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Canonical quantization of non-Abelian gauge theories in the axial gauge

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We explore canonical quantization in the axial gauge, with special reference to the problems of (i) additional gauge fixing; and (ii) the infrared infinities which occur in eliminating the dependent variables. We show that the freedom inherent in (i) permits the removal of (ii), resulting in a Hamiltonian with no explicit infinities, generating the proper equations of motion. This does not mean, however, that all matrix elements of the Hamiltonian will be finite. In particular, the bare-vacuum-expectation value of H still contains an infrared divergence.

I. INTRODUCTION

The Coulomb gauge in non-Abelian gauge theories has been coming under attack recently, principally on charges of ambiguity[1] and of discontinuity[2] in the time evolution of the potentials. It is still not clear whether these apparent difficulties render the Coulomb gauge inappropriate for the formulation of non-Abelian theories, or whether they can in fact be used somehow to achieve a better understanding of the physical phenomena involved.[3] If one inclines to the former view, then one is led almost inevitably to consider the class of so-called linear gauges, $n_\mu A^\mu = 0$ with n_μ spacelike, since only for them does the source of ambiguity pointed out by Gribov fail to make an immediate appearance. It is the purpose of this paper to examine the canonical quantization of non-Abelian gauge theories in the axial gauge, $A_3^a = 0$.

The gauge $A_3^a = 0$ in non-Abelian gauge theories was introduced by Arnowitt and Fickler[4] because the constraint equations were thereby considerably simplified. Shortly thereafter, the problem of quantization in the axial gauge was studied briefly by Schwinger,[5] who commented that his procedure was incomplete for two reasons (i) The Hamiltonian density contained a nontrivial infinity, and (ii) the gauge was not completely specified, since the freedom still existed to make gauge transformations independent of z. In fact, these two problems are related because the infinity is proportional to the generator of the transverse gauge transformations, $Q_\perp^a(x,y)$.

More recently, Mandelstam[6] observed that the condition for finiteness of the energy density is violated by the bare vacuum:

$$Q_\perp^a(x,y)\,|\,0\rangle_{\text{bare}} \neq 0,$$

and set about to construct a modified vacuum state for which the energy density would be finite, and which consequently would represent an infinite improvement over the perturbation theory vacuum. In the Abelian case this problem is easily solved, but is much less tractable in the non-Abelian case. Mandelstam conjectured that there is a close relationship between rendering the energy density finite and understanding confinement.[7]

The approach in this paper will be somewhat different. Instead of imposing $Q_\perp^a = 0$ as a condition on the states, we shall make it into an operator equation. This represents a constraint on the system which one does not ordinarily have the freedom to impose. However, in this case, the additional freedom comes precisely from the ability to fix the gauge further by making transverse gauge transformations.

Before proceeding, a word about boundary conditions: If the general gauge transformation is represented by a matrix $U(x,y,z,t)$, then the transverse subset is given by those independent of z: $U_\perp = U(x,y,t)$. One can eliminate these altogether (and hence also the possibility discussed in the preceding paragraph) by imposing the boundary condition

$$\lim_{|\vec{x}|\to\infty} U(\vec{x},t) = 1$$

and considering what happens as $|\vec{x}|$ proceeds to infinity in the z direction. However, there seems to be no compelling reason for imposing this boundary condition other than for convenience, especially since the gauge transformations and the potentials they act on are not observable. Instead, we shall impose boundary conditions on the field strengths and fermion currents,

$$\lim_{|\vec{x}|\to\infty} F^a_{\mu\nu} = 0, \tag{1.1a}$$

$$\lim_{|\vec{x}|\to\infty} J^a_\mu = 0, \tag{1.1b}$$

where, as usual,

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu \tag{1.2}$$

and

$$J^a_\mu = \bar{\psi}\gamma_\mu \tfrac{1}{2}\lambda^a \psi. \tag{1.3}$$

Here the f^{abc} are the structure constants of the gauge group of the theory.

Since the Hamiltonian density [cf. Eq. (2.8) below] contains the terms

$$\tfrac{1}{2}(\vec{E}^a\cdot\vec{E}^a+\vec{B}^a\cdot\vec{B}^a)+A_k^aJ_k^a,$$

these boundary conditions are motivated by the desire to avoid infrared infinities in the Hamiltonian.

One can deduce from Eq. (1.1a) that A_μ^a must tend to a pure gauge transformation as $|\vec{x}|$ tends to infinity. Hence, one would expect that by means of a gauge transformation, A_μ^a could be made to vanish as $|\vec{x}|$ tends to infinity, and that thereafter, only gauge transformations tending to one at infinity would be allowed. However, in the axial gauge the plane $z\to\infty$ is disconnected from the plane $z\to-\infty$. Thus, for certain $F_{\mu\nu}^a$, it may happen that A_μ^a tends to different gauge transformations as $z\to\pm\infty$.[8] It would then be impossible to make A_μ^a vanish at both $z=\infty$ and $z=-\infty$ while maintaining the axial-gauge condition. So, especially in the axial gauge, one must be careful about concluding too hastily that no further gauge fixing is necessary.

In the remainder of the paper, our strategy will be as follows: Starting from the Lagrangian, we arrive at the Hamiltonian by following the usual canonical rules augmented with a cavalier integration by parts. We then proceed to eliminate dependent variables by imposing both $A_3^a=0$ and a further gauge condition which has the effect of fixing the transverse gauge transformations as well. The resulting Hamiltonian is a more complicated function of the independent canonical variables than the naive axial-gauge Hamiltonian, and violates manifest translation invariance in the z direction. However, we are able to justify this form of the Hamiltonian by showing, first, that the correct equations of motion are generated, and second, that the explicit infrared infinities in the energy density have been eliminated, with one exception. The remaining infinity is due to our failure to fix the gauge transformations depending only on time, and this can be removed via a redefinition of the dependent variable A_0^a.

II. CONSTRUCTING THE HAMILTONIAN

The Lagrange density is

$$\mathcal{L}=-\tfrac{1}{4}F_{\mu\nu}^aF^{\mu\nu a}+gJ_\mu^aA^{\mu a}+\bar{\psi}(i\partial_\mu\gamma^\mu-m)\psi, \tag{2.1}$$

from which follow the equations of motion

$$\partial_\mu F^{\mu\nu a}=-gJ^{\nu a}-gf^{abc}F^{\nu\mu b}A_\mu^c \tag{2.2}$$

and

$$(i\partial_\mu\gamma^\mu-m)\psi=-gA_\mu^a\gamma^\mu(\tfrac{1}{2}\lambda^a)\psi. \tag{2.3}$$

The canonical momenta are given by

$$\pi_j^a\equiv\frac{\delta\mathcal{L}}{\delta\dot{A}_j^a}=-(F^{0j})^a=F_{0j}^a. \tag{2.4}$$

Not all of these will be independent, however, once we fix the gauge.

Using the canonical definition

$$\mathcal{H}=\sum_{a,j}\pi_j^a\dot{A}_j^a+i\psi^\dagger\dot{\psi}-\mathcal{L} \tag{2.5}$$

with

$$\dot{A}_j^a=\pi_j^a+\nabla_jA_0^a-gf^{abc}A_0^bA_j^c, \tag{2.6}$$

and integrating once by parts, we find, with the aid of the constraint equation

$$\sum_{j=1}^3\nabla_j\pi_j^a=-g(J^0)^a+gf^{abc}\sum_{j=1}^3A_j^c\pi_j^b, \tag{2.7}$$

the following expression for $\mathcal{H}$:

$$\mathcal{H}=\sum_{j=1}^3\pi_j^a\pi_j^a+\tfrac{1}{4}F_{\mu\nu}^aF^{\mu\nu a}+\psi^\dagger(i\vec{\alpha}\cdot\vec{\nabla}+m)\psi+g\sum_{j=1}^3J_j^aA_j^a. \tag{2.8}$$

This is not yet in canonical form since we have to specify which variables are to be eliminated by means of the constraint equation (2.7).

III. AXIAL GAUGE

We begin the process of gauge specification by setting

$$A_3^a=0. \tag{3.1}$$

It is then convenient to rewrite the constraint equation (2.7) as

$$\nabla_3F_{30}^a=gJ_0^a+\sum_{j=1}^2(\nabla_j\delta^{ab}-gf^{abc}A_j^c)\pi_j^b\equiv\sigma^a. \tag{3.2}$$

Following Schwinger,[5] we adopt the symmetric definition of the inverse derivative:

$$F_{30}^a=\frac{1}{2}\left[\int_{-\infty}^z dz'\sigma^a(z')-\int_z^\infty dz'\sigma^a(z')\right]\equiv\int_{-\infty}^\infty dz'\epsilon(z-z')\sigma^a(z'). \tag{3.3}$$

We can then partially rewrite the Hamiltonian in terms of canonical variables:

$$\mathcal{H}=\tfrac{1}{2}\sum_{j=1}^2\pi_j^a\pi_j^a+\tfrac{1}{2}F_{30}^aF_{30}^a+\tfrac{1}{4}F_{ij}^aF_{ij}^a+\tfrac{1}{2}F_{3j}^aF_{3j}^a+\psi^\dagger(i\vec{\alpha}\cdot\vec{\nabla}+m)\psi+g\sum_{j=1}^2J_j^aA_j^a, \tag{3.4}$$

where F^a_{30} is given by Eq. (3.3). It is already possible, however, to see that further gauge conditions are necessary. To begin with, from Eq. (3.3) we have

$$\lim_{z\to\pm\infty} F^a_{30} = \pm\tfrac{1}{2}Q^a_\perp , \tag{3.5}$$

$$Q^a_\perp \equiv \int_{-\infty}^{\infty} dz'\sigma^a(z'), \tag{3.6}$$

which violates Eq. (1.1a) unless $Q^a_\perp = 0$.

Furthermore, we can use the fact that

$$\int_{-L}^{L} dz'dz''dz\, f(z')\epsilon(z-z')\epsilon(z-z'')g(z'') = \tfrac{1}{2}L\int_{-L}^{L} dz'f(z')\int_{-L}^{L} dz''g(z'') - \tfrac{1}{2}\int_{-L}^{L} dz'dz''f(z')\,|z'-z''|\,g(z'') \tag{3.7}$$

to see that the term $\frac{1}{2}F^a_{30}F^a_{30}$ in the Hamiltonian density will, when integrated on z, contain the infinite piece

$$\lim_{L\to\infty} \tfrac{1}{4}LQ^a_\perp(x,y)Q^a_\perp(x,y) \tag{3.8}$$

unless $Q^a_\perp = 0$. This infinity is nontrivial in the sense that it will show up in the equations of motion.

Our method of enforcing the condition $Q^a_\perp = 0$ is as follows: In the spirit of the axial gauge, we construct a linear functional of A_2:

$$\Phi^a_2(x,y,t) \equiv \int_{-\infty}^{\infty} dz\, f_0(z)A^a_2(x,y,z,t). \tag{3.9}$$

Here $f_0(z)$ is an arbitrary real integrable function of z:

$$\int_{-\infty}^{\infty} dz\, f_0(z) \equiv \lambda_0 < \infty. \tag{3.10}$$

Now Φ^a_2 can be made to vanish by making a gauge transformation independent of z, leaving A^a_3 undisturbed. We therefore impose the condition

$$\Phi^a_2(x,y,t) = 0. \tag{3.11}$$

In order to continue the canonicalization of H, we must identify the momentum conjugate to Φ^a_2 and solve for it using the constraint equation (3.2). It is convenient to imagine that f_0 is one of a set of complete orthonormal functions $\{f_n\}$,

$$\sum_{n=0}^{\infty} f_n(z)f_n(z') = \delta(z-z'), \tag{3.12a}$$

$$\int_{-\infty}^{\infty} dz\, f_n(z)f_m(z) = \delta_{nm}, \tag{3.12b}$$

which in addition have the property of being integrable:

$$\int_{-\infty}^{\infty} dz\, f_n(z) \equiv \lambda_n < \infty. \tag{3.13}$$

For example, harmonic-oscillator wave functions have these properties. In that case we would have

$$\lambda_{2n} = \frac{[(2n)!]^{1/2}}{2^n n!}\lambda_0, \tag{3.14a}$$

$$\lambda_{2n+1} = 0, \tag{3.14b}$$

but the explicit forms of the f_n or the λ_n will not be needed in what follows.

Using Eq. (3.12a), we can expand our variables in a series in f_n:

$$v(x,y,z,t) = \sum_n v^{(n)}(x,y,t)f_n(z), \tag{3.15}$$

where v stands for any of the relevant fields, and

$$v^{(n)} = \int_{-\infty}^{\infty} dz\, f_n(z)v(z). \tag{3.16}$$

At the present level of gauge fixation, we regard the $A^{a(n)}_j$, $n\neq 0$, and the corresponding $\pi^{a(n)}_j$, as independent variables. However, since $A^{a(0)}_2 = \Phi^a_2 = 0$, we must solve for $\pi^{a(0)}_2$ from Eq. (3.2). As we have already pointed out [cf. Eq. (3.5)], Eq. (3.2) together with the boundary condition (1.1a) implies that $Q^a_\perp = 0$. From Eq. (3.6), we can represent $Q^a_\perp$ as

$$Q^a_\perp = \sum_{n=0}^{\infty} \lambda_n\sigma^{a(n)} = 0, \tag{3.17}$$

where the λ_n are defined by Eq. (3.13). The $\sigma^{a(n)}$ are given by

$$\sigma^{a(n)} = gJ^{a(n)}_0 + \sum_{j=1}^{2}\nabla_j\pi^{a(n)}_j - gf^{abc}\sum_{j=1}^{2}\sum_{m_1,m_2} c_{nm_1m_2}A^{c(m_1)}_j\pi^{b(m_2)}_j, \tag{3.18}$$

where

$$c_{nm_1m_2} = \int dz\, f_n(z)f_{m_1}(z)f_{m_2}(z) \tag{3.19}$$

is totally symmetric in the indices (nm_1m_2). From (3.12a) we have

$$\sum_n \lambda_n f_n(z) = 1, \tag{3.20}$$

which implies

$$\sum_n \lambda_n c_{nm_1m_2} = \delta_{m_1m_2}. \tag{3.21}$$

We then have directly from Eq. (3.17)

$$\nabla_2\pi_2^{a(0)}=\frac{1}{\lambda_0}\Big[gf^{abc}\sum_n A_j^{c(n)}\pi_j^{b(n)}-\sum_n\lambda_n(gJ_0^{a(n)}+\nabla_1\pi_1^{a(n)})-\sum_{n\neq 0}\lambda_n\nabla_2\pi_2^{a(n)}\Big]. \tag{3.22}$$

Note that $\pi_2^{b(0)}$ does not appear in the first term on the right-hand side of Eq. (3.22) because $A_2^{c(0)}=0$. Therefore we can solve for $\pi_2^{a(0)}$ from Eq. (3.22) by defining the inverse of ∇_2 in the same way that the inverse of ∇_3 was defined in Eq. (3.3).

Let us examine the form of the term $\frac{1}{2}F_{30}^aF_{30}^a$ in the Hamiltonian. If we define

$$g_m(z)=\frac{1}{2}\left(\int_{-\infty}^{z}dz'-\int_{z}^{\infty}dz'\right)f_m(z'), \tag{3.23}$$

we find that

$$F_{30}^a=\sum_m\sigma^{a(m)}g_m(z). \tag{3.24}$$

Furthermore, we define a matrix g_{nm} by

$$g_m(z)=\sum_n g_{mn}f_n(z), \tag{3.25}$$

$$g_{mn}\equiv\int dz\,g_m(z)f_n(z)=-g_{nm}, \tag{3.26}$$

where the antisymmetry of g is established by integration by parts and the use of

$$\lim_{z\to\pm\infty}g_m(z)=\pm\tfrac{1}{2}\lambda_m,$$

which follows from Eq. (3.23).

The Hamiltonian density integrated over z now takes the form

$$\mathcal{H}(x,y,t)=\tfrac{1}{2}\sum_n\pi_1^{a(n)}\pi_1^{a(n)}+\tfrac{1}{2}\sum_{n\neq 0}\pi_2^{a(n)}\pi_2^{a(n)}+\tfrac{1}{2}\pi_2^{a(0)}\pi_2^{a(0)}+\tfrac{1}{4}\sum_{i,j=1}^{2}F_{ij}^{a(n)}F_{ij}^{a(n)}+\tfrac{1}{2}\sum_{j=1}^{2}F_{3j}^{a(n)}F_{3j}^{a(n)}$$
$$+\tfrac{1}{2}\sum_{n,m,m'}(\tilde{g}_{nm}\sigma^{a(m)})(\tilde{g}_{nm'}\sigma^{a(m')})+\text{“}\psi^\dagger(i\vec{\alpha}\cdot\vec{\nabla}+m)\psi\text{”}+g\sum_{j=1}^{2}\sum_n J_j^{a(n)}A_j^{a(n)}, \tag{3.27}$$

where we have made use of

$$\sigma^{a(0)}=-\frac{1}{\lambda_0}\sum_{n\neq 0}\lambda_n\sigma^{a(n)} \tag{3.17$'$}$$

to replace g_{nm} by

$$\tilde{g}_{nm}\equiv g_{nm}-\frac{\lambda_m}{\lambda_0}g_{n0}. \tag{3.28}$$

In Eq. (3.27), $\pi_2^{a(0)}$ is a dependent variable whose functional form is obtained from Eq. (3.22), and "$\psi^\dagger(i\vec{\alpha}\cdot\vec{\nabla}+m)\psi$" is shorthand for

$$\int dz\,\psi^\dagger(i\vec{\alpha}\cdot\vec{\nabla}+m)\psi.$$

We now show that, as expected, the infinity which was present in the $(F_{30})^2$ term of H [cf. Eq. (3.8)] has been eliminated by the additional gauge specification. We have

$$\sum_n\tilde{g}_{nm}\tilde{g}_{nm'}\equiv T_{mm'}$$
$$=\sum_n g_{nm}g_{nm'}-\frac{\lambda_m}{\lambda_0}\sum_n g_{n0}g_{nm'}-\frac{\lambda_{m'}}{\lambda_0}\sum_n g_{nm}g_{n0}+\frac{\lambda_m\lambda_{m'}}{\lambda_0^2}\sum_n g_{n0}g_{n0}, \tag{3.29}$$

$$\sum_n g_{nm}g_{nm'}=\int dz_1dz_2f_m(z_1)\Big[\sum_n g_n(z_1)g_n(z_2)\Big]f_{m'}(z_2),$$

but

$$\sum_n g_n(z_1)g_n(z_2)=\frac{1}{4}\left(\int_{-L}^{z_1}dz'\int_{-L}^{z_2}dz''+\int_{z_1}^{L}dz'\int_{z_2}^{L}dz''-\int_{-L}^{z_1}dz'\int_{z_2}^{L}dz''-\int_{z_1}^{L}dz'\int_{-L}^{z_2}dz''\right)\delta(z'-z'')$$
$$=\tfrac{1}{2}L-\tfrac{1}{2}|z_1-z_2|\,, \tag{3.30}$$

where we have cut off the z' and z'' integrals at $\pm L$ in order to display explicitly the infinity that would have been present in the absence of gauge fixing. From Eq. (3.30) we find

$$\sum_n g_{nm}g_{nm'}=\lambda_m\lambda_{m'}(\tfrac{1}{2}L)-\tfrac{1}{2}\int dz_1dz_2 f_m(z_1)\,|z_1-z_2|\,f_{m'}(z_2)\,,$$

and putting this into Eq. (3.29) we have

$$T_{mm'}=-\tfrac{1}{2}\int_{-\infty}^{\infty}dz_1dz_2\left[f_m(z_1)-\frac{\lambda_m}{\lambda_0}f_0(z_1)\right]|z_1-z_2|\left[f_{m'}(z_2)-\frac{\lambda_{m'}}{\lambda_0}f_0(z_2)\right], \tag{3.31}$$

in which the explicit infinity has canceled. It does not necessarily follow, however, that all matrix elements of H are free of infrared singularities. This point is illustrated in Appendix C.

The alert reader will have observed that although the $(F_{30})^2$ term in H has now been rendered finite, a new infinity has popped up in the $[\pi_2^{a(0)}]^2$ term, since $\pi_2^{a(0)}$ is of the form

$$\pi_2^{a(0)}(x,y,t)=\frac{1}{2}\left(\int_{-\infty}^{y}dy'-\int_{y}^{\infty}dy'\right)\times\xi^a(x,y',t), \tag{3.32}$$

and therefore integration on y will produce the infinite term

$$\tfrac{1}{4}L\,Q_\xi^a(x)Q_\xi^a(x),$$

where

$$Q_\xi^a(x)=\int dy\,\xi^a(y).$$

The reason for this is clear: Although we have restricted the gauge sufficiently to eliminate gauge transformations depending on y and z, we still have to contend with those depending only on x and t. It is, furthermore, clear that the elimination of gauge transformations depending on x will simply be a replay of the procedure of eliminating those depending on y. That is, we impose the condition

$$\Phi_1^a(x,t)\equiv\int dy\,dz\,h_0(y,z)A_1^a(x,y,z,t)=0,$$

where, for convenience we can take, for example

$$h_0(y,z)=f_0(y)f_0(z).$$

It will then be possible to expand the fields in a double series:

$$v^{(n)}(x,y,t)=\sum_m v^{(n,m)}(x,t)f_m(y),$$

and, if we integrate Eq. (3.22) over all y, the left-hand side will vanish, leaving an equation of the form

$$\nabla_1\pi_1^{a(0,0)}=(\text{a function of canonical variables}),$$

which is the analog of Eq. (3.22) in this case. Putting this into the Hamiltonian will eliminate the infinity in the $(\pi_2^{a(0)})^2$ term at the cost of producing another one in the $(\pi_1^{a(0,0)})^2$ term. At this point only the gauge transformations depending on time will remain.

Rather than exhibit all of this explicitly, which will only complicate the Hamiltonian considerably to no particular advantage, let us short-circuit the problem by considering henceforth the simpler case of two spatial dimensions. Then the Hamiltonian of Eq. (3.27) contracts to

$$\mathcal{H}(y,t)=\tfrac{1}{2}\sum_{n\neq0}\pi^{a(n)}\pi^{a(n)}+\tfrac{1}{2}\pi^{a(0)}\pi^{a(0)}+\tfrac{1}{2}\sum_n F_{32}^{a(n)}F_{32}^{a(n)}+\tfrac{1}{2}\sum_{m,m'}\sigma^{a(m)}T_{mm'}\sigma^{a(m')}+\text{“}\psi^\dagger(i\vec{\alpha}\cdot\vec{\nabla}+m)\psi\text{”}+g\sum_n J^{a(n)}A^{a(n)}, \tag{3.33}$$

where the vestigial subscripts on A_2, π_2, and J_2 have been dropped.

In the next section, we shall demonstrate that the Hamiltonian of Eq. (3.33) correctly reproduces the equations of motion given by Eqs. (2.2) and (2.3). Since there is still an infinity contained in the $(\pi^{a(0)})^2$ term, there will be a corresponding infinity in the equations of motion which is entirely associated with the definition of A_0. We shall then show that the final freedom to make gauge transformations depending on time allows us simultaneously redefine A_0 and to modify the Hamiltonian so that the infinity completely disappears.

IV. EQUATIONS OF MOTION

The purpose of this section is to verify that the correct equations of motion follow from the Hamiltonian Eq. (3.33). These equations (not all of which are actually equations of motion) fall naturally into three classes:

(i) equations of motion for the canonical variables,
(ii) constraint equations for A_0,
(iii) equations of motion for the dependent variables.

We shall tackle each of these in turn.

Of course the basic commutator that is supposed to generate all of these equations is

$$[A^{a(n)}(y,t),\pi^{b(m)}(y',t)]=i\delta^{ab}\delta^{nm}\delta(y-y'). \tag{4.1}$$

From Eq. (4.1) it follows directly that

$$[\pi^{a(0)}(y),A^{d(m')}(y')]=\frac{i\lambda_{m'}}{\lambda_0}\delta^{ad}\delta(y-y') - \frac{ig}{\lambda_0}f^{adc}A^{c(m')}(y')\epsilon(y-y') \tag{4.2}$$

and

$$[\sigma^{a(n)}(y),A^{d(m)}(y')]=-\delta^{ad}\delta^{mn}\nabla_y\delta(y-y') + igf^{adc}\delta(y-y')\left[\sum_{m_1}c_{nmm_1}A^{c(m_1)}(y)-\frac{\lambda_m}{\lambda_0}\sum_{m_1}c_{n0m_1}A^{c(m_1)}(y)\right] + \frac{ig^2}{\lambda_0}f^{abc}f^{bde}A^{e(m)}(y')\epsilon(y-y')\sum_{m_1}c_{n0m_1}A^{c(m_1)}(y), \tag{4.3}$$

where $\epsilon(y-y')\equiv\frac{1}{2}[\theta(y-y')-\theta(y'-y)]$ as in Eq. (3.3).

The commutators of Eqs. (4.2) and (4.3) can now be used to evaluate $[\mathcal{H}(y),A^{d(m)}(y')]$ where $\mathcal{H}$ is given by Eq. (3.33). Integration on y then gives the commutator of H with A:

$$\begin{aligned}[H,A^{d(m)}(y')]=&-i\pi^{d(m)}(y')+i\frac{\lambda_m}{\lambda_0}\pi^{d(0)}(y')+\frac{ig}{4\lambda_0}f^{adc}\left(\int_{-\infty}^{y'}dy-\int_{y'}^{\infty}dy\right)\{\pi^{a(0)}(y),A^{c(m)}(y')\}\\&+i\sum_{n'}T_{mn'}\nabla_2\sigma^{d(n')}(y')+\tfrac{1}{2}igf^{adc}\sum_{n,n',m_1}T_{nn'}c_{nmm_1}\{\sigma^{a(n')}(y'),A^{c(m_1)}(y')\}\\&-\tfrac{1}{2}igf^{adc}\frac{\lambda_m}{\lambda_0}\sum_{n,n',m_1}T_{nn'}c_{n0m_1}\{\sigma^{a(n')}(y'),A^{c(m_1)}(y')\}\\&-\frac{ig^2}{4\lambda_0}f^{abc}f^{bde}\sum_{n,n',m_1}T_{nn'}c_{n0m_1}\left(\int_{-\infty}^{y'}dy-\int_{y'}^{\infty}dy\right)\{\sigma^{a(n')}(y),A^{e(m)}(y')A^{c(m_1)}(y)\}.\end{aligned} \tag{4.4}$$

What we want is something that looks a good deal simpler:

$$[H,A^d(y')]=-i\dot{A}^d(y')=-i[\pi^d(y')+\nabla_2A_0^d(y')-\tfrac{1}{2}gf^{dbc}\{A^c(y'),A_0^b(y')\}]. \tag{4.5}$$

Equation (4.4) will be the same as Eq. (4.5) [when projected onto $f_m(z)$], if we identify the quantity A_0 as

$$\begin{aligned}A_0^{d(m)}(y')=&-\frac{1}{2}\frac{\lambda_m}{\lambda_0}\left(\int_{-\infty}^{y'}dy-\int_{y'}^{\infty}dy\right)\pi^{d(0)}(y)-\sum_{n'}T_{mn'}\sigma^{d(n')}(y')\\&+\tfrac{1}{4}g\frac{\lambda_m}{\lambda_0}f^{adc}\sum_{n,n',m_1}T_{nn'}c_{n0m_1}\left(\int_{-\infty}^{y'}dy-\int_{y'}^{\infty}dy\right)\{\sigma^{a(n')}(y),A^{c(m_1)}(y)\},\end{aligned} \tag{4.6}$$

or, restoring the z' variable via $A_0^d=\sum_{m'}A_0^{d(m')}f_{m'}(z')$,

$$\begin{aligned}A_0^d(y',z')=&-\sum_{m,n}T_{mn}\sigma^{d(n)}(y')f_m(z')-\frac{1}{2\lambda_0}\left(\int_{-\infty}^{y'}dy-\int_{y'}^{\infty}dy\right)\pi^{d(0)}(y)\\&+\frac{g}{4\lambda_0}f^{adc}\sum_{n,n',m_1}T_{nn'}c_{n0m_1}\left(\int_{-\infty}^{y'}dy-\int_{y'}^{\infty}dy\right)\{\sigma^{a(n')}(y),A^{c(m_1)}(y)\}.\end{aligned} \tag{4.7}$$

A couple of remarks are in order: One sees in Eqs. (4.4), (4.5), and (4.6) the appearance of anticommutators where classically one would have the product of dynamical variables. This is of course a consequence of the Hermiticity of the Hamiltonian, which guarantees that, e.g., $\dot{A}$ will be a Hermitian operator. In general, however, we shall not worry about the order of the operator factors in the equations of motion; we shall be satisfied if, ignoring order, they agree with the classical equations (2.2) and (2.3). The particular operator ordering that emerges from our formalism then defines a particular quantum realization of the classical system governed by the Lagrange density of

Eq. (2.1).

Also, in the definition of A_0, Eq. (4.7), we see the expected infinity in the equation of motion. It occurs in the $(1/\nabla_2)\pi^{d(0)}$ term, since from Eq. (3.32), $\pi^{d(0)}$ already contains a factor of $1/\nabla_2$. The resolution of this problem will be deferred to the next section.

At this point, we are able to check the constraint equations on A_0. They are

$$\dot{A}_3 = 0$$

which implies

$$\nabla_3 A_0^d = F_{30}^d, \tag{4.8}$$

and

$$\dot{A}^{d(0)} = 0,$$

which implies

$$\nabla_2 A_0^{d(0)} = gf^{dbc}\int_{-\infty}^{\infty} dz\, f_0(z) A^c(z) A_0^b(z) - \pi^{d(0)}. \tag{4.9}$$

To check Eq. (4.8), we observe that the z dependence of A_0 is isolated in the first term on the right-hand side of Eq. (4.7). Hence

$$\nabla_3 A_0^d = -\sum_{m,n} \sigma^{d(n)} T_{nm} \frac{d}{dz} f_m(z).$$

Now

$$\sum_m \lambda_m \frac{d}{dz} f_m(z) = \int dz' \frac{d}{dz}\Big[\sum_m f_m(z') f_m(z)\Big]$$

$$= \int dz' \frac{d}{dz}\,\delta(z - z')$$

$$= 0, \tag{4.10}$$

which, together with Eq. (3.17), allows us to write

$$\nabla_3 A_0^d = \tfrac{1}{2}\sum_{m,n}\int dz_1 dz_2 \sigma^{d(n)} f_n(z_1)\,|z_1 - z_2|\, f_m(z_2)\frac{d}{dz} f_m(z)$$

$$= \tfrac{1}{2}\sum_n \int dz_1 \sigma^{d(n)} f_n(z_1)\frac{d}{dz}\,|z - z_1|$$

$$= \sum_n \int dz_1 \epsilon(z - z_1) f_n(z_1)\sigma^{d(n)}$$

$$= \int dz_1 \epsilon(z - z_1)\sigma^d(z_1) = F_{30}^d(z) \tag{4.11}$$

as required.

To check Eq. (4.9) we observe that $T_{0n} = 0$, and therefore from Eq. (4.6)

$$\nabla_2 A_0^{d(0)} = -\pi^{d(0)} + \tfrac{1}{2} g f^{adc}\sum_{n,n',m_1} T_{nn'} c_{n0m_1}\{\sigma^{a(n')}, A^{c(m_1)}\},$$

so that Eq. (4.9) will be satisfied provided

$$f^{adc}\sum_{n,n',m} T_{nn'} c_{n0m}\{\sigma^{a(n')}, A^{c(m)}\}$$

$$= f^{dbc}\int dz\, f_0(z)\{A^c(z), A_0^b(z)\}$$

$$= f^{dbc}\sum_{n,m} c_{nm0}\{A^{c(n)}, A_0^{b(m)}\}. \tag{4.12}$$

But $\lambda_m c_{nm0} = \delta_{n0}$, and $A^{c(0)} = 0$. Thus the λ_m terms of A_0 do not contribute, and we have for the right-hand side of Eq. (4.12)

$$f^{dbc}\sum_{n,m} c_{nm0}\Big\{A^{c(n)}, -\sum_{n'} T_{mn'}\sigma^{b(n')}\Big\}$$

$$= -f^{dac}\sum_{n,n',m} T_{nn'} c_{nm0}\{A^{c(m)}, \sigma^{a(n')}\}$$

$$= \text{left-hand side},$$

as required.

Having verified Eqs. (4.8) and (4.9), we return to the equations of motion for $\pi^{d(m)}$ $(m \neq 0)$. The analogs of Eqs. (4.3) and (4.4) are

$$[\pi^{a(0)}(y), \pi^{d(m)}(y')] = \frac{ig}{\lambda_0} f^{abd}\pi^{b(m)}(y')\epsilon(y - y') \tag{4.13}$$

and

$$[\sigma^{a(n)}(y), \pi^{d(m)}(y')] = -igf^{abd}\sum_{m_2} c_{nmm_2}\pi^{b(m_2)}(y')\delta(y - y') - \frac{ig^2}{2\lambda_0} f^{abc} f^{bed}\sum_{m_1} c_{nm_10}\{A^{c(m_1)}(y), \pi^{e(m)}(y')\}\epsilon(y - y'). \tag{4.14}$$

From these follows

$$[H, \pi^{d(m)}(y')] = igJ^{d(m)}(y') + i\sum_n d_{mn} F_{32}^{d(n)}(y')$$
$$- \frac{ig}{2}\sum_{n,n',m_1} T_{nn'} c_{nmm_1} f^{abd}\{\sigma^{a(n')}(y'), \pi^{b(m_1)}(y')\} - \frac{ig}{4\lambda_0} f^{abd}\Big(\int_{-\infty}^{y'} dy - \int_{y'}^{\infty} dy\Big)\{\pi^{a(0)}(y), \pi^{b(m)}(y')\}$$
$$+ \frac{ig^2}{8\lambda_0} f^{abc} f^{bed}\sum_{n,n',m_1} T_{nn'} c_{nm_10}\Big(\int_{-\infty}^{y'} dy - \int_{y'}^{\infty} dy\Big)\{\sigma^{a(n')}(y), \{A^{c(m_1)}(y), \pi^{e(m)}(y')\}\}. \tag{4.15}$$

Here d_{mn} is a matrix defined by

$$\frac{d}{dz} f_m(z) = \sum_n d_{mn} f_n(z), \quad d_{mn} = -d_{nm}. \tag{4.16}$$

What we expect is that

$$[H, \pi^{d(m)}(y')] = -i\dot{\pi}^{d(m)}(y') = -i[(\nabla_3 F_{32}^d)^{(m)} - gJ^{d(m)} + \tfrac{1}{2} g f^{dbc}\{\pi^{b(m_1)}, A_0^{c(m_2)}\} c_{mm_1m_2}]. \tag{4.17}$$

Since

$$(\nabla_3 F)^{(m)} = -\sum_n d_{mn} F^{(n)},$$

we see that Eqs. (4.15) and (4.17) will agree provided that

$$-\tfrac{1}{2} i g f^{dbc} c_{mm_1m_2}\{\pi^{b(m_1)}, A_0^{c(m_2)}\}$$

is the same as the last three terms on the right-hand side of Eq. (4.15). Showing this is a straightforward matter of inserting the three terms that make up $A_0^{c(m_2)}$ in Eq. (4.6) and verifying that they agree term by term with the relevant part of Eq. (4.15). The only difference is that from Eq. (4.6) we obtain the nested anticommutator

$$\{\{\sigma^{a(n')}(y), A^{c(m_1)}(y)\}, \pi^{e(m)}(y')\},$$

whereas Eq. (4.15) contains

$$\{\sigma^{a(n')}(y), \{A^{c(m_1)}(y), \pi^{e(m)}(y')\}\}.$$

Thus the quantum equations of motion do not preserve A_0 intact. Instead, the various operator factors that make up A_0^c arrange themselves on either side of π^e in a particular way. Nevertheless, the quantum equations do have the correct classical form when operator orderings are ignored, which is all that we have a right to demand.

We still have to look at the equations of motion for the dependent variables $\pi^{a(0)}$ and F_{30}^a. In principle this is straightforward: We simply differentiate Eqs. (3.2) and (3.22) with respect to time, and substitute therein the expressions for $\dot{A}$ and $\dot{\pi}$ that have been derived in Eqs. (4.4) and (4.15). In practice, this gets a little messy. For completeness, we shall show in Appendix A that $\pi^{a(0)}$ and F_{30} obey the correct equations of motion. We also should check the equation for $\dot{\psi}$; but ψ has been left relatively unscathed by the additional gauge fixing and we therefore deem it appropriate to leave verification of the equation of motion of the quark field as an exercise for the interested reader.

Finally, we observe that the conceptual device we have employed throughout this section, namely the replacement of the variable z by a discrete index n with the help of the basis functions $\{f_n(z)\}$, can now be dispensed with, in the sense that we can reformulate the theory in terms of the original $A^a(z)$ and $\pi^a(z)$, which will, however, obey unconventional commutation relations. If we assemble the information contained in Eqs. (4.1), (4.2), and (4.13), we can, first of all, derive

$$[\pi^{a(0)}(y), \pi^{b(0)}(y')] = \frac{ig}{\lambda_0} \epsilon(y-y') f^{abc} \times [\pi^{c(0)}(y) - \pi^{c(0)}(y')] + \frac{ig^2}{4\lambda_0^2} f^{abc} Q^c, \tag{4.18}$$

where

$$Q^a \equiv \int dy\, dz [J_0^a - \tfrac{1}{2} f^{abc}\{\pi^b, A^c\}] \tag{4.19}$$

is the total charge. Restoring the z variable by summing on n as in Eq. (3.15), we obtain the following set of commutation relations:

$$[\pi^a(y,z), \pi^b(y',z')] = \frac{ig}{\lambda_0} \epsilon(y-y') f^{abc}[\pi^c(y,z) f_0(z') - \pi^c(y',z') f_0(z)] + \frac{ig^2}{4\lambda_0^2} f^{abc} Q^c f_0(z) f_0(z'), \tag{4.20}$$

$$[A^a(y,z), \pi^b(y',z')] = i\delta^{ab}\delta(y-y')[\delta(z-z') - (1/\lambda_0) f_0(z')] + \frac{ig}{\lambda_0} f^{abc} f_0(z') A^c(y,z) \epsilon(y-y'), \tag{4.21}$$

$$[A^a(y,z), A^b(y',z')] = 0, \tag{4.22}$$

and

$$[\pi^a(y,z), \psi_{\alpha i}(y',z')] = g\frac{f_0(z)}{\lambda_0} (\tfrac{1}{2}\lambda^a)_{ik} \psi_{\alpha k}(y',z') \epsilon(y-y'), \tag{4.23}$$

where α is the Lorentz index and i is the color index of the quark field. In terms of these variables, the Hamiltonian takes the form

$$H = \int dy\, dz \left[\tfrac{1}{2}\pi^a(y,z)\pi^a(y,z) + \tfrac{1}{2}\frac{\partial}{\partial z} A^a(y,z) \frac{\partial}{\partial z} A^a(y,z) + \psi^\dagger(y,z)(i\vec{\alpha}\cdot\vec{\nabla} + m)\psi(y,z) + gJ^a(y,z) A^a(y,z)\right] - \tfrac{1}{4}\int dy\, dz_1 dz_2 \sigma^a(y,z_1) |z_1 - z_2| \sigma^a(y,z_2), \tag{4.24}$$

where

$$\sigma^a(y,z)=\bar{\sigma}^a(y,z)-\frac{1}{\lambda_0}f_0(z)\int_{-\infty}^{\infty}dz'\bar{\sigma}^a(y,z') \quad (4.25)$$

and

$$\bar{\sigma}^a(y,z)=gJ_0^a+\frac{\partial}{\partial y}\pi^a-\tfrac{1}{2}gf^{abc}\{A^c,\pi^b\}. \quad (4.26)$$

Note that it follows from Eqs. (4.20)–(4.23) that

$$\int_{-\infty}^{\infty}dz'\bar{\sigma}^a(y,z')$$

is a c number.

V. THE FINAL INFINITY

Because of the $\frac{1}{2}\pi^{a(0)}\pi^{a(0)}$ term, the Hamiltonian contains the piece

$$\hat{H}=\frac{1}{4}\frac{g^2}{\lambda_0^2}LQ^aQ^a, \quad (5.1)$$

where Q_a is given by Eq. (4.19), which is infinite as the cutoff L tends to infinity. Likewise, the quantity A_0^d which appears in the equations of motion contains the term $-(1/\lambda_0)(1/\nabla_2)\pi^{d(0)}$, which has the infinite piece

$$\hat{A}_0^d=-\frac{g}{2\lambda_0^2}LQ^d. \quad (5.2)$$

Suppose we simply subtract the infinity from H:

$$H'=H+\Delta H, \quad (5.3)$$

$$\Delta H=-\frac{1}{4}\frac{g^2}{\lambda_0^2}LQ^aQ^a. \quad (5.4)$$

It will be shown in Appendix B that the commutation rules

$$[Q^a,A^b(y,z)]=if^{abc}A^c(y,z), \quad (5.5a)$$

$$[Q^a,\pi^b(y,z)]=if^{abc}\pi^c(y,z) \quad (5.5b)$$

are valid. (The reason these are nontrivial is because of the unusual commutation rules satisfied by A and π.) Given Eq. (5.4), we see that the changes in the equations of motion are

$$\Delta\dot{\pi}^d=-\frac{1}{4}\frac{g^2}{\lambda_0^2}Lf^{abd}\{Q_a,\pi^b\} \quad (5.6a)$$

and

$$\Delta\dot{A}^d=\frac{1}{4}\frac{g^2}{\lambda_0^2}Lf^{adb}\{Q_a,A^b\}. \quad (5.6b)$$

Comparison with Eqs. (4.17) and (4.5) tells us that these changes can both be produced by adding to A_0^d the quantity

$$\Delta A_0^d=\frac{g}{2\lambda_0^2}LQ^d. \quad (5.7)$$

But from Eq. (5.2), this is precisely what is needed to render A_0^d finite as $L\to\infty$. In other words, the simple expedient of subtracting the infinity from H leaves the equations of motion satisfied, but with a new A_0^d that differs from the old one by the amount needed to make it finite. Thus the new Hamiltonian, Eq. (5.3), generates the correct equations of motion and is finite besides.

One can understand the connection between this result and the time-dependent gauge transformations in the following way. First let us introduce a matrix description of the dynamical variables

$$v\equiv\sum_a\tfrac{1}{2}\lambda^a v^a, \quad (5.8)$$

where v^a stands for either A^a or π^a. A purely time-dependent gauge transformation is given by

$$v'=U(t)vU^{-1}(t), \quad (5.9)$$

where $U(t)$ is an appropriate unitary matrix. Then

$$\dot{v}'=U\dot{v}U^{-1}+[\dot{U}U^{-1},v']. \quad (5.10)$$

Now the original Hamiltonian commutes with U, and so

$$i[H,v']=U\dot{v}U^{-1}. \quad (5.11)$$

It is therefore necessary to add a term to H in order to generate the second term on the right-hand side of Eq. (5.10). The transformation property of A_0 is

$$A_0'=UA_0U^{-1}-\frac{i}{g}\dot{U}U^{-1}, \quad (5.12)$$

so we can eliminate the infinity in A_0, Eq. (5.2), by choosing $U=e^{-ig\hat{A}_0t}$ and hence

$$\dot{U}U^{-1}=-ig\hat{A}_0=\frac{ig^2L}{2\lambda_0^2}\left(\frac{Q^d\lambda^d}{2}\right). \quad (5.13)$$

So we must find ΔH such that

$$i[\Delta H,v']=\frac{ig^2L}{8\lambda_0^2}U[Q^d\lambda^d,v^b\lambda^b]U^{-1}. \quad (5.14)$$

But

$$[Q^d\lambda^d,v^b\lambda^b]=\tfrac{1}{2}([Q^d,v^b]\{\lambda^d,\lambda^b\}-\{v^b,Q^d\}[\lambda^b,\lambda^d])$$

and

$$[Q^d,v^b]\{\lambda^d,\lambda^b\}=if^{dbc}v^c\{\lambda^d,\lambda^b\}=0$$

by the antisymmetry of f. Therefore the right-hand side of Eq. (5.14) is

$$-\frac{ig^2L}{16\lambda_0^2}U([\lambda^b,\lambda^d]\{v^b,Q^d\})U^{-1}$$

$$=\frac{i^2g^2L}{8\lambda_0^2}f^{dbc}U(\lambda^c\{v^b,Q^d\})U^{-1}.$$

Since $[Q^aQ^a, v^c] = if^{acb}\{Q^a, v^b\}$ and $[Q^aQ^a, U] = 0$, we see that the appropriate choice for H is

$$\Delta H = -\frac{g^2L}{4\lambda_0^2} Q^aQ^a, \tag{5.15}$$

which agrees with Eq. (5.4), and therefore allows us to understand the *ad hoc* modification of Eq. (5.3) as a particular choice of the last gauge freedom at our disposal.

VI. CONCLUSIONS

It has been the aim of this paper to understand the canonical quantization of non-Abelian gauge theories in the axial gauge. The twin problems, first noted by Schwinger,[5] of infinities in the Hamiltonian and additional gauge freedom, have been played off against each other to produce the final result of a Hamiltonian free from explicit infrared singularities, which generates the correct equations of motion.

Some things of a formal nature remain to be done. In the first place, at the end of Sec. III we switched abruptly from the full (3 + 1)-dimensional problem to the simpler (2 + 1)-dimensional one. It is hoped, however, that our arguments that none of the essential points have been lost in so doing have been convincing. If not, the reader is invited to go back to the end of Sec. III and work out an expanded version of the calculations of Secs. IV and V.

In the second place, when one chooses a gauge that sacrifices the manifest expression of certain symmetries, one must check the commutation rules of the relevant generators to be sure that the symmetries are there, albeit in disguised form. In our case, we have left the Poincaré group in an apparent shambles. Not only have we lost manifest Lorentz invariance, but the axial-gauge condition Eq. (3.1) violates rotational symmetry and the function $f_0(z)$ that we were forced to introduce destroys translational symmetry as well. Thus many commutators need to be checked, and, since the generators are bilinear in the fields, the commutators will tend to be more complicated than the ones encountered in Sec. IV. It is possible that one may have to add certain extra terms to the generators similar to the "t_ϕ" term that Schwinger[9] found in the radiation gauge in order to exhibit these symmetries, but one may hope that no serious violence has been done to the underlying Poincaré invariance of the theory.

There is also the phenomenological question of whether the additional gauge fixing results in modifications in the predictions of the theory. It is unlikely that anyone would want to use our Hamiltonian to develop a conventional perturbation expansion, but it is possible that the infrared properties of our Hamiltonian may be able to be understood, and may provide some insight into the long-distance behaivor of quantum chromodynamics.

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APPENDIX A

In this appendix we wish to verify the equations of motion for $\pi^{a(0)}$ and F_{30}:

$$\dot{\pi}^{a(0)} = -gJ^{a(0)} - \sum_m d_{0m} F_{32}^{a(m)} + \tfrac{1}{2} g f^{abc} c_{0m_1m_2}\{\pi^{b(m_1)}, A_0^{c(m_2)}\} \tag{A1}$$

and

$$\dot{F}_{30} = gJ_3 + gF_{30}\times A_0 + \nabla_2 F_{32} - gF_{32}\times A. \tag{A2}$$

Turning first to the verification of Eq. (A1), we have from Eq. (3.22)

$$\begin{aligned}\nabla_2\dot{\pi}^{a(0)} = \frac{1}{\lambda_0}\Big[& gf^{abc}\sum_n (\pi^{c(n)} + \nabla_2 A_0^{c(n)} - gf^{cde}c_{nm_1m_2}A^{e(m_1)}A_0^{d(m_2)})\pi^{b(n)} \\ & + \sum_{n\neq 0}(-\lambda_n\delta^{ab}\nabla_2 + gf^{abc}A^{c(n)})\Big(-gJ^{b(n)} - \sum_m d_{nm}F_{32}^{b(m)} + gf^{bde}\pi^{d(m_1)}A_0^{e(m_2)}c_{nm_1m_2}\Big) \\ & - g\sum_n \lambda_n \dot{J}_0^{a(n)}\Big].\end{aligned} \tag{A3}$$

Now the $(-\lambda_n\delta^{ab}\nabla_2)$ term is

$$\frac{1}{\lambda_0}\nabla_2\Big(-g\lambda_0 J^{a(0)} + g\sum_n \lambda_n J^{a(n)} - \lambda_0\sum_m d_{0m}F_{32}^{a(m)} + \lambda_0 gf^{ade}\pi^{d(m_1)}A_0^{e(m_2)}c_{0m_1m_2} - gf^{ade}\pi^{d(m)}A_0^{e(m)}\Big),$$

which is equal to

$$\text{expected terms} + (g/\lambda_0)\nabla_2 \sum_n \lambda_n J^{a(n)} - (g/\lambda_0) f^{ade}\nabla_2(\pi^{d(m)} A_0^{e(m)}),$$

where "expected terms" means ∇_2[right-hand side of Eq. (A1)]. We observe that

(a) $-(g/\lambda_0) f^{ade}\nabla_2(\pi^{d(m)}A_0^{e(m)}) + (g/\lambda_0) f^{abc}(\nabla_2 A_0^{c(n)})\pi^{b(n)} = -(g/\lambda_0) f^{ade}(\nabla_2\pi^{d(m)})A_0^{e(m)}$,

(b) $f^{abc}\sum_n \pi^{c(n)}\pi^{b(n)} = 0$,

(c) $(g^2/\lambda_0) f^{abc} c_{nm_1m_2}(f^{bde}A^{c(n)}\pi^{d(m_1)}A_0^{e(m_2)} - f^{cde}A^{e(m_1)}A_0^{d(m_2)}\pi^{b(n)})$

$$= (g^2/\lambda_0)c_{nm_1m_2}A^{c(n)}\pi^{d(m_1)}A_0^{e(m_2)}(f^{abc}f^{bde} + f^{adb}f^{bce})$$

$$= (g^2/\lambda_0)c_{nm_1m_2}A^{c(n)}\pi^{d(m_1)}A_0^{e(m_2)}f^{bcd}f^{aeb} ,$$

(d) $(g/\lambda_0) f^{abc}\sum_{n,m} A^{c(n)} d_{nm} F_{32}^{b(m)} = -(g/\lambda_0) f^{abc}\sum_n A^{c(n)}(d^2)_{nl}A^{b(l)} = 0$

by the symmetry of d^2 and the antisymmetry of f.

Putting all of these back into Eq. (A3), we obtain

$$\nabla_2\dot{\pi}^{a(0)} = \text{expected terms} + \mathfrak{F}^a[J] - (g/\lambda_0) f^{ade}A_0^{e(n)}(\nabla_2\pi^{d(n)} + g c_{nm_1m_2} f^{dcb}A^{c(m_2)}\pi^{b(m_1)}) , \tag{A4}$$

where

$$\mathfrak{F}^a[J] \equiv (g/\lambda_0)\nabla_2 J^{a(n)}\lambda_n - (g^2/\lambda_0) f^{abc}A^{c(n)}J^{b(n)} - (g/\lambda_0)\dot{J}_0^{a(n)}\lambda_n . \tag{A5}$$

But

$$\nabla_2\pi^{d(n)} + g c_{nm_1m_2} f^{dcb}A^{c(m_2)}\pi^{b(m_1)} = \sigma^{d(n)} - gJ_0^{d(n)}$$

[cf. Eq. (3.18)], and, using $\lambda_n\sigma^{a(n)} = 0$, we have

$$f^{ade}A_0^{e(n)}\sigma^{d(n)} = -f^{ade}T_{nn'}\sigma^{e(n')}\sigma^{d(n)} ,$$

which vanishes because $T_{nn'}$ is symmetric. Therefore,

$$\nabla_2\dot{\pi}^{a(0)} = \text{expected terms} + (g^2/\lambda_0) f^{ade}A_0^{e(n)}J_0^{d(n)} + \mathfrak{F}^a[J] . \tag{A6}$$

Now we must use current conservation in the form

$$\partial_\mu J^{\mu a} - g f^{abc}A_\mu^c J^{\mu b} = 0 . \tag{A7}$$

We integrate on z, and use the boundary condition Eq. (1.1b) to obtain

$$\dot{J}_0^{a(n)}\lambda_n - \lambda_n\nabla_2 J^{a(n)} - g f^{abc}(A_0^{c(n)}J_0^{b(n)} - A^{c(n)}J^{b(n)}) . \tag{A8}$$

Putting this back into Eq. (A6) produces the desired result. Next we turn to an examination of Eq. (A2). We start with

$$\dot{\sigma}^a = g\dot{J}^{0a} + \nabla_2\dot{\pi}^a - g f^{abc}(\dot{A}^c\pi^b + A^c\dot{\pi}^b)$$
$$= g\dot{J}^{0a} + \nabla_3(\nabla_2 F_{32}^a - g f^{abc}A^c F_{32}^b) + g f^{abc}(\sigma^b - gJ_0^b)A_0^c - g(\nabla_2 J^a - g f^{abc}A^c J^b) , \tag{A9}$$

where we have used the known values of $\dot{\pi}$ and $\dot{A}$. Using current conservation, Eq. (A7), we find

$$\dot{\sigma} = g\,\nabla_3 J_3 + \nabla_3(\nabla_2 F_{32} - g F_{32}\times A) + g\nabla_3(F_{30}\times A_0) , \tag{A10}$$

which, since $\sigma = \nabla_3 F_{30}$, is the same as Eq. (A2).

APPENDIX B

We wish to verify that

$$[Q^a, A^d(y', z')] = i f^{adc}A^c(y', z') \tag{B1}$$

and

$$[Q^a, \pi^d(y', z')] = i f^{adc}\pi^c(y', z') . \tag{B2}$$

We have

$$[Q^a, A^d(y', z')] = -\tfrac{1}{2} f^{abc}\int dy\,dz\{A^c(y,z), [\pi^b(y,z), A^d(y', z')]\} , \tag{B3}$$

which, from Eq. (4.21), is

$$[Q^a, A^d(y',z')] = \tfrac{1}{2} i f^{abc} \int dy\, dz \{A^c(y,z), \delta^{bd}\delta(y-y')[\delta(z-z') - (1/\lambda_0) f_0(z)] + (g/\lambda_0) f^{dbe} f_0(z) A^e(y',z')\epsilon(y'-y)\}\,. \tag{B4}$$

The two $f_0(z)$ terms vanish by Eq. (3.11), so we have directly

$$[Q^a, A^d(y',z')] = i f^{adc} A^c(y',z')\,,$$

as required.

Next, using Eq. (4.23), we find that

$$B^{ad} \equiv \left[\int dy\, dz\, J_0^a(y,z), \pi^d(y',z')\right] = \frac{-ig}{\lambda_0} f^{ade} f_0(z') \int_{-\infty}^{\infty} dy''\, dz''\, \epsilon(y'-y'') J_0^e(y'',z'')\,. \tag{B5}$$

Then

$$[Q^a, \pi^d(y',z')] = B^{ad} - \tfrac{1}{2} f^{abc} \int dy\, dz \{[\pi^b(y,z), \pi^d(y',z')], A^c(y,z)\} - \tfrac{1}{2} f^{abc} \int dy\, dz \{[A^c(y,z), \pi^d(y',z')], \pi^b(y,z)\}\,. \tag{B6}$$

We note that in the $[\pi,\pi]$ commutator, those terms containing $f_0(z)$ will vanish by Eq. (3.11). We are left with

$$[Q^a, \pi^d(y',z')] = B^{ad} - i f^{abd} \pi^b(y',z') + \frac{ig}{2\lambda_0} f^{adb} f^{bec} \int dy\, dz\, \epsilon(y'-y) \{A^c(y,z), \pi^e(y,z)\} f_0(z') + (i/\lambda_0) f_0(z') f^{abd} \int dz\, \pi^b(y',z)\,. \tag{B7}$$

In the last term

$$\int dz\, \pi^b(y',z) = \sum_n \lambda_n \pi^{b(n)}(y') = \sum_{n\neq 0} \lambda_n \pi^{b(n)}(y') + \int dy''\, \epsilon(y'-y'') \rho^b(y'') - \sum_{n\neq 0} \lambda_n \pi^{b(n)}(y')\,,$$

where

$$\rho^b(y'') = \tfrac{1}{2} g \int dz\, f^{bec} \{\pi^e(y'',z), A^c(y'',z)\} - g \int dz\, J_0^b(y'',z)\,.$$

Therefore,

$$\begin{aligned}[Q^a, \pi^d(y',z')] &= i f^{adb} \pi^b(y',z') + B^{ad} + \frac{ig}{2\lambda_0} f^{adb} f^{bec} \int dy\, dz\, \epsilon(y'-y) \{A^c(y,z), \pi^e(y,z)\} f_0(z') \\ &\quad + \frac{ig}{2\lambda_0} f^{abd} f^{bec} \int dy\, dz\, \epsilon(y'-y) \{\pi^e(y,z), A^c(y,z)\} f_0(z') - \frac{ig}{\lambda_0} f_0(z') f^{abd} \int dy\, dz\, \epsilon(y'-y) J_0^b(y,z) \\ &= i f^{adb} \pi^b(y',z')\,,\end{aligned}$$

as required.

APPENDIX C

In his discussion of color confinement, Mandelstam[6] points out that in the axial gauge, the expectation value of $E_3{}^2$ in the bare vacuum contains an infrared infinity. The question arises as to whether the infrared infinities that we have removed from H coincide with this infinity or not. It suffices to examine the issue in the Abelian theory, for which the relevant term in the energy density is

$$\mathcal{H}' = \int_{-L}^{L} dz_1\, dz_2\, \epsilon(z-z_1)\epsilon(z-z_2) J_0(z_1) J_0(z_2)\,, \tag{C1}$$

where J_0 is the usual fermion charge density $\psi^\dagger\psi$. In the bare vacuum [which is translation invariant as far as the fermion fields are concerned, despite the introduction of $f_0(z)$ in the gauge-field part of the Hamiltonian] the expectation value of $J_0 J_0$ has the form

$$\langle J_0(z_1) J_0(z_2)\rangle = F(z_1 - z_2)\,, \tag{C2}$$

where, for simplicity, let us take

$$F(\xi)=\theta(\xi_0-\xi)\theta(\xi_0+\xi)\quad(\xi_0>0). \tag{C3}$$

This form of F is finite at $\xi=0$ and provides an absolute infrared cutoff. To the extent that the real F deviates from this ideal form, the conclusions to be reached below can only get worse.

The modification brought about by the additional gauge fixing described in Sec. III is for the purposes of $\mathcal{H}'$ to replace J_0 in Eq. (C1) by

$$\hat{J}_0(z)=J_0(z)-\frac{1}{\lambda_0}f_0(z)\int_{-L}^{L}dz'J_0(z'). \tag{C4}$$

Let us now calculate

$$A=\left\langle\int_{-L}^{L}dz\mathcal{H}'(z)\right\rangle \tag{C5}$$

for the normal axial gauge, and again for the modified axial gauge, Eq. (C4), which we shall denote by A_M. In both cases we can use the identity of Eq. (3.7),

$$\int_{-L}^{L}dz\,\epsilon(z-z_1)\epsilon(z-z_2)=\tfrac{1}{2}L-\tfrac{1}{2}|z_1-z_2|\,, \tag{C6}$$

to write $A=B+C$ (and likewise $A_M=B_M+C_M$), where B or B_M comes from the first term on the right-hand side of (C6) and C or C_M comes from the second term. By translation invariance, one expects that A or A_M will contain a divergence proportional to L. This will still give a finite energy per unit volume. However, explicit calculation shows that as $L\to\infty$,

$$B\to 2\xi_0L^2\,,$$

while C grows only as L. Thus

$$A\to 2\xi_0L^2\,,$$

which is the divergence in the energy density noted by Mandelstam.

In the modified case, as discussed in Sec. III, we have

$$B_M=0\,.$$

However, one also finds that

$$C_M\to 2\xi_0L^2\,.$$

This extra divergence comes entirely from the cross terms

$$\int_{-L}^{L}dz'\,dz_1\,dz_2\frac{f_0(z_1)}{\lambda_0}|z_1-z_2|F(z_2-z')\,.$$

Thus

$$A_M\to 2\xi_0L^2\,,$$

so the divergence which was present in A is still there in A_M, although its location has been shifted.[10]

Note added in proof. After the completion of this work I learned of a somewhat different treatment of the same problem by Y.-P. Yao in the Abelian case [J. Math. Phys. 5, 1319 (1964)].

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[5]J. Schwinger, Phys. Rev. 130, 402 (1963).

[6]S. Mandelstam, invited talk at the Washington meeting of the APS, 1977 (unpublished).

[7]There are many other references to the use of the axial gauge. For example, W. Kummer, Acta Phys. Austriaca 14, 149 (1961); R. N. Mohapatra, Phys. Rev. D 4, 2215 (1971); W. Konetschny and W. Kummer, Nucl. Phys. B100, 106 (1975), and references therein. More recently, the axial gauge in non-Abelian gauge theories has been studied by I. Bars and M. B. Green [Phys. Rev. D 17, 537 (1978)], I. Bars and F. Green (work in preparation), and I. Bars [Yale report, 1977 (unpublished)].

[8]This observation has also been made by I. Bars and F. Green (Ref. 7).

[9]J. Schwinger, Phys. Rev. 127, 324 (1962).

[10]I am grateful to S. Mandelstam for a private communication on the points in this appendix.