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Canonical and path-integral quantizations in the $A_0 = 0$ gauge: Abelian case

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It is shown that both canonical and path-integral quantizations of an electromagnetic field coupled with point charged particles can be carried out in the $A_0 = 0$ gauge by the usual rules without fixing the gauge completely and eliminating the longitudinal degrees of freedom. A Coulomb interaction potential is obtained as an effective potential either by separating the longitudinal-mode variables in the Schrödinger differential equation or by integrating all longitudinal-mode quantum fluctuations in the Feynman path integral. There is no Faddeev-Popov ghost or infinite gauge volume factor in our path-integral treatment.

I. INTRODUCTION

Recently Gribov has pointed out the very interesting phenomenon that for non-Abelian gauge theories the transversality condition $\vec{\nabla}\cdot\vec{A}_a=0$ does not actually fix the gauge completely.[1] His observation of the gauge-fixing degeneracy has cast some doubts on the validity of the Faddeev-Popov quantization rule.[2] Thus we have to reconsider the quantization of gauge fields and find a different procedure which will not be troubled by the gauge-fixing ambiguity.

In this paper we will discuss a new approach to quantize gauge fields without following the "recipe" of Faddeev and Popov.[2] Although we consider here only an Abelian gauge field, the basic idea and techniques developed here can be applied to the quantization of non-Abelian gauge fields without facing additional fundamental obstacles.

We consider the quantization of an electromagnetic field coupled with nonrelativistic point charged particles. We choose the vector potential gauge where A_0 vanishes for all time. By this choice, gauge freedom is not completely eliminated, because we still can perform time-independent gauge transformations. We will demonstrate that both canonical and path-integral quantizations can be carried out by the usual rules without fixing the gauge completely and eliminating the longitudinal modes associated with the gauge freedom.

Before presenting our formulation, we want to emphasize that the path integral we use here is quite different from that used by Faddeev and Popov[2] as well as 't Hooft and Veltman.[3] We will call the path integral they used "Feynman history integral,"[4] while ours will be called "sequential Feynman path integral." Although the formal expressions of these two integrals look the same, they actually have very different properties. In the formulation of Feynman history integrals, one normally expands $A_\mu(x)$ as $\int d^4k\, a_\mu(k)e^{ik\cdot x}$ and performs the functional integration over the Fourier coefficient $a_\mu(k)$'s. Consequently, the differential operator

$$K_{\mu\nu}(x,y) = -(\partial^2 g_{\mu\nu} - \partial_\mu\partial_\nu)\delta^4(x-y)\,, \qquad (1.1)$$

which appears in the quadratic part of the action functional

$$S_0 = -\tfrac{1}{4}\int d^4x\, F_{\mu\nu}F^{\mu\nu}$$

$$= \tfrac{1}{2}\int d^4x\, d^4y\, A^\mu(x)K_{\mu\nu}(x,y)A^\nu(y)\,, \qquad (1.2)$$

is singular and cannot be inverted to obtain the Feynman propagator.[2,4] On the other hand, in Feynman's original formulation, the path integral is defined as the limit function of a sequence of integrals.[5] The trajectories involved in the sequential path integrals are polygonal curves resembling the paths of a particle in Brownian motion. Consequently, the quadratic operator involved in the sequential path integral for gauge fields is not a singular differential operator; instead, it is a matrix operator whose inverse is well defined and can be computed compactly. Thus, by resorting to the original definition of sequential path integrals, we will never encounter the troublesome singular operator $K_{\mu\nu}(x,y)$.

The content of this article is organized as follows: In Sec. II, we discuss the classical dynamics and the normal-mode Lagrangian in the $A_0=0$ gauge. Because we do not fix the gauge completely the Lagrangian contains many longitudinal modes which can be characterized as zero-frequency-mode particles, analogous to the translation mode of a soliton Lagrangian.[6] We discuss the canonical quantization in Sec. III and the path-integral quantization in Sec. IV. Remarkably, we obtain the Coulomb interaction potential as an effective potential either by separating the longitudinal-mode variables in the Schrödinger differential equation or by integrating all longitudinal-mode quantum fluctuations in the sequential Feynman path integral. Our result is consistent with the usual Coulomb gauge quantization. However, we do not have the fictitious Faddeev-Popov ghost or the infinite-gauge-volume factor[2] in our path-integral treatment. Additional remarks are made in Sec. V.

II. CLASSICAL ELECTRODYNAMICS IN THE $A_0=0$ GAUGE

We consider a system of the electromagnetic field coupled with some nonrelativistic point charged particles. We choose the gauge where $A_0=0$ for all time. Then the Lagrangian assumes the following form:

$$L=\int d^3x[\tfrac{1}{2}\dot{\vec{A}}^2-\tfrac{1}{2}(\vec{\nabla}\times\vec{A})^2+\vec{j}\cdot\vec{A}] + \tfrac{1}{2}\sum_a m_a\dot{\vec{r}}_a{}^2, \tag{2.1}$$

where

$$\vec{j}(\vec{x},t)=\sum_a e_a\dot{\vec{r}}_a\delta^3(\vec{x}-\vec{r}_a(t)). \tag{2.2}$$

This Lagrangian has no Lagrange multiplier and, consequently, each component of $\vec{A}$ can be varied independently and regarded as a dynamical variable. From the definitions of $\vec{E}$ and $\vec{B}$ and the Euler-Lagrange differential equations one obtains the following equations of motion:

$$\vec{\nabla}\cdot\vec{B}=0,\quad \vec{\nabla}\times\vec{E}=-\dot{\vec{B}}, \tag{2.3}$$

$$\vec{\nabla}\times\vec{B}=\vec{j}(\vec{x},t)+\dot{\vec{E}}, \tag{2.4}$$

$$m_a\ddot{\vec{r}}_a=e_a[\vec{E}(\vec{r}_a,t)-\dot{\vec{r}}_a\times\vec{B}(\vec{r}_a,t)]. \tag{2.5}$$

The remaining Maxwell equation

$$\vec{\nabla}\cdot\vec{E}(\vec{x},t)=\rho(\vec{x},t)=\sum_a e_a\delta^3(\vec{x}-\vec{r}_a(t)) \tag{2.6}$$

cannot be derived by the usual variational principle from the Lagrangian in Eq. (2.1). Gauss's law will be implemented by other means.

We choose the initial field configuration which satisfies the condition

$$\vec{\nabla}\cdot\dot{\vec{A}}(\vec{x},t_0)=-\rho(\vec{x},t_0). \tag{2.7}$$

Then the equation of motion in (2.4) and the charge conservation relation

$$\vec{\nabla}\cdot\vec{j}(\vec{x},t)+\dot{\rho}(\vec{x},t)=0 \tag{2.8}$$

imply the relation

$$\vec{\nabla}\cdot\dot{\vec{A}}(\vec{x},t)-\vec{\nabla}\cdot\dot{\vec{A}}(\vec{x},t_0)=-\rho(\vec{x},t)+\rho(\vec{x},t_0). \tag{2.9}$$

Thus the field configuration evolved at a later time also satisfies the Gauss law requirement. In Dirac's terminology, Gauss's law is implemented as a "weak" consistency condition which is valid only for the actual motion.[7]

For the purpose of field quantization it is convenient to express the Lagrangian in terms of normal-mode coordinates. We consider the field in a large box of volume $\Omega=(2L)^3$ and expand it by Fourier series:

$$\vec{A}(\vec{x},t)=\sum_{\vec{k}\lambda}\vec{q}_{\vec{k}\lambda}(t)\phi_{\vec{k}\lambda}(\vec{x}), \tag{2.10}$$

where

$$\phi_{\vec{k}\lambda}(x)=(2/\Omega)^{1/2}\times\begin{cases}\cos(\vec{k}\cdot\vec{x}), & \lambda=1,\quad \vec{k}\neq 0\\ 1/\sqrt{2}, & \vec{k}=0\\ \sin(\vec{k}\cdot\vec{x}), & \lambda=-1,\quad \vec{k}\neq 0\end{cases} \tag{2.11}$$

$$\vec{k}\in\{(\pm n\pi/L,\pm m\pi/L,l\pi/L)\,|\,l,m,n=0,1,2,\ldots\}. \tag{2.12}$$

For each $\vec{k}$ and λ there are three normal-mode coordinates. If $\vec{k}\neq 0$, the vector $\vec{q}_{\vec{k}\lambda}$ can be decomposed into a longitudinal component plus two transverse components:

$$\begin{aligned}\vec{q}_{\vec{k}\lambda}(t)&\equiv\hat{k}q^L_{\vec{k}\lambda}(t)+\hat{\epsilon}_1(\vec{k})q^{T1}_{\vec{k}\lambda}(t)+\hat{\epsilon}_2(\hat{k})q^{T2}_{\vec{k}\lambda}\\ &\equiv\vec{q}^{\,L}_{\vec{k}\lambda}(t)+\vec{q}^{\,T}_{\vec{k}\lambda}(t),\end{aligned} \tag{2.13}$$

$$\hat{\epsilon}_i(\vec{k})\cdot\hat{\epsilon}_j(\vec{k})=\delta_{ij},\quad \hat{\epsilon}_i(\vec{k})\cdot\hat{k}=0,\quad i=1=2. \tag{2.14}$$

The vector potential $\vec{A}(\vec{x},t)$ can also be decomposed into a longitudinal part and a "transverse" part (which also includes three $\vec{k}=0$ modes):

$$\vec{A}(\vec{x},t)=\vec{A}^L(\vec{x},t)+\vec{A}^T(\vec{x},t), \tag{2.15}$$

$$\vec{A}^L(\vec{x},t)=\sum_{\vec{k}\neq 0,\lambda}\hat{k}q^L_{\vec{k}\lambda}(t)\phi_{\vec{k}\lambda}(\vec{x}). \tag{2.16}$$

In general, $\vec{A}^L$ and $\vec{A}^T$ are not orthogonal to each other, but they have the following properties:

$$\vec{\nabla}\cdot\vec{A}^T=0,\quad \vec{\nabla}\times\vec{A}^L=0. \tag{2.17}$$

If we consider a time-independent gauge transformation

$$\vec{A}'(\vec{x},t)=\vec{A}(\vec{x},t)+\vec{\nabla}\Lambda(\vec{x}), \tag{2.18}$$

only the longitudinal-mode coordinates will be changed:

$$q^L_{\vec{k}\lambda}(t)'=q^L_{\vec{k}\lambda}(t)-\operatorname{sgn}(\lambda)k\Lambda_{\vec{k},-\lambda}; \tag{2.19}$$

therefore $q^L_{\vec{k}\lambda}$'s are usually regarded as the "redundant" variables associated with the gauge degrees of freedom. However, in the treatment proposed here, we will keep all longitudinal-mode variables and regard them as dynamical variables to be quantized.

We can write the Lagrangian of Eq. (2.1) in terms of the normal-mode coordinates,

$$L=\tfrac{1}{2}\sum_{\vec{k}\lambda}[(\dot{q}^L_{\vec{k}\lambda})^2+(\dot{\vec{q}}^T_{\vec{k}\lambda})^2-k^2(\vec{q}^T_{\vec{k}\lambda})^2]+\tfrac{1}{2}\sum_a m_a\dot{\vec{r}}_a^{\,2}$$
$$+\sum_a e_a\dot{\vec{r}}_a(t)\cdot\sum_{\vec{k}\lambda}[\hat{k}q^L_{\vec{k}\lambda}(t)+\vec{q}^T_{\vec{k}\lambda}(t)]\phi_{\vec{k}\lambda}(\vec{r}_a(t))\,. \qquad (2.20)$$

The corresponding Hamiltonian can be constructed as follows:

$$H=\frac{1}{2}\sum_{\vec{k}\lambda}\{[\pi^L_{\vec{k}\lambda}(t)]^2+[\vec{\pi}^T_{\vec{k}\lambda}(t)]^2+k^2[\vec{q}^T_{\vec{k}\lambda}(t)]^2\}$$
$$+\sum_a\frac{1}{2m_a}\left\{\vec{p}_a(t)-e_a\sum_{\vec{k}\lambda}[\hat{k}q^L_{\vec{k}\lambda}(t)+\vec{q}^T_{\vec{k}\lambda}(t)]\phi_{\vec{k}\lambda}(\vec{r}_a(t))\right\}^2. \qquad (2.21)$$

Notice that there are no quadratic binding potentials for the longitudinal modes. They are the "zero-frequency modes" of our dynamical system.

The fact that the longitudinal modes appear in the gauge field Lagrangian as zero-frequency-mode particles is quite analogous to the soliton case where the translation mode is the zero-frequency mode of the normal-mode Lagrangian in the one-soliton sector.[6] Both kinds of the zero-frequency modes originate from the degeneracy of the classical solutions with respect to transformations of some basic symmetries. The basic symmetries involved are translation invariance in the soliton case and gauge invariance in the gauge field cases. In both cases the zero-frequency modes cannot be trivially decoupled in the normal-mode Lagrangian and treated as completely free particles like the translation modes of a polyatomic molecule. In Sec. IV we will discuss how to handle the longitudinal modes as zero-frequency-mode particles using the sequential Feynman path integral.

III. CANONICAL QUANTIZATION

By the usual rule of making the canonical momenta into linear differential operators one can easily construct the quantum-mechanical Hamiltonian and the electric field operator:

$$H=-\frac{\hbar^2}{2}\sum_{\vec{k}\lambda}(\partial/\partial q^L_{\vec{k}\lambda})^2+\sum_{\vec{k}\lambda}\left[-\frac{\hbar^2}{2}(\partial/\partial\vec{q}^T_{\vec{k}\lambda})^2+\frac{k^2}{2}(\vec{q}^{\,T}_{\vec{k}\lambda})^2\right]$$
$$+\sum_a\frac{1}{2m_a}\left[\frac{\hbar}{i}(\partial/\partial\vec{r}_a)-e_a\sum_{\vec{k}\lambda}(\hat{k}q^L_{\vec{k}\lambda}+\vec{q}^T_{\vec{k}\lambda})\phi_{\vec{k}\lambda}(\vec{r}_a)\right]^2, \qquad (3.1)$$

$$\vec{\mathcal{E}}(\vec{x})=-\vec{\pi}(\vec{x})=i\hbar\sum_{\vec{k}\lambda}\phi_{\vec{k}\lambda}(\vec{x})[\hat{k}(\partial/\partial q^L_{\vec{k}\lambda})+(\partial/\partial\vec{q}^T_{\vec{k}\lambda})]\,. \qquad (3.2)$$

In ordianry quantum mechanics, wave functions are considered to be vectors in the Hilbert space $\mathcal{H}$ spanned by the eigenfunctions of the Hamiltonian operator. However, if we consider an arbitrary state vector $|\Psi\rangle$, the electric field observed might not satisfy Gauss's law. Thus Dirac proposed that a weak equation like Gauss's law should be imposed as a subsidiary condition on the state vector[7]

$$[\vec{\nabla}\cdot\vec{\mathcal{E}}(\vec{x})-\rho(\vec{x})]|\Psi_M\rangle=0,\quad |\Psi_M\rangle\in\mathcal{H}_M\subset\mathcal{H}\,. \qquad (3.3)$$

This condition can also be expressed as follows:

$$\frac{\hbar}{i}\sum_{\vec{k},\lambda}k\,\mathrm{sgn}(\lambda)\phi_{\vec{k},-\lambda}(\vec{x})\frac{\partial}{\partial q^L_{\vec{k}\lambda}}\Psi_M$$
$$=\sum_{a,\vec{k},\lambda}e_a\phi_{\vec{k}\lambda}(\vec{x})\phi_{\vec{k}\lambda}(\vec{r}_a)\Psi_M\,. \qquad (3.4)$$

The solution of this differential equation is

$$\Psi_M(\{q^L_{\vec{k}\lambda}\},\{\vec{q}^T_{\vec{k}\lambda}\},\{\vec{r}_a\},t)=f(\{q^L_{\vec{k}\lambda}\},\{\vec{r}_a\})\Psi'(\{\vec{q}^T_{\vec{k}\lambda}\},\{\vec{r}_a\},t)\,, \qquad (3.5)$$

where

$$f(\{q^L_{\vec{k}\lambda}\},\{\vec{r}_a\})\equiv\exp\left\{\frac{i}{\hbar}\sum_{\vec{k}\neq0,\lambda}\frac{\mathrm{sgn}(\lambda)}{k}q^L_{\vec{k}\lambda}\times\sum_a e_a\phi_{\vec{k},-\lambda}(\vec{r}_a)\right\}. \qquad (3.6)$$

Now we would like to examine how the Hamiltonian H defined in Eq. (3.1) operates on the wave function Ψ_M.

First we consider the kinetic energy operator of the longitudinal modes

$$T^L\Psi_M=\tfrac{1}{2}\sum_{\vec{k}\neq0,\lambda}\frac{1}{k^2}\left[\sum_a e_a\phi_{\vec{k}\lambda}(\mathrm{r}_a)\right]^2\Psi_M$$
$$=\sum_{\vec{k}\neq0,\lambda}\left[\sum_{a<b}\frac{e_ae_b}{k^2}\phi_{\vec{k}\lambda}(\vec{r}_a)\phi_{\vec{k}\lambda}(\vec{r}_b)+\frac{1}{2}\sum_a\frac{e_a^{\,2}}{k^2}\phi_{\vec{k}\lambda}^{\,2}(\vec{r}_a)\right]\Psi_M$$
$$=\sum_{a<b}\frac{1}{4\pi}\frac{e_ae_b}{|\vec{r}_a-\vec{r}_b|}\Psi_M+\begin{pmatrix}\text{infinite self-energies}\\ \text{of the point charges}\end{pmatrix}\Psi_M\,. \qquad (3.7)$$

Here we have used the following relation:

$$\sum_{\vec{k}\neq0,\lambda}\frac{1}{k^2}\phi_{\vec{k}\lambda}(\vec{r}_a)\phi_{\vec{k}\lambda}(\vec{r}_b)\xrightarrow[\Omega\to\infty]{}\int\frac{d^3k}{(2\pi)^3}\frac{\cos\vec{k}\cdot(\vec{r}_a-\vec{r}_b)}{k^2}$$
$$=\frac{1}{4\pi}\frac{1}{|\vec{r}_a-\vec{r}_b|}\,. \qquad (3.8)$$

Thus we obtain the Coulomb interaction potential as an effective potential produced by the operator T^L. The self-energies of the point charges are time independent and will be subtracted away in the subsequent definition of the Hamiltonian H.

The operator H_a is not manifestly gauge invariant because it contains the gauge-dependent longitudinal-mode coordinates $q^L_{\vec{k}\lambda}$. However, using the relation

$$\frac{\hbar}{i}\frac{\partial}{\partial\vec{r}_a}f=e_a\sum_{\vec{k}\neq 0,\lambda}\hat{k}q^L_{\vec{k}\lambda}\phi_{\vec{k}\lambda}(\vec{r}_a)f$$
$$=e_a\vec{A}^L(\vec{r}_a)f, \tag{3.9}$$

one can verify that all longitudinal-mode coordinates cancel in the operation of $H_a(f\Psi')$:

$$H_a(f\Psi')=\frac{1}{2m_a}\left\{-\hbar^2\nabla_a^2-e_a\frac{\hbar}{i}[(\vec{\nabla}_a\cdot\vec{A}(\vec{r}_a))+2\vec{A}(\vec{r}_a)\cdot\vec{\nabla}_a]+e_a^2A^2(\vec{r}_a)\right\}(f\Psi')$$
$$=\frac{f}{2m_a}\left[\frac{\hbar}{i}\vec{\nabla}_a-e_a\vec{A}^T(\vec{r}_a)\right]^2\Psi' \quad (\vec{\nabla}_a\equiv\partial/\partial\vec{r}_a)\ . \tag{3.10}$$

Consequently, one can define the following gauge-invariant effective Hamiltonian:

$$H_{\text{eff}}\equiv\sum_{a<b}\frac{1}{4\pi}\frac{e_ae_b}{|\vec{r}_a-\vec{r}_b|}+\sum_{\vec{k},\lambda}\tfrac{1}{2}[-\hbar^2(\partial/\partial\vec{q}^T_{\vec{k}\lambda})^2+k^2(\vec{q}^T_{\vec{k}\lambda})^2]+\sum_a\frac{1}{2m_a}\left[\frac{\hbar}{i}\vec{\nabla}_a-e_a\sum_{\vec{k},\lambda}\vec{q}^T_{\vec{k}\lambda}\phi_{\vec{k}\lambda}(\vec{r}_a)\right]^2, \tag{3.11}$$

such that

$$H(f\Psi')=f(\{q^L_{\vec{k}\lambda}\},\{\vec{r}_a\})H_{\text{eff}}\Psi'(\{\vec{q}^T_{\vec{k}\lambda}\},\{\vec{r}_a\},t)\ . \tag{3.12}$$

This is the main result of our canonical treatment. Several remarks may be made:

(1) Equation (3.12) and the Schrödinger equation $H\Psi=i\hbar(\partial/\partial t)\Psi$ imply that a state vector $|\Psi_M\rangle$ which initially belonged to the subspace $\mathcal{H}_M$ always evolves within the invariant subspace $\mathcal{H}_M$.

(2) Our effective Hamiltonian is consistent with the usual Hamiltonian obtained by the Coulomb gauge quantization. However, quantization in the $A_0=0$ gauge looks more attractive because the rule of constructing the Hamiltonian operator H in Eq. (3.1) is simpler and more natural.

(3) The quantization rule

$$\pi^L_{\vec{k}\lambda}\to\frac{\hbar}{i}(\partial/\partial q^L_{\vec{k}\lambda}),\quad \vec{\pi}^T_{\vec{k}\lambda}\to\frac{\hbar}{i}(\partial/\partial\vec{q}^T_{\vec{k}\lambda}) \tag{3.13}$$

is equivalent to the following canonical commutation relation for the Heisenberg field operators:

$$[\pi_i(\vec{x},t),A_j(\vec{y},t)]=\frac{\hbar}{i}\delta_{ij}\delta^3(\vec{x}-\vec{y})\ . \tag{3.14}$$

IV. PATH-INTEGRAL QUANTIZATION

In this section we discuss how to quantize the electromagnetic field in the $A_0=0$ gauge by the sequential Feynman path integral. We will concentrate our effort on handling the integrations of the longitudinal-mode variables.

Given a classical Lagrangian $L(\{\vec{A},\dot{\vec{A}}\},\{\vec{r}_a,\dot{\vec{r}}_a\})$ in Eq. (2.1) one can formally write down the quantum-mechanical Green's function as a path integral over all values of the coordinates at each time, subject to the boundary conditions of the initial and final configurations:

$$K(\{\vec{A}''(\vec{x})\},\{\vec{r}''_a\},t'';\{\vec{A}'(\vec{x})\},\{\vec{r}'_a\},t')=\int_{\vec{A}'}^{\vec{A}''}\mathcal{D}\vec{A}(x,t)\prod_a\int_{\vec{r}'_a}^{\vec{r}''_a}\mathcal{D}\vec{r}_a(t)\exp\left[\frac{i}{\hbar}\int_{t'}^{t''}dt\,L(\{\vec{A},\dot{\vec{A}}\},\{\vec{r}_a,\dot{\vec{r}}_a\})\right]. \tag{4.1}$$

However, as Feynman and Hibbs noted before, it is much easier to evaluate the path integral after transforming the problem into normal-mode coordinates.[8] Such a transformation gives us

$$K(\{\vec{q}''_{\vec{k}\lambda}\},\{\vec{r}''_a\},t'';\{\vec{q}'_{\vec{k}\lambda}\},\{\vec{r}'_a\},t')\equiv K_{t''t'}=\prod_a\int_{\vec{r}'_a}^{\vec{r}''_a}\mathcal{D}\vec{r}_a(t)\prod_{\vec{k}\lambda}\int_{\vec{q}'_{\vec{k}\lambda}}^{\vec{q}''_{\vec{k}\lambda}}\mathcal{D}\vec{q}_{\vec{k}\lambda}(t)\exp\left[\frac{i}{\hbar}\int_{t'}^{t''}dtL(\{\vec{q}_{\vec{k}\lambda},\dot{\vec{q}}_{\vec{k}\lambda}\},\{\vec{r}_a,\dot{\vec{r}}_a\})\right], \tag{4.2}$$

where $L(\{\vec{q}_{\vec{k}\lambda}, \dot{\vec{q}}_{\vec{k}\lambda}\}, \{\vec{r}_a, \dot{\vec{r}}_a\})$ was given in Eq. (2.20) and

$$\vec{A}'(\vec{x}) = \sum_{\vec{k}\lambda} (\hat{k} q^{L'}_{\vec{k}\lambda} + \vec{q}^{\,T'}_{\vec{k}\lambda}) \phi_{\vec{k}\lambda}(\vec{x}) \tag{4.3}$$

$$\vec{A}''(\vec{x}) = \sum_{\vec{k}\lambda} (\hat{k} q^{L''}_{\vec{k}\lambda} + \vec{q}^{\,T''}_{\vec{k}\lambda}) \phi_{\vec{k}\lambda}(\vec{x}) \tag{4.4}$$

are the initial and final field configurations, respectively. The path integral in Eq. (4.2) can be handled more easily because the coordinates of different normal modes do not mix with each other in the quadratic part of the normal-mode Lagrangian.

For a given set of the particle trajectories $\{\vec{r}_a(t) \,|\, a = 1, 2, \ldots, n\}$, the path integral in Eq. (4.2) contains the following subintegral:

$$K_L([J_{\vec{k}\lambda}]; q^{L''}_{\vec{k}\lambda}, t''; q^{L'}_{\vec{k}\lambda}, t') \equiv \int_{q^{L'}_{\vec{k}\lambda}}^{q^{L''}_{\vec{k}\lambda}} \mathfrak{D} q^{L}_{\vec{k}\lambda}(t) \times \exp\left\{ \frac{i}{\hbar} \int_{t'}^{t''} dt [\tfrac{1}{2} (\dot{q}^{L}_{\vec{k}\lambda})^2 + J_{\vec{k}\lambda}(t) q^{L}_{\vec{k}\lambda}(t)] \right\}, \tag{4.5}$$

where

$$J_{\vec{k}\lambda}(t) \equiv \sum_a e_a \hat{k} \cdot \dot{\vec{r}}_a(t) \phi_{\vec{k}\lambda}(\vec{r}_a(t)) . \tag{4.6}$$

This is precisely the path integral of a zero-frequency-mode particle subject to an external force $J_{\vec{k}\lambda}(t)$. This type of path integral was also encountered in quantizing the translation mode of solitons.[6] As we discussed in Appendix A of Ref. 6, this path integral can be rigorously evaluated with the following result:

$$K_L([J_{\vec{k}\lambda}]; q^{L''}_{\vec{k}\lambda}, t''; q^{L'}_{\vec{k}\lambda}, t') = \frac{1}{[2\pi i \hbar (t'' - t')]^{1/2}} \exp\left(\frac{i}{\hbar} S_{\vec{k}\lambda} \right), \tag{4.7}$$

where

$$S_{\vec{k}\lambda} = \tfrac{1}{2} (q^{L''}_{\vec{k}\lambda} - q^{L'}_{\vec{k}\lambda})^2 / (t'' - t') + \int_{t'}^{t''} dt\, J_{\vec{k}\lambda}(t) [q^{L''}_{\vec{k}\lambda} (t - t') + q^{L'}_{\vec{k}\lambda} (t'' - t)] / (t'' - t') - \frac{1}{2} \int_{t'}^{t''} dt \int_{t'}^{t''} d\tau\, J_{\vec{k}\lambda}(t) g_0(t; \tau) J_{\vec{k}\lambda}(\tau), \tag{4.8}$$

$$g_0(t; \tau) = \begin{cases} (t'' - \tau)(t - t')/(t'' - t'), & \text{if } t < \tau \\ (t'' - t)(\tau - t')/(t'' - t'), & \text{if } t > \tau . \end{cases} \tag{4.9}$$

This result is valid for an arbitrary "external" force applying to the zero-frequency-mode particle. In our case the external force is determined by the positions and velocities of the charged particles through Eq. (4.6). One can rewrite the force $J_{\vec{k}\lambda}$ in the following form:

$$J_{\vec{k}\lambda}(t) = \sum_a \frac{e_a}{k} \operatorname{sgn}(\lambda) \dot{\phi}_{\vec{k}, -\lambda}(\vec{r}_a(t)) . \tag{4.10}$$

Using the relation

$$\frac{\partial^2}{\partial t \partial \tau} g_0(t; \tau) = \delta(t - \tau) - \frac{1}{t'' - t'} , \tag{4.11}$$

one can integrate by parts both integrals of Eq. (4.8) and obtain

$$\int_{t'}^{t''} dt \int_{t'}^{t''} d\tau\, J_{\vec{k}\lambda}(t) g_0(t; \tau) J_{\vec{k}\lambda}(\tau) = \int_{t'}^{t''} dt \frac{1}{k^2} \left[\sum_a e_a \phi_{\vec{k}, -\lambda}(\vec{r}_a(t)) \right]^2 - \frac{1}{t'' - t'} \frac{1}{k^2} \left[\sum_a e_a \int_{t'}^{t''} dt \phi_{\vec{k}, -\lambda}(\vec{r}_a(t)) \right]^2 , \tag{4.12}$$

$$\int_{t'}^{t''} dt\, J_{\vec{k}\lambda}(t) [q^{L''}_{\vec{k}\lambda} (t - t') + q^{L'}_{\vec{k}\lambda} (t'' - t')] / (t'' - t') = \sum_a \frac{e_a \operatorname{sgn}(\lambda)}{k} \left[\phi_{\vec{k}, -\lambda}(\vec{r}_a(t'')) q^{L''}_{\vec{k}\lambda} - \phi_{\vec{k}, -\lambda}(\vec{r}_a(t')) q^{L'}_{\vec{k}\lambda} - \frac{1}{t'' - t'} (q^{L''}_{\vec{k}\lambda} - q^{L'}_{\vec{k}\lambda}) \int_{t'}^{t''} dt\, \phi_{\vec{k}, -\lambda}(\vec{r}_a(t)) \right] . \tag{4.13}$$

Thus after the integration by parts,

$$S_{\vec{k}\lambda} = \frac{1}{2} \frac{1}{t'' - t'} \left[q^{L''}_{\vec{k}\lambda} - q^{L'}_{\vec{k}\lambda} - \sum_a e_a \frac{\operatorname{sgn}(\lambda)}{k} \int_{t'}^{t''} dt\, \phi_{\vec{k}, -\lambda}(\vec{r}_a(t)) \right]^2 + \sum_a \frac{e_a \operatorname{sgn}(\lambda)}{k} [q^{L''}_{\vec{k}\lambda} \phi_{\vec{k}, -\lambda}(\vec{r}_a'') - q^{L'}_{\vec{k}\lambda} \phi_{\vec{k}, -\lambda}(\vec{r}_a')] - \tfrac{1}{2} \int_{t'}^{t''} dt \frac{1}{k^2} \left[\sum_a e_a \phi_{\vec{k}, -\lambda}(\vec{r}_a(t)) \right]^2 . \tag{4.14}$$

Now one can take the product of all longitudinal-mode subintegrals and rewrite it in the following form:

$$\prod_{\vec{k}\neq 0,\lambda} K_L([J_{\vec{k}\lambda}]; q^{L''}_{\vec{k}\lambda}, t''; q^{L'}_{\vec{k}\lambda}, t')$$

$$= \left\{ \prod_{\vec{k}\lambda} \frac{1}{[2\pi i\hbar(t''-t')]^{1/2}} \exp\left[\frac{i}{\hbar} \frac{1}{2(t''-t')} \left(q^{L''}_{\vec{k}\lambda} - q^{L'}_{\vec{k}\lambda} - \sum_a \frac{e_a \operatorname{sgn}(\lambda)}{k} \int_{t'}^{t''} dt \phi_{\vec{k},-\lambda}(\vec{r}_a(t)) \right)^2 \right] \right\}$$
$$\times f(\{q^{L''}_{\vec{k}\lambda}\}, \{\vec{r}''_a\}) f^*(\{q^{L'}_{\vec{k}\lambda}\}, \{\vec{r}'_a\})$$
$$\times \exp\left\{ -\frac{i}{\hbar} \frac{1}{2} \sum_{\vec{k}\lambda} \frac{1}{k^2} \int_{t'}^{t''} dt \left[\sum_a e_a \phi_{\vec{k}\lambda}(\vec{r}_a(t)) \right]^2 \right\}, \tag{4.15}$$

where the function f is the same as that defined in Eq. (3.6). The exponent of the last multiplicative factor in (4.15) is proportional to the eigenvalue of T^L in the differential equation (3.7). If one also drops off the infinite self-energies of the point charges, one can replace the last exponential factor by the following factor effectively:

$$\exp\left[-\frac{i}{\hbar} \int_{t'}^{t''} dt V_c(\{\vec{r}_a(t)\}) \right] = \exp\left(-\frac{i}{\hbar} \sum_{a<b} e_a e_b \int_{t'}^{t''} dt \frac{1}{4\pi |\vec{r}_a(t) - \vec{r}_b(t)|} \right). \tag{4.16}$$

Thus the Coulomb interaction potential is obtained as an effective potential by integrating all longitudinal-mode quantum fluctuations in the path-integral expression of Eq. (4.2).

In general a Feynman path integral is a quantum-mechanical Green's function which governs the time evolution of an arbitrary Schrödinger wave function by the integral equation

$$\Psi(\{q^{L''}_{\vec{k}\lambda}\}, \{\vec{q}^{T''}_{\vec{k}\lambda}\}, \{\vec{r}''_a\}, t'') = \prod_{a,\vec{k}\lambda} \int d\vec{r}'_a d\vec{q}^{T'}_{\vec{k}\lambda} dq^{L'}_{\vec{k}\lambda} K_{t''t'} \Psi(\{q^{L'}_{\vec{k}\lambda}\}, \{\vec{q}^{T'}_{\vec{k}\lambda}\}, \{\vec{r}'_a\}, t'). \tag{4.17}$$

As we discussed in Sec. III, the initial wave function must also satisfy Gauss's law of Eq. (3.4) and assume the explicit form of Eq. (3.5):

$$\Psi_{t'} = f(\{q^{L'}_{\vec{k}\lambda}\}, \{\vec{r}'_a\}) \Psi'(\{\vec{q}^{T'}_{\vec{k}\lambda}\}, \{\vec{r}'_a\}, t'). \tag{4.18}$$

From Eq. (4.15), one finds that the Green's function $K_{t''t'}$ contains a factor of $f^*(\{q^{L'}_{\vec{k}}\}, \{\vec{r}'_a\})$ which cancels the f factor of $\Psi_{t'}$. Substituting the result of the path-integral evaluation of $K_{t''t'}$ into Eq. (4.17) and interchanging certain integration orders one obtains

$$\Psi_{t''} = f(\{q^{L''}_{\vec{k}\lambda}\}, \{\vec{r}''_a\}) \prod_{a,\vec{k}\lambda} \int_{-\infty}^{\infty} d\vec{r}'_a d\vec{q}^{T'}_{\vec{k}\lambda} \int_{\vec{r}'_a}^{\vec{r}''_a} \mathfrak{D}\vec{r}_a(t) \int_{\vec{q}^{T'}_{\vec{k}\lambda}}^{\vec{q}^{T''}_{\vec{k}\lambda}} \mathfrak{D}\vec{q}^T_{\vec{k}\lambda}(t)$$
$$\times \exp\left[\frac{i}{\hbar} \int_{t'}^{t''} dt L_{\text{eff}}(\{\vec{q}^T_{\vec{k}\lambda}, \dot{\vec{q}}^T_{\vec{k}\lambda}\}, \{\vec{r}_a, \dot{\vec{r}}_a\}) \right] \Psi'(\{(\vec{q}^{T'}_{\vec{k}\lambda}\}, \{\vec{r}'_a\}, t')$$
$$\times \int_{-\infty}^{\infty} dq^{L'}_{\vec{k}\lambda} \frac{1}{[2\pi i\hbar(t''-t')]^{1/2}} \exp\left\{ \frac{i}{\hbar} \frac{1}{2(t''-t')} \left[q^{L''}_{\vec{k}\lambda} - q^{L'}_{\vec{k}\lambda} - \sum_a \frac{e_a \operatorname{sgn}(\lambda)}{k} \int_{t'}^{t''} dt \phi_{\vec{k},-\lambda}(\vec{r}_a(t)) \right]^2 \right\}, \tag{4.19}$$

where

$$L_{\text{eff}} \equiv \sum_{\vec{k},\lambda} \tfrac{1}{2}[(\dot{\vec{q}}^T_{\vec{k}\lambda})^2 - k^2(\vec{q}^T_{\vec{k}\lambda})^2] + \tfrac{1}{2} \sum_a m_a \dot{\vec{r}}_a{}^2$$
$$+ \sum_{a,\vec{k},\lambda} e_a \dot{\vec{r}}_a(t) \cdot \vec{q}^T_{\vec{k}\lambda}(t) \phi_{\vec{k}\lambda}(\vec{r}_a(t)) - V_c(\{\vec{r}_a(t)\}). \tag{4.20}$$

In Eq. (4.19) each integral of $\int dq^{L'}_{\vec{k}\lambda}$ is a Gaussian integral whose value is equal to one and also is independent of the choice of the particle trajectories $\{\vec{r}_a(t)\}$. After the integrations of all longitudinal-mode variables Eq. (4.19) can be written as follows:

$$\Psi_{t''} = f(\{q^{L''}_{\vec{k}\lambda}\}, \{\vec{r}''_a\}) \Psi'(\{\vec{q}^{T''}_{\vec{k}\lambda}\}, \{\vec{r}''_a\}, t''), \tag{4.21}$$

$$\Psi'(\{\vec{q}^{T''}_{\vec{k}\lambda}\}, \{\vec{r}''_{\vec{k}\lambda}\}, t'') = \prod_{a,\vec{k},\lambda} \int d\vec{r}'_a d\vec{q}^{T'}_{\vec{k}\lambda} K^{(\text{eff})}_{t''t'}$$
$$\times \Psi'(\{\vec{q}^{T'}_{\vec{k}\lambda}\}, \{\vec{r}'_a\}, t'), \tag{4.22}$$

where

$$K^{(\mathrm{eff})}_{t''t'} \equiv K_{\mathrm{eff}}(\{\vec{q}^{\,T''}_{\vec{k}\lambda}\},\{\vec{r}''_a\},t'';\{\vec{q}^{\,T'}_{\vec{k}\lambda}\},\{\vec{r}'_a\},t')$$

$$\equiv \prod_{a,\vec{k},\lambda}\int_{\vec{r}'_a}^{\vec{r}''_a}\mathfrak{D}\vec{r}_a(t)\int_{\vec{q}^{\,T'}_{\vec{k}\lambda}}^{\vec{q}^{\,T''}_{\vec{k}\lambda}}\mathfrak{D}\vec{q}^{\,T}_{\vec{k}\lambda}(t)\times\exp\left(\frac{i}{\hbar}\int_{t'}^{t''}dt\,L_{\mathrm{eff}}\right) \quad (4.23)$$

is an effective Feynman path integral which does not contain any longitudinal-mode variables. These equations summarize the result of our path-integral treatment.

The effective Feynman path integral we obtained in Eq. (4.23) is actually identical to the path-integral amplitude written by Feynman and Hibbs in Eq. (9.44) of their textbook.[8] In their treatment, the Coulomb interaction potential $V_c(\{\vec{r}_a(t)\})$ is obtained by imposing the Coulomb gauge condition and solving the time component of the vector potential *classically*. In our path-integral treatment, the Coulomb interaction potential is obtained as an effective potential by integrating all longitudinal-mode variables.

The result obtained in this section is also consistent with the previous result of the canonical quantization treatment. Equation (4.21) implies that the Schrödinger wave function which initially satisfied Gauss's law at t' also satisfies Gauss's law at all later time. Furthermore, if one begins with the effective Hamiltonian in Eq. (3.11), one can also deduce the same effective Feynman path integral in Eq. (4.23).

In the present approach a correct evaluation of the Feynman path integral in Eq. (4.5) is crucial to the success of the whole scheme. In particular, the kernel $g_0(t;\tau)$ given in Eq. (4.9) has played a very important role in obtaining the instantaneous Coulomb interaction potential. As we discussed in Appendix A of Ref. 6, this kernel can be considered as the continuum limit of certain inverse matrix elements:

$$\lim_{N\to\infty}\Delta t(\sigma_N^{-1})_{ij} = g_0(t'+i\Delta t;\,t'+j\Delta t),$$
$$\Delta t = (t''-t')/N\,, \quad (4.24)$$

where σ_N is an $(N-1)\times(N-1)$ Jacobi matrix. Using the method invented by Montroll[9] one can directly compute $(\sigma_N^{-1})_{ij}$ and verify the compact expression of Eq. (4.9). On the other hand, if one evaluates the path integral in Eq. (4.5) by expanding the trajectories as $\int d\nu\, e^{i\nu t}a(\nu)$ and integrating over the Fourier coefficients (as one normally does in the Feynman history integrals), one would not obtain the same formulas as our Eqs. (4.7), (4.8), and (4.9). In particular, the kernel obtained in the Fourier expansion method is infrared divergent:

$$g_0'(t;\tau) = \int \frac{d\nu}{2\pi}\frac{e^{-i\nu(t-\tau)}}{\nu^2+i\epsilon}\,. \quad (4.25)$$

This has been known as the "zero-mode problem" in the soliton quantization.[6,10] In our opinion, this infrared-singularity problem is actually created by the manipulations introduced in the Fourier integral expansion method of path-integral evaluations.

V. CONCLUDING REMARKS

We have demonstrated that both canonical and path-integral quantizations of an Abelian gauge field can be carried out in the $A_0=0$ gauge by the usual rules without fixing the gauge completely and eliminating the longitudinal modes. In the present approach, the problem of the gauge-fixing degeneracy or uniqueness seems to be an irrelevant question. Thus the quantization procedure developed here can be generalized to the non-Abelian cases without facing the dilemma of the gauge-fixing ambiguity pointed out by Gribov.[1] Nevertheless, in the non-Abelian gauge field cases, we will face certain technical difficulties which might be very difficult to solve. In the canonical quantization approach, the analog of Gauss's law differential equation in (3.3) cannot be easily solved in the non-Abelian cases. In the path-integral quantization approach the longitudinal-mode variables can no longer be integrated compactly as in the Abelian case, because they also couple with the transverse modes in the cubic and quartic terms of the Lagrangian. Maybe these couplings can be handled perturbatively by the zero-mode Feynman rule developed in Ref. 6. (Conceivably, no Faddeev-Popov ghost will emerge in such a type of path-integral treatment.) We will investigate this interesting problem later.

Note added. After the completion of the present paper, the author was informed that the problem of field quantization in the $A_0=0$ gauge without complete gauge fixing was also discussed in the recent work of Willemsen.[11] However, he did not exhibit the Coulomb interaction potential which we obtained in both Eqs. (3.7) and (4.15). The recent work of Senjanović[12] also deals with the same kind of problem.

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