, $\vec{\sigma}$, described by an angle $\phi(\vec{r})$. All physical quantities are periodic in ϕ.

The interaction between sites is of the form $\vec{\sigma}(r) \cdot \vec{\sigma}(r+1)$ or $\cos[\phi(\vec{r}) - \phi(\vec{r}')]$. Thus for an anisotropic lattice the action is defined as

$$\mathcal{A} = -\sum_r \beta_t \cos[\phi(\vec{r}) - \phi(\vec{r}+\hat{n}_t)] - \sum_r \beta_z \cos[\phi(\vec{r}) - \phi(\vec{r}+\hat{n}_z)]. \tag{4.1}$$

Evidently the action is minimized by configurations in which all the spins are parallel as for the Ising model, although this state is not unique. However, unlike the Ising case the degeneracy is infinite corresponding to the continuously variable direction of magnetization. The fact that the de-

generate states are infinitely close to one another makes the two systems essentially different. This will be evident when we discuss the long-range correlations in the system.

To construct a time-continuum limit we can follow the method used for the Ising case. The same result can be obtained from a simpler and more familiar argument. Let us first allow $\beta_t \to \infty$. We also introduce a time lattice spacing a which in the present case behaves like $1/\beta_t$. The reason for rescaling the time direction is again to make the theory finite as $\beta_t \to \infty$.

We write the time-like term as

$$-\beta_t \sum_{\vec{r}} \cos a\left(\frac{\phi(\vec{r}) - \phi(\vec{r}+\hat{n}_t)}{a}\right). \tag{4.2}$$

As $\beta_t \to \infty$ the important values of $\phi(\vec{r})$ and $\phi(\vec{r}+\hat{n}_t)$ will be those for which $\phi(\vec{r}) - \phi(\vec{r}+\hat{n}_t) \to 0$ as $1/\beta_t$, which is equal to a. Thus we replace (4.2) by

$$-\beta_t \left[1 - \frac{a^2}{2}\left(\frac{\partial\phi}{\partial t}\right)^2\right]. \tag{4.3}$$

We may of course ignore the constant term. The sum over space-time location may be approximated by

$$\sum_{\vec{r}} = \frac{1}{a}\int dt \sum_z .$$

Thus the timelike terms in a are replaced by

$$(\beta_t a)\int dt \sum_z \frac{1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2. \tag{4.4}$$

The space terms can be written

$$-\sum_{\vec{r}} a\,\frac{\beta_z}{a}\cos[\phi(\vec{r}) - \phi(\vec{r}+\hat{n}_z)]. \tag{4.5}$$

If $\beta_z\beta_t$ is allowed to remain finite as $\beta_t \to \infty$ then β_z/a is a constant λ and (4.5) becomes

$$-\lambda\int dt \sum_z \cos[\phi(\vec{r}) - \phi(\vec{r}+\hat{n}_z)]. \tag{(4.6}$$

Thus the full action is

$$\mathcal{A} = +\int dt \sum_z \left\{\frac{\phi(z)^2}{z} - \lambda\cos[\phi(z) - \phi(z+1)]\right\}. \tag{4.7}$$

In this case the trajectories in β_z, β_t space corresponding to time-continuum limits are hyperbolas

$$\beta_z\beta_t = \lambda. \tag{4.8}$$

Finding the Hamiltonian from (4.7) is just the usual problem of passing from the Lagrangian to the Hamiltonian in mechanics except that the time variable is Euclidean. Thus we obtain

$$H = \sum_z \left\{\frac{L^2(z)}{2} - \lambda\cos[\phi(z) - \phi(z+1)]\right\}, \tag{4.9}$$

where L is the canonical momentum conjugate to ϕ. Quantum mechanically it satisfies

$$[L, e^{i\phi}] = \pm e^{\pm i\phi}. \tag{4.10}$$

Since the variables ϕ are periodic it follows that L has a discrete spectrum which consists of all the integer numbers.

B. Large-λ behavior

For λ large the term $-\lambda\Sigma\cos[\phi(z) - \phi(z+1)]$ forces the field ϕ to be very smooth. For the low-energy configurations the field differences at neighboring sites will be so small that we may expand the cosine. Thus

$$H = \sum \frac{L(z)^2}{2} + \frac{\lambda[\phi(z) - \phi(z+1)]^2}{2} - \frac{\lambda[\phi(z) - \phi(z+1)]^4}{4!} + \cdots . \tag{4.11}$$

For the lowest-energy states it is a good approximation to truncate the series after the second term. To see this explicitly a change of variables will be useful. Define

$$U = \lambda^{1/4}\phi, \qquad P = \lambda^{-1/4}L. \tag{4.12}$$

Evidently U and P are conjugate variables. In terms of the new variables H becomes

$$\lambda^{-1/2}H = \frac{1}{2}\left\{\sum P(z)^2 + [U(z) - U(z+1)]^2 - \frac{1}{12\lambda^{1/2}}[U(z) - U(z+1)]^4 + \cdots\right\}. \tag{4.13}$$

In this form it is plausible that the quartic and higher terms can be neglected for the lowest-energy states. It is also evidently useful to rescale the energy so that $\lambda^{-1/2}H$ becomes the new energy.

Let us consider the ground-state correlations (vacuum expectation values) defined by

$$\langle 0 | e^{i\phi(0)}e^{-i\phi(z)} | 0\rangle , \tag{4.14}$$

which can be written as

$$\left\langle 0 \left| \exp\left\{\frac{i}{\lambda^{1/4}}[U(0) - U(z)]\right\} \right| 0\right\rangle . \tag{4.15}$$

We will approximate this quantity by the corresponding free field value defined by truncating (4.13). For a free field we can write

$$\left\langle 0\left|\exp\left\{\frac{i}{\lambda^{1/4}}[U(0)-U(z)]\right\}\right|0\right\rangle = \exp\left\{-\frac{1}{2\lambda^{1/2}}\langle 0|[U(0)-U(z)]^2|0\rangle\right\}, \quad (4.16)$$

and for large z one easily finds

$$\langle 0|[U(0)-U(z)]^2|0 = \text{const}\times \ln|z| .$$

Thus the correlation function behaves like

$$\langle e^{i\phi(0)}e^{-i\phi(r)}\rangle \sim |z|^{-c/2\sqrt{\lambda}} . \quad (4.17)$$

Thus for large λ the correlation decays as a power of the distance.[4] This type of behavior is somewhat unusual. On the one hand, the fact that the correlation goes to zero implies a lack of infinite range order or what is equivalent, no spontaneous magnetization. On the other hand, the order is not of the usual short-range type which decays as an exponential. Note that the power behavior depends on λ.

C. The small-λ behavior

For the limit $\lambda \to 0$ the dominant term in H is

$$\sum_z \frac{L(z)^2}{2} \quad (4.18)$$

and the ground state is the product state annihilated by each $L(z)$:

$$L(z)|0\rangle = 0 \quad (\text{all } z) . \quad (4.19)$$

For this ground state the correlation function is identically zero for any nonvanishing separation. This is because $\exp^{i\phi(0)}$ increases the value of $L(0)$ by one unit which is not compensated by $\exp[-i\phi(z)]$ if $z \neq 0$.

For small λ we may use perturbation theory to compute the correlation function. To get a nonvanishing contribution to the correlation the perturbation must act at least z times. Accordingly the correlation will behave like

$$\lambda^z = e^{-|\ln\lambda| z} . \quad (4.20)$$

For this phase the correlation decays in the conventional way of a disordered system. The energy spectrum for small λ consists of a ground state and massive excitations created by applying the operators $\exp[i\phi(z)]$.

D. Kinks

We have seen that the $\lambda \gg 1$ phase is not characterized by a nonvanishing order parameter. We will now show that a dual order parameter exists which also vanishes for $\lambda \gg 1$ but which is nonzero for $\lambda \gg 1$. The existence of this dual order can be used to characterize the phase transition.

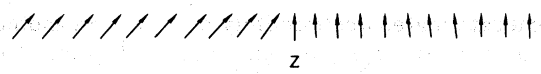

FIG. 33. The kink creation operator rotates all the spins from $-\infty$ up to site z by the same angle f.

Thus define

$$K_f(z_0) = \exp\left[if\sum_{z<z_0} L(z)\right] . \quad (4.21)$$

Let us consider the action of this operator on the *classical* ground state for which $\phi(z)=0$. Since $L(z)$ and $\phi(z)$ are conjugate, K_f rotates all the spins for $z<z_0$ by angle f (see Fig. 33).

We shall now show that

$$\langle 0|K_f(z)|0\rangle \quad (4.22)$$

is a suitable parameter to describe the phase transition and since it is related with the dual of the XY model, we call it the dual order parameter. Let us consider first the large-λ phase. We calculate the correlation function

$$\langle 0|K_f(s)K_f^*(z)|0\rangle = C_f(z-s) , \quad (4.23)$$

where $z>s$, and we want to compute this function in the limit $|z-s| \gg 1$. In order to carry out this calculation it is useful to make a spin wave expansion of the field $\phi(z)$. Let $a^+(k), a^-(k)$ be the creation and destruction operators of a spin wave of momentum k and let $\omega_k \sim |k|$ be the dispersion relation for small k. Since

$$P(z) = \phi(z) = \lambda^{1/4}L(z)$$

we get the following expansions for $\phi(z)$ and $\dot{\phi}(z)$:

$$\phi(z) = \int \frac{dk}{\sqrt{\omega_k}}[a^+(k)e^{ikz} + a^-(k)e^{-ikz}] ,$$
$$\dot{\phi}(z) = -i\int dk\sqrt{\omega_k}\,[a^+(k)e^{ikz} - a^-(k)e^{-ikz}] . \quad (4.24)$$

The magnitude we want to compute is

$$C_f(z-s) = \left\langle 0\left|\exp\left[\frac{\text{if}}{\lambda^{1/4}}\sum_{s\le x\le z}\dot{\phi}(x)\right]\right|0\right\rangle, \quad (4.25)$$

which is the same as

$$C_f(z-s) = \exp\left\{-\frac{f^2}{2\lambda^{1/2}}\left\langle 0\left|\left[\sum_{s\le x\le z}\dot{\phi}(x)\right]^2\right|0\right\rangle\right\}_{\text{free field}} \quad (4.26)$$

for the same arguments given above.

After some algebra one finds that the leading contribution to

$$\left\langle 0\left|\left[\sum_{s\le x\le z}\dot{\phi}(x)\right]^2\right|0\right\rangle_{\text{free}} \quad (4.27)$$

as $L=|z-s|$ to infinity is

$$\left\langle 0\left|\left[\sum_{0\le x\le L}\phi(x)\right]^2\right|0\right\rangle_{\text{free}} \simeq 8\ln\frac{\pi L}{2}\ ; \tag{4.28}$$

then the correlation function $C_f(L)$ falls off as

$$C_f(L)=\exp\left(-\frac{4f^2}{\lambda^{1/2}}\ln\frac{\pi L}{2}\right)$$

$$=\text{const}\times L^{-4f^2/\lambda^{1/2}} \tag{4.29}$$

in leading terms in L. This means that the order parameter $\langle 0|K_f(z)|0\rangle$ is exactly zero in this phase since

$$\langle 0|K_f|0\rangle^2=\lim_{L\to\infty} C_f(L)=0. \tag{4.30}$$

However, the behavior of the correlation function is not of the type for a disordered phase, i.e., exponential decay.

Let us finally consider the small-λ phase where $\langle 0|K_f|0\rangle$ is finite. Since the ground state for $\lambda=0$ verifies that $L|0\rangle=0$ for all lattice sites, it is clear that

$$\langle 0|K_f|0\rangle=1 \tag{4.31}$$

at $\lambda=0$. Once again we may ask the question: Is it nonzero only at $\lambda=0$ or is there a finite neighborhood of $\lambda=0$ where $\langle 0|K_f|0\rangle$ has a nonvanishing value? This question can be answered by computing $\langle 0|K_f|0\rangle$ in perturbation theory.

The calculation gives the result

$$\langle 0|K_f|0\rangle=1-\tfrac{1}{2}\lambda^2(1-\cos f)+O(\lambda^4) \tag{4.32}$$

for the lowest nontrivial order in perturbation theory. Since the coefficient of λ^2 is finite we argue again, the same as in the Ising model, that $\langle 0|K_f|0\rangle$ is not only nonzero at $\lambda=0$ but it has a nonvanishing value at a finite neighborhood of $\lambda=0$. This result makes it impossible to have a first-order phase transition in λ at $\lambda=0$ as is the case of the one-dimensional Ising model as a function of temperature.

As a conclusion we summarize the results just obtained drawing a qualitative picture of both phases. For large λ the system has massless spin waves and exhibits an almost-ordered phase in terms of the original system. In this phase the system has heavy kinks. On the other hand, for small λ the spin waves become massive and the kinks become massless and, what is much more important, they condense giving rise to a nonvanishing dual order parameter $\langle 0|K_f|0\rangle$ which characterizes the phase.

V. ABELIAN GAUGE THEORY

A. The model

As in Sec. III a simple cubic three-dimensional lattice replaces space. Time is continuous. The degrees of freedom are attached to the links[14] and consist of planar rotators or phase angles $\phi(\vec{r},\hat{n})$. The conjugate variables $L(\vec{r},\hat{n})$ $[=-L(\vec{r}+\hat{n},-\hat{n})]$ have integer spectrum.

The Hamiltonian for this system is given by the sum of two terms which we call electric and magnetic,

$$H_{\text{electric}}=\sum_{\text{links}}\frac{g^2}{2a}L(\vec{r},\hat{n})^2\ , \tag{5.1}$$

where g is a dimensionless coupling constant and a is the lattice spacing in some arbitrary units.

The magnetic term is a sum of interactions, each associated with an elementary square box of the lattice. Let us consider a given box as shown in Fig. 34. The sides of the box are labeled 1, 2, 3, 4 and are thought of as oriented. The magnetic interaction for this box is given by

$$-\frac{1}{ag^2}\cos[\phi(1)+\phi(2)+\phi(3)+\phi(4)]. \tag{5.2}$$

Thus the Hamiltonian is

$$H=H_e+H_m$$

$$=\sum_{\text{links}}\frac{g^2}{2a}L(\vec{r},\hat{n})^2$$

$$-\sum_{\text{boxes}}\frac{1}{g^2a}\cos[\phi(1)+\cdots+\phi(4)]\ . \tag{5.3}$$

B. Gauge invariance

For each site of the lattice we can define a gauge transformation which rotates the phase angles of all six links radiating from that site. Thus

$$e^{i\phi(\vec{r}_0,\hat{n})}\to e^{i\phi(\vec{r}_0,\hat{n})+i\lambda}\ . \tag{5.4}$$

These transformations are expressed as unitary

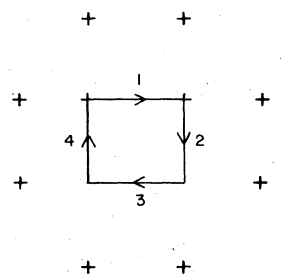

FIG. 34. The magnetic terms are associated with the boxes.

operators

$$G(r_0)=\exp\left[i\lambda\sum_{i=1}^{6}L(\vec{r}_0,\hat{n}_i)\right]\ . \tag{5.5}$$

The Hamiltonian is invariant under these transformations. Furthermore, as in the Z_2 gauge theory we will consider the physical space of states to consist of the gauge-invariant states. From (5.5) we see that a physical state is defined by

$$\sum_{i=1}^{6}L(\vec{r}_0,n_i)\,|\psi\rangle=0. \tag{5.6}$$

Later we will see that this is the lattice form of Gauss's law $\nabla\cdot\vec{E}=0$. It is easy to show that for any gauge-invariant state the expectation value of $e^{i\phi(\vec{r},\hat{n})}$ vanishes.

Moreover, the two-point correlation function

$$\langle 0\,|e^{i[\phi(0,n)-\phi(\vec{R},\hat{n})]}\,|0\rangle$$

vanishes. The vacuum is defined to be a gauge-invariant state $|0\rangle$ where we have $|0\rangle=G\,|0\rangle$ for all G. Thus

$$\begin{aligned}\langle 0\,|e^{i\phi(0,\hat{n})}e^{-i\phi(\vec{R},\hat{n})}\,|0\rangle&=\langle 0\,|G_\lambda^{-1}(0)e^{i\phi(0,\hat{n})}G_\lambda(0)G_\lambda^{-1}(0)e^{-i\phi(\vec{R},\hat{n})}G_\lambda(0)\,|0\rangle\\&=\langle 0\,|e^{i[\phi(0,\hat{n})+\lambda]}e^{-i\phi(\vec{R},\hat{n})}\,|0\rangle\end{aligned} \tag{5.7}$$

since L and ϕ commute at different lines. Then

$$\langle 0\,|e^{i\phi(0,\hat{n})}e^{-i\phi(\vec{R},\hat{n})}\,|0\rangle=e^{i\lambda}\langle 0\,|e^{i\phi(0,\hat{n})}e^{-i\phi(\vec{R},\hat{n})}\,|0\rangle \tag{5.8}$$

for all values of λ. Equation (5.8) implies that[15]

$$\langle 0\,|e^{i\phi(0,\hat{n})}e^{-i\phi(\vec{R},\hat{n})}\,|0\rangle=0\ . \tag{5.9}$$

This means that the two-point correlation function is not the right object to look at. Again, as in the Z_2 gauge theory, the correct magnitude is the loop integral, namely

$$\exp\left(i\sum_{\Gamma}\phi\right), \tag{5.10}$$

where $\sum_\Gamma\phi$ means the sum of all the angles ϕ over all the links lying along the closed loop Γ.

Following Wilson's criterion[3] we calculate

$$\left\langle 0\left|\exp\left(i\sum_{\Gamma}\phi\right)\right|0\right\rangle, \tag{5.11}$$

which is known as the loop integral. We shall show that this function is able to distinguish between two phases. First of all as the operator (5.10) is gauge invariant (5.11) may not be zero. Furthermore, we shall show that in the small coupling phase it decays as $e^{-\text{perimeter}}$ and in the large-coupling phase it goes as $e^{-\text{area}}$ in agreement with the results already discussed in the Z_2 gauge theory. In addition we shall see a remarkable parallel between the behavior of the 3+1 Abelian gauge theory and the 1+1 XY ferromagnet.

C. Small-g phase

As in the XY model we have to rescale the variables in order to describe the small-g phase. Thus we define the components of the vector potential by

$$A(\vec{r},\hat{n})=\frac{1}{ag}\phi(\vec{r},\hat{n}) \tag{5.12}$$

and the conjugate electric field $E(\vec{r},\hat{n})$,

$$E(\vec{r},\hat{n})=\frac{g}{a^2}L(\vec{r},\hat{n}), \tag{5.13}$$

so that

$$[A(\vec{r},\hat{n}),E(\vec{r}',n')]=\frac{i}{a^3}\delta_{\hat{n},\hat{n}'}\delta_{\vec{r},\vec{r}'}\ . \tag{5.14}$$

The Hamiltonian now takes the form

$$\begin{aligned}H=a^3\sum_{\text{links}}\frac{E^2}{2}(\text{link})+a^3\sum_{\text{boxes}}\frac{1}{2a^2}\left(\textstyle\sum_\Gamma A\right)^2\\-a^3\sum_{\text{boxes}}g^2\frac{a^4}{4!}\frac{\left(\sum_\Gamma A\right)}{a^4}+O(g^4),\end{aligned} \tag{5.15}$$

where Γ is an elementary box of the lattice. For the long wavelengths we can approximate

$$\begin{aligned}&a^3\sum\to\int d^3x\ ,\\&\frac{1}{a}\sum_{\Gamma}A\to\vec{\nabla}\times\vec{A}\ .\end{aligned} \tag{5.16}$$

The long-wavelength physics is approximated by the continuum field theory

$$H=\tfrac{1}{2}\int a^3x[E^2(x)+(\vec{\nabla}\times\vec{A})^2]+O(g^2a^4). \tag{5.17}$$

Thus the first two terms define conventional free field electrodynamics and the higher-order nonrenormalizable terms are believed to be unimportant for distances very much larger than the lattice spacing.

We now would like to calculate the loop integral

$$C_\Gamma=\left\langle 0\left|T\left[\exp\left(ig\oint_\Gamma\vec{A}\cdot d\vec{l}\right)\right]\right|0\right\rangle \tag{5.18}$$

for a circular path Γ of radius R.

It can be easily proved that

$$C_\Gamma = \exp\left\{-\frac{g^2}{2}\left\langle 0\left|\int_\Gamma\int_\Gamma [\vec{A}(l)\cdot d\vec{l}][\vec{A}(l')\cdot d\vec{l}']\right|0\right\rangle\right\}, \tag{5.19}$$

where $\vec{A}(\vec{l})$ indicates the vector potential at the point $\vec{l}$ along the closed curve.

Since C_Γ is gauge invariant we choose to evaluate (5.19) in the Feynman gauge. The free field propagator is

$$\langle A^\mu(x)A^\nu(y)\rangle = \frac{1}{2\pi^2}\frac{g^{\mu\nu}}{|\vec{x}-\vec{y}|^2} \quad . \tag{5.20}$$

The integrals can be most easily evaluated defining angular coordinates on the circle. The final result is

$$C_\Gamma = \exp\left(-\frac{g^2}{4\pi^2}I\right), \tag{5.21}$$

where I is the integral

$$I = \tfrac{1}{4}\int_0^{2\pi}\int_0^{2\pi}\frac{\cos(\theta_1-\theta_2)}{\sin^2[(\theta_1-\theta_2)/2]}\,d\theta_1 d\theta_2 \quad . \tag{5.22}$$

However, I is singular since the integrand has singularities at $\theta_1-\theta_2 = 0, 2\pi$. Thus the integral has to be cut off by constraining both angles and angular differences to be larger than a/R. With this cutoff procedure the integral I can be expressed as

$$I = +2\ln\frac{R}{a} + \frac{2\pi R}{a} + \text{const} \tag{5.23}$$

for $R \gg a$.

Thus the loop integral has the asymptotic form

$$C_\Gamma(R) = \frac{\text{const}}{(R/a)^{(g^2/2\pi^2)}}\exp\left(-\frac{g^2 R}{2\pi a}\right) \quad . \tag{5.24}$$

This is essentially a perimeter law modified by a falling power at large distances. This is very similar to the behavior of the XY model. In both cases the asymptotic behavior at large distances of the correlation function is a power-law modification of an ordered-phase behavior.

D. Large-g phase

To study the $g \gg 1$ phase we write the Hamiltonian

$$H' = \frac{a}{g^2}H = \sum_{\text{links}}\frac{L^2}{2}(\vec{r},\hat{n}) - \frac{1}{g^4}\sum_{\text{boxes}}\cos\left(\sum_{\text{box}}{}_\Gamma\,\phi\right). \tag{5.25}$$

In the limit $g \to \infty$ the vacuum state $|0\rangle$ is determined by the first term in the Hamiltonian and therefore satisfies

$$L(r,n)|0\rangle = 0 \quad \text{all links.} \tag{5.26}$$

The second term will be treated as a perturbation and it will excite boxes with circulating electric flux. This phase is very similar to the small-λ phase of the Z_2 gauge theory. Here we have stable lines of electric flux whose energy is proportional to their length.

We shall now calculate the loop integral in this phase. As in the Z_2 case the lowest contribution in powers of $1/g^4$ is proportional to $(1/g^4)^N$, where N is the number of boxes of the minimal surface bounded by the loop.

Thus

$$C_\Gamma = \text{const}\times\left(\frac{1}{g^4}\right)^A = \text{const}\times e^{-CA\ln g} \quad , \tag{5.27}$$

where A is the area of the loop.

We have seen that for weak coupling the correlation function behaves as $e^{-\text{perimeter}}$ while here it is $e^{-\text{area}}$. These two behaviors characterize two different phases of the theory which must be separated by a phase transition.

E. Monopoles

The loop integral characterizing the two phases can be rewrittin, for small g, as

$$\left\langle 0\left|\exp\left(ig\iint\vec{B}\cdot d\vec{\sigma}\right)\right|0\right\rangle, \tag{5.28}$$

where the integral indicates the total magnetic flux passing through the loop. The phase transition is caused by the increasing large scale fluctuations of the magnetic flux which randomize the integral for arbitrary large loops. The source of these fluctutations can be traced to the condensation of magnetic monopoles.[16]

We shall first discuss what is a magnetic monopole on a lattice. Following Dirac[17] we define a magnetic monopole of the field configuration created by an infinitely thin solenoid with one end placed at infinity. To define a monopole on a lattice[18] we embed the solenoid along the Z axis with the finite end at the center of a cube.

The monopole is described by the classical vector potential $A_{\text{cl}}(\vec{r}-\vec{r}_0)$, where $\vec{r}_0$ is the position of the monopole. The classical lattice monopole, Fig. 35, is defined by assigning a phase to each link according to the rule

$$\phi(\vec{r},\hat{n}) = g\int_{\vec{r}}^{\vec{r}+n}\vec{A}_{\text{cl}}\cdot d\vec{l} \quad . \tag{5.29}$$

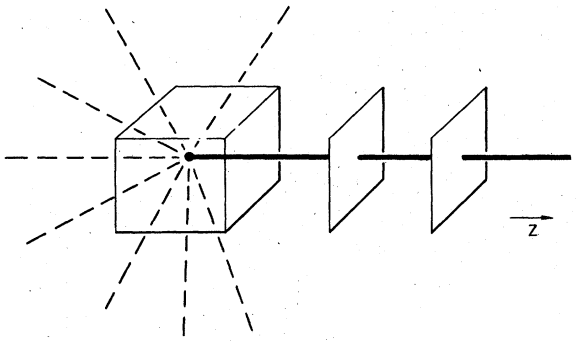

FIG. 35. The classical lattice monopole.

Let us consider the magnetic energy of the classical lattice monopole. Since the field is static there is no electric contribution to the energy ($E=\dot{A}$). The magnetic energy is

$$E_{\text{mag}}=\frac{1}{ag^2}\sum_{\text{boxes}}\left[1-\cos\left(\sum\phi\right)\right], \tag{5.30}$$

where we have subtracted the energy of the classical vacuum. The $(1/g)\sum\phi$ is the flux F of the classical field through the box. Let us divide the energy into two parts, a term for those boxes through which the solenoid passes and all others. For the first class of boxes the flux per box is just the total monopole charge μ. The energy stored in those boxes is

$$\frac{L}{ag^2}[1-\cos(\mu g)], \tag{5.31}$$

where L is the length of the solenoid in lattice units and it is infinite.

Accordingly the magnetic monopole energy can be finite if

$$\mu g=2n\pi. \tag{5.32}$$

This is the famous Dirac quantization relation[17] which in the lattice formulation expresses the condition of the energy of the monopole to be finite.

In order to define the monopole condensate phase ($g\gg 1$) it is convenient to introduce a monopole creation operator $M^+(\vec{r}_0)$,

$$M^+(\vec{r}_0)=\exp\left[i\sum_{\text{links}}\phi_{\text{cl}}(\text{link})L(\text{link})\right]. \tag{5.33}$$

This operator has the effect of translating the phase of every link by an amount ϕ_{cl}. Thus it also shifts the magnetic flux on each box by F_{cl}. Acting on the vacuum of the samll g phase it creates a state with a monopole with position $\vec{r}_0$.

Let us consider $\langle 0|M^+(\vec{r})|0\rangle$ in the weak-coupling vacuum. For this purpose we write

$$M^+=\exp\left[i\int\vec{A}_{\text{cl}}^{(r)}\cdot\vec{E}(r)d^3r\right].$$

The electric field can be expanded in creation and annihilation operators for free photons and the expectation value can be computed to be

$$\langle 0|M^+(\vec{r})|0\rangle=\exp\left[-\tfrac{1}{2}\int\int A_{\text{cl}}^i(\vec{r})A_{\text{cl}}^j(r')\times\langle E^i(r)E^j(r')\rangle d^3r\,d^3r'\right]. \tag{5.34}$$

Explicit calculations with free fields show the exponent to be logarithmically divergent of the volume. This result is analogous to the behavior of the kink creation operator in the XY model.

Next let us consider the behavior of this quantity in the $g\gg 1$ phase. For $g=\infty$ the vacuum state satisfies

$$L|0\rangle=0 \quad \text{all links}$$

and therefore

$$\langle 0|M^+(0)|0\rangle=1. \tag{5.35}$$

For weak coupling the order parameter $\langle 0|M^+(0)|0\rangle$ vanishes, but for $g\to\infty$ we have seen that it is equal to one. To show that it is connected to a phase transition we have to show that the order parameter is nonzero for some range of the coupling constant. To test this statement we have to compute the derivatives of $\langle 0|M^+|0\rangle$ with respect to $1/g^4$ by strong-coupling perturbation theory. If these derivatives are finite we can say that in the neighborhood of $g=\infty$ the ground-state expectation value of M^+ is nonzero. Indeed

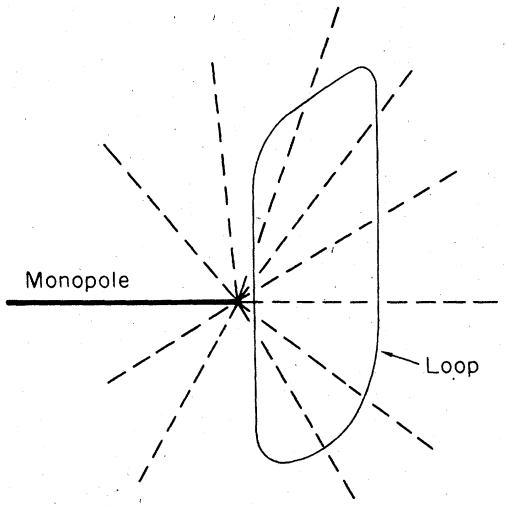

FIG. 36. A loop and the flux of a nearby monopole.

strong-coupling perturbation expansion gives the result

$$\langle 0|M^{+}|0\rangle = 1 - \frac{1}{64g^{8}} \sum_{\text{boxes}} [1 - \cos F(\text{box})] + O(g^{-16}) \ . \tag{5.36}$$

This result shows that for $g \gg 1$ the system looks like a monopole condensate with a finite monopole density. In the large-g phase the loop integral C_Γ has the asymptotic behavior

$$C_\Gamma \sim \exp(-\text{area}) \ .$$

We can now understand this result as an effect due to the monopole condensate. In fact monopoles near the loop will change the phase of the loop integral in about π per monopole (Fig. 36). In the phase where the monopoles form a condensate, they will change the loop's phase wildly or, what is the same, they will randomize the loop integral. The total effect will be to make the loop integral fall off very fast. Thus, the phase in which monopoles form a condensate is the disordered phase of the system.

ACKNOWLEDGMENT

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