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Topological excitations in the Abelian Higgs model

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A lattice version of the Abelian Higgs model is studied in arbitrary Euclidean dimension. Using an exact duality transformation, the theory is rewritten in terms of its topological excitations. The dual form of the theory specifies in a simple way all the allowed topological excitations as well as their interactions. The combination of the scalar Higgs field and the Abelian gauge field produces excitations found neither in the pure gauge theory nor in the pure scalar theory (XY model). In three dimensions, for example, we find finite vortex strings terminating on monopoles, as well as closed vortex loops. Implications of these singularities for the critical behavior of the theory are briefly discussed.

I. INTRODUCTION

In recent years, there has arisen a growing awareness of the importance of topologically stable solutions of nonlinear field theories. Indeed, topological excitations have been shown to be of central importance in the physics of fluids and solids,[1] elementary particles,[2] and cosmology.[3] The existence of these excitations follows from very general features of the theory, in particular, its internal symmetries and its space-time dimensionality.

These excitations can have profound effects on the behavior of the theory. In the first place, since they are collective excitations of the original fields of the theory, they can appear as real concrete objects. An example of this is vortex formation in superfluid He^4 which may be regarded as a topological excitation of a theory with a global U(1) symmetry.[4] In the second place, these excitations can, under certain circumstances, induce phase transitions. Examples of this phenomenon are the phase transition of the two-dimensional XY model,[5] the Ising-model interface phase transition,[6] and the confinement mechanism in three-dimensional compact photodynamics first described by Polyakov.[7]

In this paper, we will analyze the topological excitations of a locally U(1)-invariant theory which may be thought of as a lattice version of the Abelian Higgs model. Our approach will be to use an exact duality transformation[8] which will allow us to write the Abelian Higgs model as a function of its topological excitations. In this dual form of the model, the partition function will be expressed as a functional of certain integer-valued fields which represent topological excitations of the original scalar and gauge vector fields. In this way, we will be able to express both the form of the excitations and their interactions. In a later paper we will describe the effects of these excitations on certain correlation functions of the theory, and will discuss possible phase transitions arising from the presence of these objects.[9]

Our interest in this model has several sources. First, the theory represents a kind of hybrid of the models discussed in Ref. 8. In particular, it is a combination of the simplex numbers $s=1$ (XY model) and $s=2$ (compact photodynamics) theories. Since these two models have qualitatively different topological excitations, it is interesting to see the form of the excitations which emerge from the hybrid theory. From a less formal and more physical point of view, the model is interesting for at least three reasons. First, when properly treated, the model has much in common with models of spin-glasses which are currently of great interest to solid state physicists.[10] Second, in their seminal paper on topological singularities as elementary particles, Nielsen and Olesen[11] considered the Abelian gauge field coupled to a charged, scalar field in a Higgs-type potential. The symmetry of our model is the same as theirs, except that our vector gauge fields are compact. As we shall see, this modifies their results in an interesting way. A third motivation follows if we recall Polyakov's result for compact photodynamics in three dimensions.[7] He showed that the point monopoles of that theory produced confinement in the sense that the expectation value of Wilson's loop integral fell like e^{-A}, where A is some minimum area enclosed by the gauge loop. It is interesting to ask how this result and its interpretation is changed when the gauge fields are coupled to matter. The simplest such theory is the one we will study in this paper.

In the next section, we will introduce the model and show that in a naive continuum limit it becomes the Abelian Higgs model in which the radial degree of freedom has been completely frozen. In Sec. III we introduce an exact duality transformation which lets us write the model in terms of certain integer-valued fields. In Sec. IV we show how to identify these integer-valued fields as the topological excitations of the original theory by introducing a slightly modified form of the Abelian Higgs model for which this identification is reasonably straightforward. Some comments about our results and some comparisons with related models are presented in Sec. V.

II. THE MODEL

Consider a hypercubical lattice in d dimensions. Associated with each site j of the lattice is a two-dimensional spin $S(j)=e^{i\chi(j)}$. (For notational simplicity we will not indicate the vector nature of the lattice vector j.) Associated with each link of the lattice is another spin $U_\mu(j)=e^{i\theta_\mu(j)}$, where a link is defined by lattice site j and a direction μ. These spins interact according to the lattice Lagrangian

$$\mathcal{L}=\frac{\kappa}{2}\sum_l S(j)U_\mu^\dagger(j)S^\dagger(j-\hat{\mu})+\frac{\beta}{2}\sum_p U_\mu(j)U_\nu(j+\hat{\mu})U_\mu^\dagger(j+\hat{\nu})U_\nu^\dagger(j)+\text{H.c.}\,, \tag{1}$$

where the first sum runs over all links of the lattice and the second sum runs over all plaquettes. The partition function (generating functional) of the theory is

$$\begin{aligned} Z&=\int_{-\pi}^{\pi}\delta\theta_\mu(j)\delta\chi(j)e^{\mathcal{L}}\\ &=\int_{-\pi}^{\pi}\delta\theta_\mu(j)\delta\chi(j)\\ &\quad\times\exp\left[\kappa\sum_l\cos[\Delta_\mu\chi(j)-\theta_\mu(j)]+\beta\sum_p\cos\left(\frac{1}{(d-2)!}\epsilon_{\mu\nu\beta_1\cdots\beta_{\alpha-2}}\epsilon_{\beta_1\cdots\beta_{\alpha-2}\rho\sigma}\Delta_\rho\theta_\sigma(j)\right)\right], \end{aligned} \tag{2}$$

where ϵ is the totally antisymmetric symbol in d dimensions, and Δ_μ denotes a discrete difference, e.g., $\Delta_\mu\chi(j)=\chi(j)-\chi(j-\hat{\mu})$. Note that the term proportional to β is just the usual action for the pure gauge field theory on a lattice.

This Lagrangian is invariant under a local U(1) rotation:

$$S(j)\to R(j)S(j),\quad U_\mu(j)\to R^\dagger(j-\hat{\mu})\,U_\mu(j)\,R(j) \tag{3a}$$

or, in terms of the angles,

$$\chi(j)\to\chi(j)+\Lambda(j),\quad \theta_\mu(j)\to\theta_\mu(j)+\Delta_\mu\Lambda(j) \tag{3b}$$

where $R(j)=e^{i\Lambda(j)}$. This is just a compact version of the local symmetry of scalar QED.

To obtain the correspondence with the Abelian Higgs theory, we temporarily restore the lattice spacing a. The lattice Lagrangian then becomes

$$a^{d-4}\sum\kappa\cos[a\Delta_\mu\chi(j)-\theta_\mu(j)]+\frac{\beta}{2}\cos\left(\frac{1}{(d-2)!}\epsilon_{\mu\nu\beta_1\cdots\beta_{\alpha-2}}\epsilon_{\beta_1\cdots\beta_{\alpha-2}\rho\sigma}\Delta_\rho\theta_\sigma(j)\right). \tag{4}$$

Define $\theta_\mu\equiv aA_\mu$, and consider the naive continuum limit $a\to 0$. We use the replacements

$$\begin{aligned} &\chi(j)\to\chi(x)\,,\\ &\theta_\mu(j)\to aA_\mu(x)\,,\\ &\Delta_\mu\to a\nabla_\mu\,,\\ &\sum\to a^{-d}\int d^dx\,. \end{aligned} \tag{5}$$

Expanding the cosines to second order yields a constant term plus

$$-\int d^dx\left[\frac{\kappa}{2}(\nabla_\mu\chi-A_\mu)^2+\frac{\beta}{4[(d-2)!]^2}F_{\mu\nu}{}^2\right], \tag{6}$$

where

$$F_{\mu\nu}\equiv\epsilon_{\mu\nu\beta_1\cdots\beta_{d-2}}\epsilon_{\beta_1\cdots\beta_{d-2}\rho\sigma}\nabla_\rho A_\sigma\,.$$

We would like to expose the relationship of this expression to the broken-symmetry phase of the Abelian Higgs model, whose Lagrangian density is conventionally written (in the Euclidean continuum) as

$$\mathcal{L}=\tfrac{1}{2}(D_\mu\phi)^\dagger D^\mu\phi+V(\phi^\dagger\phi)+\tfrac{1}{4}F_{\mu\nu}{}^2\,, \qquad (7)$$

where

$$F_{\mu\nu}\equiv\partial_\mu A_\nu-\partial_\nu A_\mu\,,$$

$$D_\mu\phi\equiv(\partial_\mu-ieA_\mu)\phi\,,$$

and we take the potential

$$V(x)\equiv\frac{\lambda}{4}(x-R^2)^2\,.$$

Writing $\phi=\rho e^{i\chi}$, the Lagrangian density becomes

$$\mathcal{L}=\tfrac{1}{2}\rho^2(\partial_\mu\chi-eA_\mu)^2+\tfrac{1}{2}(\partial_\mu\rho)^2+V(\rho^2)+\tfrac{1}{4}F_{\mu\nu}{}^2\,. \qquad (8)$$

We expect that the field ϕ will suffer spontaneous symmetry breakdown, and the ground state of the theory will have $\langle\rho^2\rangle=R^2$. We may formally freeze out the radial vibrations by considering the limit $\lambda\to\infty$. Then $(\partial_\mu\rho)^2=0$, and the Lagrangian simplifies to[12,13]

$$\mathcal{L}=\tfrac{1}{2}R^2(\partial_\mu\chi-A_\mu)^2+\frac{1}{4e^2}F_{\mu\nu}{}^2\,, \qquad (9)$$

where we have rescaled the potential $A_\mu\to(1/e)A_\mu$. The correspondence with the naive continuum limit of our original expression Eq. (6) is now apparent, and we should identify $\kappa=R^2$, $\beta=[(d-2)!/e]^2$.

Now, one might think, because our theory possesses the gauge symmetry Eq. (3), that it is a trivial theory since, choosing $\Lambda(j)=-\chi(j)$, we apparently have the theory of a free vector boson of mass $m=eR$. However, this is not correct since χ is an angular variable and consequently need not be single valued. Thus $\Lambda=-\chi$ cannot be a true gauge transformation in general, since a multivalued gauge function Λ may well induce physical singularities in the field strength $F_{\mu\nu}$. Alternatively, if we choose to admit such singular gauge transformations, then A_μ itself takes on the significance of an angular variable, i.e., A_μ becomes compact. In this case, its equation of motion is

$$(\Box+m^2)A_\mu=0 \pmod{2\pi}\,. \qquad (10)$$

The singularities allowed thereby are, of course, the topological excitations of this theory, just as monopoles arise in Polyakov's discussion of compact photodynamics for $d=3$, or for that matter, just as vortices arise from the solution of

$$\Box\chi=0 \pmod{2\pi} \qquad (11)$$

in his discussion of the $d=2$ XY model.[7]

The naive equivalence which we have demonstrated between our lattice theory and the continuum Abelian Higgs model does not guarantee that the continuum limit of the lattice theory is the Abelian Higgs model when that limit is taken via the renormalization group. Another way to say this is that the large-distance behavior of our theory and the continuum Abelian Higgs model may not be the same, at least not for all values of κ and β. This possibility is also associated with the following important observation: From Eq. (5) we see that the phase of the lattice Higgs field $\chi(j)$ retains its identity as a phase angle in the continuum, since $\chi(j)\to\chi(x)$. On the other hand, the lattice gauge angle $\theta_\mu(j)$ becomes $aA_\mu(x)$ as $a\to0$, so that the finite range of $\theta_\mu(j)$ is mapped into an infinite range for $A_\mu(x)$. So even in this naive limit, the gauge fields in some sense are no longer compact. Hence it is not clear that the theory will still retain the full effects of the multivalued gauge transformations alluded to earlier, and so, one might fear that the topological excitations associated with the compact nature of $\theta_\mu(j)$ will disappear. This could well lead to a difference in the long-range behavior of the lattice and continuum theories. We will comment further on this problem in Sec. V.

III. THE DUALITY TRANSFORMATION

We now discuss the dual form of our model in various dimensions. In what follows we will always assume periodic (spherical) boundary conditions. Other boundary conditions may induce a background field which in certain cases can change the physics. This will be discussed elsewhere.[9]

A. Two dimensions

In two dimensions, the partition function, Eq. (2), is simply

$$Z=\int\delta\chi(j)\delta\theta_\mu(j)\exp\left\{\sum\kappa\cos[\Delta_\mu\chi(j)-\theta_\mu(j)]+\beta\cos[\epsilon_{\mu\nu}\Delta_\mu\theta_\nu(j)]\right\}. \qquad (12)$$

Using the Fourier expansion

$$e^{\alpha\cos x}=\sum_{n=-\infty}^{\infty}I_n(\alpha)\,e^{inx}\,, \qquad (13)$$

we obtain

$$Z=\sum_{\{n(j),\,m_\mu(j)\}}\prod I_{n(j)}(\beta)I_{m_\mu(j)}(\kappa)\int_{-\pi}^{\pi}\delta\chi(j)\delta\theta_\mu(j)\exp\left(\sum\left\{in(j)\epsilon_{\mu\nu}\Delta_\mu\theta_\nu(j)+im_\mu(j)[\Delta_\mu\chi(j)-\theta_\mu(j)]\right\}\right), \qquad (14)$$

where the first product sign denotes the fact that we have one $I_{n(j)}(\beta)$ for each plaquette, and one $I_{m_\mu(j)}(\kappa)$ for each link. Carrying out the angular integrations, we obtain the constraint equations on the integers $m_\mu(j)$, $n(j)$:

(1) $\Delta_\mu m_\mu(j) = 0$,

(2) $m_\mu(j) - \epsilon_{\mu\nu}\,\Delta_\nu n(j) = 0$.

The first equation is exactly as in the $d=2$ XY model; its general solution is a curl

$$m_\mu(j) = \epsilon_{\mu\nu}\,\Delta_\nu\,\psi(j)\,, \qquad (15)$$

where $\psi(j)$ is an integer-valued scalar field which is naturally defined on the vertices of the dual lattice. (The dual lattice is obtained by shifting the original lattice by half a lattice spacing in each direction.) Inserting this into the second equation, we find that

$$\epsilon_{\mu\nu}\,\Delta_\mu[\psi(j) - n(j)] = 0\,. \qquad (16)$$

In two dimensions, this means simply that the gradient of $\psi(j) - n(j)$ vanishes, so $\psi(j) - n(j)$ is independent of j, and it is easy to see that without loss of generality, we can choose it to be zero. Thus the partition function becomes (up to an overall constant) exactly

$$Z = \sum_{\{\psi(j)\}} \prod I_{\psi(j)}(\beta)\, I_{\epsilon_{\mu\nu}\Delta_\nu\psi(j)}(\kappa)\,. \qquad (17)$$

The original, continuous angular fields have been replaced by integer-valued fields on the dual lattice. A primary virtue of this Fourier analysis is its simplicity in the low-temperature limit, $\beta, \kappa \gg 1$. Keeping the leading terms in this limit and neglecting field-independent constants, Z becomes

$$Z = \sum_{\{\psi(j)\}} \exp\left\{\sum - \frac{1}{2\kappa}\,[\Delta_\mu\,\psi(j)]^2 - \frac{1}{2\beta}\,\psi(j)^2 + O(\kappa^{-3}, \beta^{-3})\right\}. \qquad (18)$$

This is a useful low-temperature representation for Z for two reasons. First, the quadratic terms are most important for $\beta, \kappa \gg 1$, quartic and higher-order terms are effectively less important by an extra factor of β^{-1} or κ^{-1}. Second, this observation implies that it is reasonable to try to expand the Lagrangian in these terms. This should provide us with a systematic algorithm for exploring smaller and smaller values of κ and β. Note also the fact that κ and β appear in the denominator of the coefficient of the quadratic term. This is a typical effect of a duality transformation: High temperatures of the original representation are mapped into low temperatures of the dual representation, and vice versa.

We have argued that the expression in Eq. (18) is a quantitatively good approximation to the full partition function, Eq. (12), at low temperatures. But it is also qualitatively useful outside the low-temperature domain. First we note that the symmetries of the full partition function and the quadratic approximation are the same. This means that the topological excitations implied by these two forms are the same (see also Sec. IV). Second, Eq. (18) is useful for at least a qualitative determination of the critical properties of the system. Whether the theories defined by Eqs. (17) and (18) (or their analog in higher dimensions) lie in the same universality class is not clear. But experience with similar models, notably the XY model, suggests that quadratic forms such as (18) are a good guide to at least the most general features of the phase transitions. This point will be discussed further in Ref. 9. For these reasons we shall often restrict our attention to the simple quadratic approximations rather than the full theories. This should raise no conceptual difficulties: Higher-order terms can always be included in a straightforward way.

We now want to write Z in a form which displays explicitly the topological excitations. This can be done by using the identity valid for arbitrary $f(z)$,

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-2\pi i k z} f(z)\, dz\,. \qquad (19)$$

This allows us to replace the sum over integer-valued fields by continuous fields plus sources,

$$Z = \sum_{\{p(j)\}} \int d\psi(j) \exp\Big(\sum - \frac{1}{2\kappa}\{[\Delta_\mu\,\psi(j)]^2 + m^2\psi(j)^2\} + 2\pi i p(j)\psi(j)\Big)\,, \qquad (20)$$

where we set $m^2 \equiv \kappa/\beta$.

Next, we may carry out the $\psi(j)$ integrations, and we find

$$Z = Z_0 \sum_{\{p_j\}} \exp\Big[-\kappa\, 4\pi^2 \sum_{j,k} p(j)\, D(j-k; m^2)\, p(k)\Big]\,, \qquad (21)$$

where Z_0 is a massive-spin-wave partition function [i.e., it is Z with all $p(j) = 0$], and $D(j-k; m^2)$ is the lattice Green's function in two dimensions satisfying

$$[-\Delta_\mu(j)^2 + m^2]\, D(j-k; m^2) = \delta_{jk}\,. \qquad (22)$$

We will show in Sec. IV that the integers $p(j)$ may be thought of as the vortices of the original fields, so we may describe the system represented by this partition function as a theory of massive spin waves plus vortices. This is analogous to the well-known result of the Kosterlitz-Thouless treatment of the $d=2$ XY model,[5] which is a certain $m^2=0$ limit of our result. The contribution of the vortices to Z can be thought of as that of a gas of charges interacting through a short-range potential. Note that these vortices do not interact with the massive spin waves whose partition function is Z_0. This is a result of retaining only the quadratic form of Z, Eq. (18). Higher-order terms will induce interactions between the spin waves and vortices.

The occurrence of the mass m leads to significant differences from the two-dimensional XY model ($m=0$). The long-range, Coulomb (logarithmic) potential of the XY model is replaced here by an exponentially decreasing potential

$$D(j;m^2)\sim \frac{e^{-m|j|}}{(m|j|)^{1/2}} \quad \text{as } |j|\to\infty , \tag{23}$$

and, equally important the self-energy $\mu\equiv D(0;m^2)$ remains finite in the infinite-volume limit. In the $d=2$ XY model, the volume divergence of the self-energy gives rise to a neutrality condition: the only allowed configurations of vortices are those with total overall vorticity zero. In our case, on the other hand, there is no neutrality condition for this (Yukawa) gas of vortices.

Note that our solution makes plausible the assumption of Callan *et al.*,[12] that the $d=2$ Abelian Higgs model can be approximately described by an ideal gas of vortex points (instantons). Specifically, this will be a good approximation when the density of instantons is sufficiently small so that the mean separation between them is large compared with the range m^{-1} of the interaction potential. This will be true at sufficiently low temperatures when $e^{-\beta\mu}$ is small. Note that the ideal gas assumption may also be reasonable at higher temperatures since the strength of the interaction effectively decreases like κ.

The short-range nature of the vortex-vortex interaction, and the lack of neutrality are also crucial for a discussion of the possible phases of the system. We will deal with these questions elsewhere.[9] We turn now to a discussion of the topological excitations which one finds in the three-dimensional case.

B. Three dimensions

In three dimensions, we have (suppressing site labels)

$$\begin{aligned} Z &= \int\int \delta\chi\,\delta\theta_\sigma \exp\left[\sum \frac{\beta}{2}\cos(\epsilon_{\mu\nu\lambda}\epsilon_{\lambda\rho\sigma}\Delta_\rho\theta_\sigma)+\kappa\cos(\Delta_\sigma\chi-\theta_\sigma)\right] \\ &= \sum_{\{n_{\mu\nu},m_\mu\}}\prod I_{n_{\mu\nu}}(\beta)I_{m_\sigma}(\kappa)\int \delta\chi\delta\theta_\sigma \exp[\tfrac{1}{2}n_{\mu\nu}\epsilon_{\mu\nu\beta}\epsilon_{\beta\rho\sigma}\Delta_\rho\theta_\sigma+m_\sigma(\Delta_\sigma\chi-\theta_\sigma)]\,. \end{aligned} \tag{24}$$

Performing the χ integrations, we obtain the constraint

$$\Delta_\sigma m_\sigma = 0\,. \tag{25}$$

Since the divergence vanishes, m_σ is a curl of another field A_β,

$$m_\sigma = \epsilon_{\sigma\rho\beta}\Delta_\rho A_\beta\,. \tag{26}$$

A_β is defined on links of the dual lattice as discussed in Ref. 8. Performing the θ_σ integrations gives the second constraint

$$\tfrac{1}{2}\epsilon_{\beta\rho\sigma}\epsilon_{\beta\mu\nu}\Delta_\rho n_{\mu\nu}+m_\sigma=0\,, \tag{27}$$

or, in terms of A_β,

$$\epsilon_{\sigma\beta\rho}\Delta_\beta(\tfrac{1}{2}\epsilon_{\beta\mu\nu}n_{\mu\nu}-A_\beta)=0\,. \tag{28}$$

The curl being zero, the quantity in parentheses must be a gradient of another field, S, defined on the vertices of the dual lattice

$$\tfrac{1}{2}\epsilon_{\beta\mu\nu}n_{\mu\nu}-A_\beta=-\Delta_\beta S \tag{29a}$$

or

$$n_{\mu\nu}=\epsilon_{\mu\nu\beta}(A_\beta-\Delta_\beta S)\,. \tag{29b}$$

Now, recall that the functional sum in Eq. (24) is over all distinct sets of $n_{\mu\nu}$ and m_σ which satisfy Eqs. (25) and (27). But fixing $\{n_{\mu\nu},m_\sigma\}$ does not unambiguously define A_β and S. A_β is defined only up to a gradient, i.e., m_σ is unchanged by the replacement $A_\beta\to A_\beta+\Delta_\beta\Lambda$. $n_{\mu\nu}$ will be unchanged also if, at the same time, $S\to S+\Lambda$. Thus the dual theory possesses a gauge invariance similar to the original theory except that the fields A_β and S are discrete and integer valued rather than continuous. In terms of them, we have

$$Z=\sum_{\{n_{\mu\nu},m_\mu\}}\prod I_{\epsilon_{\mu\nu\lambda}(A_\lambda-\Delta_\lambda S)}(\beta)\,I_{\epsilon_{\sigma\rho\lambda}\Delta_\rho A_\lambda}(\kappa)\,, \tag{30}$$

where the sum is understood to be over n's and m's satisfying the representation Eqs. (26) and (29).

As before, we may replace the sum over integer-valued fields with an integration over continuous fields plus a sum over certain sources:

$$Z=\int \delta n_{\mu\nu}\,\delta m_{\mu}\sum_{\{K_{\mu\nu},L_{\mu}\}}\prod I_{\epsilon_{\mu\nu\lambda}(A_\lambda-\Delta_\lambda S)}(\beta)\, I_{\epsilon_{\sigma\rho\lambda}\Delta_\rho A_\lambda}(\kappa)\exp\Big[\sum i\,2\pi(K_{\mu\nu}n_{\mu\nu}+L_{\mu}m_{\mu})\Big]\,. \tag{31}$$

We recall the representations Eqs. (26) and (29b) for $n_{\mu\nu}$ and m_μ. Inserting them into the exponents, we can write the exponential as

$$\begin{aligned}\sum K_{\mu\nu}n_{\mu\nu}+L_{\mu}m_{\mu}&=\sum K_{\mu\nu}\epsilon_{\mu\nu\lambda}(A_\lambda-\Delta_\lambda S)+L_\mu\epsilon_{\mu\rho\lambda}\Delta_\rho A_\lambda\\ &=\sum(\epsilon_{\mu\nu\lambda}K_{\mu\nu}-\epsilon_{\mu\rho\lambda}\Delta_\rho L_\mu)A_\lambda+(\epsilon_{\mu\nu\lambda}\Delta_\lambda K_{\mu\nu})S\\ &\equiv\sum J_\lambda A_\lambda+QS\,,\end{aligned} \tag{32}$$

where we have used periodic boundary conditions and summation by parts. We may now formally rewrite the partition function as

$$Z=\sum_{\{J_\lambda,Q\}}{}'\int{}' dA_\lambda dS\prod I_{\epsilon_{\mu\nu\lambda}(A_\lambda-\Delta_\lambda S)}(\beta)\,I_{\epsilon_{\sigma\rho\lambda}\Delta_\rho A_\lambda}(\kappa)\exp\Big[\sum 2\pi i(J_\lambda A_\lambda+QS)\Big] \tag{33a}$$

$$\simeq\sum_{\{J_\lambda,Q\}}{}'\int{}'\delta A_\lambda\delta S\exp\Big[\sum-\frac{1}{4\beta}(A_\lambda-\Delta_\lambda S)^2-\frac{1}{4\kappa}(\epsilon_{\sigma\rho\lambda}\Delta_\rho A_\lambda)^2+i2\pi(J_\lambda A_\lambda+QS)\Big]\,, \tag{33b}$$

where the last form is, up to overall constants, the quadratic approximation, accurate for $\beta,\kappa\gg 1$, as discussed in the last subsection.

The integration and summation variables have been changed from $\{m_{\mu\nu},n_\mu\}$ to $\{A_\lambda,S\}$ and $\{K_{\mu\nu},L_\mu\}$ to $\{J_\lambda,Q\}$, respectively. In making this change we must recognize that the real degrees of freedom of the theory are the m's, n's, K's, and L's. The gauge invariance of the theory in terms of its dual variables discussed above requires, in the usual way, that we specify gauge conditions when integrating and summing over the A's, S's, J's, and Q's in order to avoid overall divergences from extraneous summations. This is indicated by the primes in the last equation. Note also that the gauge choices must be made in tandem: For example, if we choose $A_0=0$, we must also fix $J_0=0$. That all of this formal manipulation, in fact, leads to the correct result can be seen by going back to the expression Eq. (24), and choosing a gauge in the summation before introducing the sources. A complete discussion of the procedure for the $d=3$ XY model (an $m^2=0$ limit of our theory) is given in Ref. 8.

We will show in the next section that the J_λ's and Q's may be regarded as the topological excitations of the original theory. But if they are to represent real excitations, then any configuration of J_λ's and Q's must be gauge invariant. From Eq. (32) we can write J_λ and Q in terms of $K_{\mu\nu}$ and L_μ, from which it is apparent that

$$\Delta_\lambda J_\lambda(j)=Q(j)\,, \tag{34}$$

which ensures that any configuration of J's and Q's will have a gauge-invariant meaning (see also Ref. 8).

How can we understand the appearance of these topological excitations? Recall what is already known about certain related theories. Had we started with no gauge potential θ_μ, we would be dealing with the pure XY model which, for $d=3$, has topological singularities given by closed vortex loops, represented by a *conserved* topological current J_μ. Had we begun with the pure gauge theory, compact photodynamics, we would have found the instantons to be magnetic monopoles Q. The topological singularities of our Abelain Higgs model are a combination of these two types, namely, closed vortex loops and open strings terminating on monopoles. Notice that there are no free monopoles. That this is the correct interpretation is shown in the next section.

We now want to consider the quadratic form of Z, Eq. (33b) and carry out the Gaussian functional integral. The gauge choice $S=0$ (and $Q=0$) is particularly simple. We have

$$\begin{aligned}Z&=\sum_{\{A_\lambda\}}\exp\Big[\sum-\frac{1}{4\beta}A_\lambda^2-\frac{1}{4\kappa}(\epsilon_{\sigma\rho\lambda}\Delta_\rho A_\lambda)^2\Big]=\sum_{\{J_\lambda\}}\int_{-\infty}^{\infty}dA_\lambda\exp\Big\{\sum-\frac{1}{4\kappa}[(\epsilon_{\sigma\rho\lambda}\Delta_\rho A_\lambda)^2+m^2A_\lambda^2]+i2\pi J_\lambda A_\lambda\Big\}\\ &=Z_0\sum_{\{J\}}\exp\Big[\sum_{j,k}-4\pi^2\kappa J_\lambda(j)D_{\lambda\mu}(j-k;m^2)J_\mu(k)\Big]\,,\end{aligned} \tag{35}$$

where we have restored the lattice site indices in the last line. Here, Z_0 is the partition function for the massive vector field with no topological excitations, and $D_{\lambda\mu}(j\,;m^2)$ is the three-dimensional lattice Green's functions defined by

$$D_{\lambda\mu}(j-k;m^2) \equiv \left[\delta_{\lambda\mu} + \frac{\Delta_\lambda\Delta_\mu}{m^2}\right]_j D(j-k;m^2)\,,$$
$$(-\Delta_\mu{}^2 + m^2)\,D(j-k;m^2) = \delta_{jk}\,. \tag{36}$$

Thus

$$Z = Z_0 \sum_{\{J\}} \exp\left[\sum_{j,k} -\kappa\pi^2\left(J_\lambda(j)\,J_\lambda(k) + \frac{1}{m^2}\,Q(j)Q(k)\right) \times D(j-k;m^2)\right]\,, \tag{37}$$

where

$$Q(j) \equiv \Delta_\lambda J_\lambda(j)\,.$$

This appears to be a theory of a massive "vector" particle in the presence of certain topologically induced sources. [Remember that $d=3$, so the angular momentum corresponds to O(2).] Having evaluated the Gaussian integral, we see that the interactions between the topological excitations are short range since

$$D(j;m^2) \sim \frac{e^{-|j|m}}{m\,|j|} \quad \text{as } |j| \to \infty\,. \tag{38}$$

This is to be contrasted with the XY model in three dimensions in which pieces of the vortex rings interact via a power-law potential.

Before turning to four and higher dimensions, it is amusing to note the following relationship between the topological excitations of the two- and three-dimensional theories. Two dimensions may be thought of as a slice through three dimensions. In the $d=3$ XY model, the topological excitations are vortex rings. The intersection of a ring with a plane is two points, or a vortex-antivortex pair. This immediately leads to the neutrality condition for the $d=2$ XY model. In the $d=3$ Abelian Higgs theory, we have, in addition to closed rings, open, finite vortex strings terminating on monopoles. Slicing through a line with a plane, we have a single-vortex penetration with no compensating antivortex. We therefore lose the requirement of overall neutrality in the $d=2$ theory, as discussed in the last subsection.

C. Four and higher dimensions

In four and higher dimensions the analysis proceeds in essentially the same way as in three dimensions, only the indices are more complicated. Since four dimensions occupies a special place in our view of space-time, we will discuss this case explicitly.

For $d=4$ the partition function has the form (again suppressing site indices)

$$Z = \int \delta\chi\delta\theta_\sigma \exp\left[\sum \frac{\beta}{2}\cos(\tfrac{1}{2}\epsilon_{\mu\nu\alpha\beta}\epsilon_{\alpha\beta\rho\sigma}\Delta_\rho\theta_\sigma) + \kappa\cos(\Delta_\sigma\chi - \theta_\sigma)\right]$$
$$= \sum_{\{n_{\mu\nu},\,m_\mu\}} \prod I_{n_{\mu\nu}}(\beta)\,I_{m_\sigma}(\kappa) \int \delta\chi\delta\theta_\sigma \exp\left\{i\left[\sum \tfrac{1}{4}n_{\mu\nu}\,\epsilon_{\mu\nu\alpha\beta}\epsilon_{\alpha\beta\rho\sigma}\Delta_\rho\theta_\sigma + m_\sigma(\Delta_\sigma\chi - \theta_\sigma)\right]\right\}. \tag{39}$$

Integrating over χ and θ_σ implies the following constraints:

$$(1)\quad \Delta_\sigma m_\sigma = 0\,, \tag{40a}$$

$$(2)\quad \tfrac{1}{4}\,\Delta_\rho n_{\mu\nu}\,\epsilon_{\mu\nu\alpha\beta}\,\epsilon_{\alpha\beta\rho\sigma} + m_\sigma = 0\,. \tag{40b}$$

The divergence condition requires that m_σ be a curl,

$$m_\sigma = \tfrac{1}{2}\,\epsilon_{\sigma\rho\alpha\beta}\Delta_\rho A_{\alpha\beta} \quad (A_{\alpha\beta} = -A_{\beta\alpha})\,. \tag{41}$$

Then the curl condition becomes

$$\epsilon_{\sigma\rho\alpha\beta}\Delta_\rho(\tfrac{1}{2}\epsilon_{\mu\nu\alpha\beta}n_{\mu\nu} - A_{\alpha\beta}) = 0\,. \tag{42}$$

This requires that the quantity in parentheses be a gradient

$$\tfrac{1}{2}\epsilon_{\alpha\beta\mu\nu}n_{\mu\nu} - A_{\alpha\beta} = \tfrac{1}{2}\epsilon_{\alpha\beta\rho\sigma}\,\epsilon_{\rho\sigma\mu\nu}\,\Delta_\mu S_\nu\,. \tag{43a}$$

So,

$$n_{\mu\nu} = \epsilon_{\mu\nu\alpha\beta}(\tfrac{1}{2}A_{\alpha\beta} + \Delta_\alpha S_\beta)\,, \tag{43b}$$

which is analogous to Eq. (29) for the three-dimensional case. As before, ambiguities in the definition of the integers $A_{\alpha\beta}$ and S_β may be understood as a gauge symmetry:

$$S_\beta \to S_\beta + \Lambda_\beta ,$$
$$A_{\alpha\beta} \to A_{\alpha\beta} + \Delta_\alpha \Lambda_\beta - \Delta_\beta \Lambda_\alpha . \tag{44}$$

Following the arguments of the last subsection we may write the partition function as

$$Z = \sum_{\{n_{\mu\nu}, m_\sigma\}} \prod I_{\epsilon_{\mu\nu\alpha\gamma}[(1/2)A_{\alpha\gamma} + \Delta_\alpha S_\gamma]}(\beta) I_{(1/2)\epsilon_{\sigma\rho\alpha\gamma}\Delta_\rho A_{\alpha\gamma}}(\kappa)$$
$$= \sum_{\{J_{\rho\sigma}, Q_\sigma\}}{}' \int{}' \delta A_{\rho\sigma} \delta S_\sigma \prod I_{\epsilon_{\mu\nu\alpha\gamma}[(1/2)A_{\alpha\gamma} + \Delta_\alpha S_\gamma]}(\beta) I_{(1/2)\epsilon_{\sigma\rho\alpha\gamma}\Delta_\rho A_{\alpha\gamma}}(\kappa) \exp\left[2\pi i \sum \tfrac{1}{2} J_{\rho\sigma} A_{\rho\sigma} + Q_\sigma S_\sigma\right], \tag{45}$$

where the prime indicates that we must fix a gauge, and the integer-values sources (which are the topological excitations) satisfy

$$J_{\rho\sigma} = -J_{\sigma\rho}, \quad \Delta_\rho J_{\rho\sigma} = Q_\sigma . \tag{46a}$$

Note that this implies that Q_σ is conserved:

$$\Delta_\sigma Q_\sigma = 0 . \tag{46b}$$

The interpretation of these topological singularities is given by a simple generalization from three dimensions. In the pure gauge theory in four dimensions, the singularities Q would be monopole loops, i.e., "world lines" of monopole-antimonopole pairs. In the XY model in four dimensions, the currents $J_{\rho\sigma}$ would be conserved, i.e., $\Delta_\rho J_{\rho\sigma} = 0 = \Delta_\sigma J_{\rho\sigma}$, and would represent *closed* orientable surfaces (spheres and spheres with handles). In addition to these closed surfaces, the model considered here has excitations which are slices through these surfaces (or windows on the surface) which are bounded by the monopole loop. In three dimensions current lines terminate on monopoles, and in four dimensions a surface terminates on monopole loops. If we think of one dimension as time, these topological excitations represent the following events: At some time a monopole-antimonopole pair is created. As they separate, they are connected by a string. At some later time, the pair comes together and annihilates. Because the string itself has dynamical degrees of freedom, one also has events in which a string loop is created, evolves, and annihilates. This latter event sweeps out a closed surface in space-time.

In the quadratic approximation the partition function in the $S_\sigma = 0$ gauge is

$$Z = \sum_{\{J_{\rho\sigma}\}} \int_{-\infty}^{\infty} \delta A_{\rho\sigma} \exp\left\{\sum -\frac{1}{4\kappa} [(\epsilon_{\tau\lambda\rho\sigma} \Delta_\lambda A_{\rho\sigma})^2 + m^2 A_{\rho\sigma}{}^2] + \pi i J_{\rho\sigma} A_{\rho\sigma}\right\} . \tag{47}$$

One can show from this form that the antisymmetric tensor $A_{\rho\sigma}$ represents a massive spin-1 field. As before, we could carry out the Gaussian integration. The result is an expression similar to Eq. (35). It has a factor Z_0 representing the partition function of a massive spin wave with two polarization indices. This is multiplied by the partition function for the topological excitations described above, whose elements interact through a Yukawa-type potential of range m^{-1}.

IV. THE DUAL FIELDS AS TOPOLOGICAL EXCITATIONS

In this section, we explore a periodic Gaussian Abelian Higgs model primarily in order to motivate the interpretation of the integer-valued sources which arise in the formulation of the dual theory as topological singularities of the original theory. A similar periodic Gaussian model but analogous to the XY model was introduced by Berezinskii[14] and Villain.[15] For that system, the periodic Gaussian model was shown to be a good low-temperature approximation to the XY model (which, when expressed in its dual representation, reproduces exactly the quadratic approximation to the dual formulation of the full model) and to be a convenient vehicle for describing the topological excitations of the system. Our periodic Gaussian model possesses precisely the same characteristics *vis a vis* the Abelian Higgs model. (Other authors who have also considered analogous periodic Gaussian models include those of Refs. 8 and 16.)

The periodic Gaussian model arises by approximating the cosine as

$$e^{\beta \cos x} \approx e^{\beta} \sum_{n=-\infty}^{\infty} \exp\left[\frac{-\beta}{2}(x + 2\pi n)^2\right] , \tag{48}$$

and allowing x on the right-hand side to range over $(-\infty, \infty)$. (Villain,[15] in fact, generalized this to allow the coefficient of the quadratic term to be temperature dependent in a certain way which, he argued, might be useful at both low and high temperatures. The form given here is valid at low temperatures. See also Jose *et al.*[16])

Consider, for example, the two-dimensional case. The partition function for our periodic Gaussian model is

$$Z=\tilde{\sum_{\{a_\mu,b\}}}\int_{-\infty}^{\infty}\delta\chi\delta\theta_\mu \exp\left[-\frac{\kappa}{2}\sum(\Delta_\mu\chi-\theta_\mu+2\pi a_\mu)^2\right]\times\exp\left(-\frac{\beta}{2}\sum(\epsilon_{\mu\nu}\Delta_\mu\theta_\nu+2\pi b)^2\right), \tag{49}$$

where we are to sum over integers a_μ and b. The periodicity of the original problem does not require summing independently over both b and a_μ. Indeed, even if we choose a gauge to define the functional integrals over χ and θ_μ, summing independently over a_μ and b would render Z defined in Eq. (49) infinite. In fact, this infinity is not directly connected with the local gauge symmetry of the original model. Even the periodic Gaussian analog of the $d=2$ XY model contains a similar (although somewhat simpler) summation redundancy. In the present case, the redundancy can be eliminated by noting that we can first shift $\theta_\mu\to\theta_\mu+2\pi a_\mu$ and then redefine $b\to b-2\pi\epsilon_{\mu\nu}\Delta_\mu a_\nu$. Thereby a_μ completely disappears from the problem, and so without loss of generality we can set $a_\mu\equiv 0$. However, our formal manipulations can be carried through regardless of this redundancy, and so we shall not now place any specific constraint on the sum over a_μ and b. The tilde over the summation sign reminds us that such a restriction is required in principle. Later in this section we will eliminate the summation redundancy in a different way to show that when $d=2$ it is not necessary to explicitly make $F_{\mu\nu}$ compact. Because of the local gauge invariance, the expression above is also infinite, due to the integrals over χ and θ_μ, and a gauge choice must be made to render it finite. But again, for the formal manipulations we are concerned with, we can ignore this infinity and imagine dividing at the end by the infinite volume of the gauge group. One may of course choose a gauge from the beginning and after carrying out analogous manipulations one will be led to the same conclusions.

Next we introduce the Fourier transform for each exponential using

$$\exp\left[-\frac{\beta}{2}(x+2\pi n)^2\right]=\frac{1}{(2\pi\beta)^{1/2}}\int_{-\infty}^{\infty}dy\exp\left[-\frac{1}{2\beta}y^2+iy(x+2\pi n)\right]. \tag{50}$$

Then we have (up to multiplicative constants)

$$Z=\tilde{\sum_{\{a_\mu,b\}}}\int\delta t_\mu\delta s\int\delta\chi\,\delta\theta_\mu\exp\left[\sum-\frac{1}{2\kappa}t_\mu^{\,2}+it_\mu(\Delta_\mu\chi-\theta_\mu+2\pi a_\mu)\right]\exp\left[\sum-\frac{1}{2\beta}s^2+is(\epsilon_{\mu\nu}\Delta_\mu\theta_\nu+2\pi b)\right]. \tag{51}$$

Carrying out the integrations over χ and θ_μ yields the constraints

$$\prod_j\delta(\Delta_\mu t_\mu(j))\,\delta(t_\nu(j)+\epsilon_{\mu\nu}\Delta_\mu s(j)). \tag{52}$$

[Notice that there are many redundant δ functions in this set of constraints. Thus Z is proportional to some infinite power of $\delta(0)$. This just reflects the fact that we did not specify the gauge and represents once again the infinite gauge volume.] These constraints are satisfied as follows: The first condition implies that t_μ can be written as a curl

$$t_\mu=\epsilon_{\mu\nu}\Delta_\nu\psi. \tag{53a}$$

Then the second condition is equivalent to

$$\Delta_\mu(s+\psi)=0, \tag{53b}$$

so that $s(j)+\psi(j)$ must be independent of the lattice site j. Without loss of generality, we may choose $s+\psi=0$, and so, up to an overall (infinite) factor, we get

$$Z=\tilde{\sum_{\{a_\mu,b\}}}\int\delta\psi\exp\left[-\frac{1}{2\kappa}\sum(\Delta_\nu\psi)^2\right]\exp\left(-\frac{1}{2\beta}\sum\psi^2\right)\times\exp\left[\sum-i2\pi(b+\epsilon_{\mu\nu}\Delta_\mu a_\nu)\psi\right]. \tag{54}$$

Thus the source of the ψ field is the integer

$$p=\epsilon_{\mu\nu}\Delta_\mu a_\nu+b. \tag{55}$$

[Compare these expressions with Eq. (20).] If we think of a_ν as the "integer" part of the gauge invariant $\Delta_\mu\chi-\theta_\mu$ and of b as the circulation of the vector potential, then we see that p represents the vorticity of the original fields.

As mentioned earlier, it is redundant to sum over both a_μ and b. It is interesting to note, in particular, that by shifting $\theta_\mu\to\theta_\mu+2\pi c_\mu$, we may choose c_μ such that $\epsilon_{\mu\nu}\Delta_\mu c_\nu+b=0$. This eliminates b from the summand in Eq. (49), and thus it is unnecessary to make the gauge field compact in two dimensions to obtain these vortices as topological singularities. This can be understood by remembering that in two dimensions the pure gauge fields have no dynamical degrees of freedom. In higher dimensions, on the other hand, additional nontrivial excitations result from the compact nature of $F_{\mu\nu}$, as we shall see. (Note in addition, that even after eliminating b, it is still redundant to sum over all a_μ. In particular, since the summand depends only on p we must not sum over a_μ's which differ from each other only by a gradient.)

We will now sketch the similar construction in three dimensions. For $d=3$ we write the periodic quadratic partition function as

$$Z=\tilde{\sum_{\{a_\mu, b_\lambda\}}}\int_{-\infty}^{\infty}\delta\chi\,\delta\theta_\mu \exp\left[-\sum\frac{\kappa}{2}(\Delta_\mu\chi-\theta_\mu+2\pi a_\mu)^2\right]\exp\left[-\sum\frac{\beta}{2}(\epsilon_{\lambda\mu\nu}\Delta_\mu\theta_\nu+2\pi b_\lambda)^2\right]. \tag{56}$$

As before, the sum over all values of both a_μ and b_λ is redundant as indicated by the tilde. However, in this case, we *cannot* eliminate b_λ since an arbitrary shift of $\theta_\nu\to\theta_\nu+2\pi c_\nu$ gives $b_\lambda\to b_\lambda+\epsilon_{\lambda\mu\nu}\Delta_\mu c_\nu$. One cannot always find a set of c_ν to cancel b_λ, because the second term is divergenceless whereas the first term, in general, is not. As before, we carry along this redundant summation for now. Next, Fourier transforming gives

$$Z=\tilde{\sum_{\{a_\mu, b_\lambda\}}}\int\delta t_\mu\,\delta s_\lambda\int\delta\chi\,\delta\theta_\mu \exp\left[\sum-\frac{1}{2\kappa}t_\mu{}^2+it_\mu(\Delta_\mu\chi-\theta_\mu+2\pi a_\mu)\right]\exp\left[\sum-\frac{1}{2\beta}s_\lambda{}^2+is_\lambda(\epsilon_{\lambda\mu\nu}\Delta_\mu\theta_\nu+2\pi b_\lambda)\right]. \tag{57}$$

Performing the integrations over χ and θ_ν gives the Dirac δ-function constraints

$$\prod_j\delta(\Delta_\mu t_\mu(j))\,\delta(t_\nu(j)+\epsilon_{\mu\nu}\Delta_\mu s_\lambda(j)). \tag{58}$$

This implies, first, that t_μ is a curl, $t_\mu=\epsilon_{\mu\rho\sigma}\Delta_\rho A_\sigma$, and, secondly, that $s_\lambda-A_\lambda$ is a gradient, $A_\lambda-s_\lambda\equiv\Delta_\lambda S$. Note also that the δ-function constraints (58) are redundant. As before, the ambiguities in the definition of A_λ and S may be understood as a gauge symmetry $A_\lambda\to A_\lambda+\Delta_\lambda\Lambda$, $S\to S+\Lambda$. Thus we have

$$\begin{aligned}Z=\tilde{\sum_{\{a_\mu, b_\lambda\}}}\int'\delta A_\lambda\,\delta S&\exp\left[\sum-\frac{1}{2\kappa}(\epsilon_{\mu\rho\sigma}\Delta_\rho A_\sigma)^2\right]\\&\times\exp\left[\sum-\frac{1}{2\beta}(A_\lambda-\Delta_\lambda S)^2\right]\\&\times\exp\{\sum i\,2\pi[a_\mu\epsilon_{\mu\nu\lambda}\Delta_\nu A_\lambda\\&\qquad+b_\lambda(A_\lambda-\Delta_\lambda S)]\},\end{aligned} \tag{59}$$

where we have dropped an overall infinite factor from the redundant δ functions, and the prime on the integral indicates that we must fix a gauge when integrating over A_λ and S. We can now perform summation by parts to write the last factor in the form [cf. Eq. (32)]

$$e^{2\pi i(J_\lambda A_\lambda+QS)}, \tag{60a}$$

where

$$\begin{aligned}J_\lambda&\equiv b_\lambda+\epsilon_{\lambda\mu\lambda}\Delta_\mu a_\nu,\\Q&\equiv\Delta_\lambda b_\lambda.\end{aligned} \tag{60b}$$

Note that the relation $Q=\Delta_\lambda J_\lambda$ follows, as expected for gauge invariance. We are thus led to the physical interpretation that the current J_λ is the (suitably defined) integer part of $F_{\mu\nu}$ plus the curl (vorticity) of the "integer part" of the angle $\Delta_\mu\chi-\theta_\mu$. Its divergence Q is the monopole density.

Similar considerations in higher dimensions motivate the general interpretation of the various integer-valued sources as topological excitations of the original fields.

V. COMMENTS

Using an exact duality transformation, we have identified the topological excitations of the lattice Abelian Higgs model, and have shown what their interactions are. We have some comments to make about these results:

(1) To better understand these excitations, it may be useful to compare our model with related models. This has been done to some extent in the text. Here we summarize these comments. To begin with, note that the familiar XY model is an $m\to 0$ of our model. This can be understood intuitively since, for $\beta\gg\kappa$, the energy is minimized for a vanishing field strength ($F_{\mu\nu}=\Delta_\mu\theta_\nu-\Delta_\nu\theta_\mu=0$). This means that θ_μ is a pure gauge (of the form $\Delta_\mu\Lambda$) and so, shifting $\chi\to\chi-\Lambda$, we recover the partition function for the XY model. In this limiting case ($m=0$), there are several striking differences from the finite-mass case.

First, as we mentioned in Sec. III A, in two dimensions the self-energy of a vortex grows for small m like $-\ln m$, so when $m=0$ single vortices at low temperatures are no longer allowed. On the other hand, a vortex-antivortex pair has finite energy even for $m=0$, so in this limit the only finite-energy configurations are those which respect overall neutrality. In addition, the interaction between vortices is no longer short range when $m^2=0$. Indeed the potential grows logarithmically as the separation between vortices increases. It is attractive between a vortex (say $p>0$) and an antivortex ($p<0$) and binds them at low enough temperatures.

In higher dimensions ($d\geq 3$), there are no monopoles or monopole currents [the $Q_{\alpha_1\cdots\alpha_{d-3}}$ in, for instance, Eqs. (33) and (45)] at $m=0$, and the topological current density $J_{\mu_1\mu_2\cdots\mu_{d-2}}$ is conserved. The short-range interaction between topological excitations in these dimensions also disappears when $m=0$, and is supplanted by a power-law potential.

It is also worthwhile to compare our model with the pure gauge theory. For example, in three dimensions, the pure gauge theory has topological singularities which are isolated monopoles. When the gauge fields are coupled in a gauge-invariant manner to the Higgs matter fields, the medium attaches a vortex string to the monopoles. Thus monopoles can no longer exist in isolation, but appear only as monopole-antimonopole pairs attached by strings.

(2) Notice that, by the exact duality transformation, we have succeeded in establishing an equivalence between theories. The original partition function is expressed in terms of phases $\chi(j)$ and gauge fields $\theta_\mu(j)$, whereas the transformation reexpresses the same quantity in terms of continuous "spin waves" $(\psi(j); A_\mu(j), S(j); \ldots)$ and topological excitations $(p(j); J_\mu(j), Q(j); \ldots)$. [Recall Eqs. (20), (33), and (45).] The two languages provide exact alternative descriptions of the same theory. This is one of the most intriguing aspects of the duality transformation, and it is interesting to ask whether a similar transformation can be found for a non-Abelian symmetry.

(3) In Ref. 9 we will describe what we expect the phases of our model to be, but we wish to make a few comments here. First we note the resemblance of the Abelian Higgs model to the theory of spin-glasses.[10] A naive model of a spin-glass has the Hamiltonian

$$H=\sum J(i,j)\,\vec{S}(i)\cdot\vec{S}(j)\,,$$

where $\vec{S}(i)$ are spin vectors coupled by a random variable $J(i,j)$. In the simplest version of the model the spins are Ising spins $S(i)=\pm 1$, and $J(i,j)$ is a nearest-neighbor coupling which has some probability p to be ferromagnetic $J(i,j)=+J$ and probability $1-p$ to be antiferromagnetic $J(i,j)=-J$. Consider in particular the case $p=\frac{1}{2}$. Then the theory has a local discrete invariance:

$$\vec{S}(k)\to-\vec{S}(k)\,,$$

$$J(k,j)\to-J(k,j)\,,$$

for some fixed k, and j a nearest neighbor of k. The local gauge invariance of electromagnetism is quite analogous; it is a continuous, U(1) or R_1, generalization thereof. Given the similarity between the models, one might expect some relationship between the phases of the two systems.

Second, we note that the behavior of the expectation value of the Wilson loop integral,[17] $\Gamma \equiv \langle \exp(i\oint\theta^\mu dx^\mu)\rangle$ changes dramatically when the Higgs fields are added to the pure gauge theory. We find that for all temperatures and all dimensions, $\Gamma\sim e^{-P}$ for large loops, where P is the perimeter of the loop. This is to be contrasted with the pure gauge theory in which Γ is sometimes falls like e^{-P}, and sometimes like e^{-P^2}, the difference being used as a signal for a phase transition. This result is perhaps not unexpected, and follows from the fact that Γ represents a "quark" loop whose charge is equal to the charge of the Higgs particle. Loop integrals associated with fractional charges have a quite different behavior.[9] We also remark that the different behavior of Γ in the pure gauge theory and in the Higgs theory can be related to the different topological excitations of the two theories. See Ref. 9 for a further discussion of these points.

Finally, we return to a point mentioned in Sec. II, namely, the behavior of our theory in the continuum limit. The question we raised there was whether in the continuum limit defined by the renormalizaion group the theory would retain all the interesting topological excitations we see on the lattice. One possible answer is that there are different phases of the models which have different continuum limits. In particular, we believe that for $d\geq 3$ and fixed m^2, there are two phases as a function of κ for finite κ. At high temperatures (κ small) we expect to find large-distance behavior which is strongly affected by the topological excitations, and therefore an appropriate continuum theory in which the excitations survive, while at low temperatures we expect these excitations not to be as important for the large-distance structure, and to disappear in the continuum theory. This point is also discussed in Ref. 9.

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