The Schwarzschild radial coordinate as a measure of proper distance

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(Received 28 December 1977)

It is shown that when time is measured in a Schwarzschild field by radially falling or rising geodesic clocks, the usual Schwarzschild radial coordinate R, defined by $ds^2 = dR^2/(1-2M/R) - (1-2M/R)dT^2 + R^2 d\Omega^2$, has the physical significance that it is a measure of proper distance between two events that occur simultaneously relative to the radially moving geodesic clocks, the two events lying on the same radial coordinate line.

I. INTRODUCTION

One of the underlying principles of general relativity is the freedom of choice of coordinates in the mathematical description of laws and physical quantities. However, certain coordinates may be "preferred" over other coordinates in the sense that they are simpler or are directly related to physical quantities, or both. For example, in the absence of gravitation, Minkowski coordinates are regarded as being measured by clocks and rods and are preferable to more complicated coordinates that are mathematical constructs in flat space-time.

It is well known that when expressed in terms of the usual Schwarzschild radial coordinate R, defined by

$$ds^2 = \frac{dR^2}{1-2M/R} - (1-2M/R)dT^2 + R^2 d\Omega^2 , \tag{1}$$

there are expressions for important physical quantities in the spherically symmetric case that take a much simpler form than they do in terms of other coordinates, such as isotropic or Kruskal coordinates. For example, the invariant area A of a sphere defined by $R =$ constant is given by

$$A = 4\pi R^2 . \tag{2}$$

Also, for a radially moving geodesic particle, the second derivative of R with respect to the proper time τ of the particle takes the inverse-square form

$$d^2R/d\tau^2 = -M/R^2 . \tag{3}$$

And in a Reissner-Nordström field given by the metric

$$ds^2 = \frac{dR^2}{1-2M/R+Q^2/R^2} - (1-2M/R+Q^2/R^2)dT^2 + R^2 d\Omega^2 , \tag{4}$$

the effective gravitating mass of the source that influences the motion of test particles is[1]

$$M_{\text{eff}} = M - Q^2/R . \tag{5}$$

We shall here show that the Schwarzschild radial coordinate R has a further direct physical significance, namely, that it is a measure of the proper distance between two simultaneous events, where simultaneity is determined by the times recorded on radially falling or rising geodesic clocks.

II. SIMULTANEITY AND PROPER DISTANCE

Consider two events lying on the same radial coordinate line with radial coordinates R_1 and R_2. Let these two events be simultaneous relative to the Schwarzschild coordinate T in Eq. (1). The proper distance L between these two events is

$$L = \int_{x^4 = \text{const}} ds = \int_{R_1, \, T=\text{const}}^{R_2} dR/(1-2M/R)^{1/2} , \tag{6}$$

where in evaluating the integral, the coordinate T is held fixed along the radial line joining the two events. Note that the square root in this expression for the proper distance becomes imaginary when $R < 2M$. This is linked to difficulties of interpretation inside the Schwarzschild radius.

The expression for proper distance in Eq. (6) depends on the simultaneity determined by the Schwarzschild coordinate T. This simultaneity is, of course, not absolute. It depends on the coordinate system. Therefore the same is true of the above definition of proper distance. Indeed, the Schwarzschild coordinate T enters twice: once in the initial statement that the events under consideration are simultaneous with respect to it, and once in the integral in Eq. (6), which is evaluated for constant T. Thus the situation itself and the integral formula for the proper distance are both coordinate dependent. In what follows we shall employ a corresponding definition of the proper distance between two events that are simultaneous for a different time coordinate, and will elaborate on the concept in the Conclusion.

III. A PHYSICAL TIME COORDINATE

The difficulties with the proper distance L in Eq. (6) and the metric in Eq. (1) when $R \leq 2M$ can be attributed to the use of the Schwarzschild coordinate T as a measure of time. This T is given by coordinate clocks that are not standard clocks. We shall show that these difficulties are alleviated when time is measured by radially falling or rising geodesic clocks that are standard clocks.

Consider radially moving geodesic standard clocks that are either falling from, or rising to, some maximum radius $R_i > 2M$. For brevity we shall speak only of falling clocks, but, as will be seen, the expressions to be developed will hold for rising clocks as well. Let the clocks be released from R_i at uniform intervals of T and be adjusted such that the time of each clock at the instant of release is equal to the (proper) time recorded on a standard clock that remains fixed at R_i. Thus one can envision a sort of clock factory fixed at R_i that, at regular intervals of time, drops clocks that are synchronized with a master clock in the clock factory. The time at any event in the manifold is then taken to be equal to the time on the radially falling geodesic clock that is coincident with this event, so that the time coordinate has the physical significance that it equals the time τ recorded by a geodesic, radially falling standard clock under the stated initial conditions. Of course with this procedure one coordinatizes only that portion of the manifold for which $R \leq R_i$, but for $R > R_i$ one can always use the Schwarzschild coordinates (R, T) with no difficulties.

To find the transformation between our operationally defined time coordinate τ and the Schwarzschild time coordinate T, we start with the well-known timelike solution to the radial geodesic equations,

$$dR/d\tau = \epsilon(2M/R - 2M/R_i)^{1/2}, \tag{7a}$$

$$dT/d\tau = \frac{(1-2M/R_i)^{1/2}}{1-2M/R}, \tag{7b}$$

where $\epsilon = -1$ or $+1$ depending on whether the geodesic clock is, respectively, falling from, or rising to, the maximum radius R_i. Equations (7a) and (7b) can be combined to give

$$d\tau/dR = (1-2M/R_i)^{1/2} dT/dR - \epsilon\frac{(2M/R - 2M/R_i)^{1/2}}{1-2M/R}. \tag{8}$$

Integrating Eq. (8) term by term between the limits R and R_i and making use of the requirement that at the instant of release the time on the falling clock is to be equal to the time recorded on a clock fixed at R_i, i.e., $\tau_i = (1-2M/R_i)^{1/2} T_i$, we obtain

$$\tau = (1-2M/R_i)^{1/2} T - \epsilon\int_{R_i}^{R} \frac{(2M/y - 2M/R_i)^{1/2}}{1-2M/y}\, dy. \tag{9}$$

This is the desired transformation between the coordinates (R, T) and (R, τ).

In terms of τ, the metric of Eq. (1) takes the form

$$ds^2 = \frac{1}{(1-2M/R_i)}[\epsilon dR - (2M/R - 2M/R_i)^{1/2} d\tau]^2 - d\tau^2 + R^2 d\Omega^2. \tag{10}$$

This form of the line element is analytic for all $R > 0$. The presence of the divergence of the integrand in Eq. (9) at the Schwarzschild radius is to be expected since a coordinate singularity can only be removed by a singular transformation. For $R > 0$, Eq. (9) holds everywhere except at $R = 2M$. Consequently, at these places the line element in Eq. (10) satisfies the Einstein field equations. Therefore, by continuity, it must satisfy the field equations also at the Schwarzschild radius. The metric form in Eq. (10) is similar to, but different from, the Eddington[2]-Finkelstein[3] metric

$$ds^2 = dR^2 - dT^{*2} + (2M/R)(dR - dT^*)^2 + R^2 d\Omega^2 \tag{11}$$

obtained from the transformation

$$T = T^* + 2M\ln(R/2M - 1). \tag{12}$$

Consider now two events that are simultaneous relative to our operational time coordinate τ. If we again take the proper distance between these two simultaneous events as

$$L = \int_{x^4 = \text{const}} ds,$$

we find from Eq. (10),

$$L = \int_{R_1}^{R_2} \frac{dR}{(1-2M/R_i)^{1/2}} = \frac{R_2 - R_1}{(1-2M/R_i)^{1/2}}. \tag{13}$$

Thus proper distance between events that occur at the same time τ is proportional to the Schwarzschild coordinate difference $R_2 - R_1$ between the events. Moreover, when the clock factory from which the clocks are dropped is situated at $R_i = \infty$, the proper distance between the two τ-simultaneous events becomes just $R_2 - R_1$. However, the limit $R_i \to \infty$ involves difficulties with infinite transit times, which we shall discuss at the end of the next section.

IV. A PHYSICAL SPACE COORDINATE

With the transformation of Eq. (9) leading to the metric form in Eq. (10), one has a situation in which the time-keeping clocks are moving relative to the space markers that measure constant R's. It might be more "natural" to have the space markers coincide with the time-keeping clocks. One would then have a radially moving reference system in which the time of an event is measured by the clock coincident with the event, and the space coordinate of the event is determined by the particular one of the falling clocks that happens to be at the event, i.e., by "how many" clocks this particular clock is away from some arbitrarily chosen clock which corresponds to the spatial origin. If the clock factory stamped the dropped clocks with consecutive numbers, the spatial coordinate would thus be the difference between the numbers stamped on the event clock and the spatial origin clock.

Let the particular clock that is serving as the spatial origin be released from R_i at the Schwarzschild time coordinate T_{0i}, and let successive "space marker" clocks be released from R_i at successive equal intervals $\Delta\tau_i$. The distance ρ to the Nth clock released after the origin clock is then defined to be

$$\rho = aN, \tag{14}$$

where the constant a is some scaling factor. Further, the Schwarzschild time coordinate T_i when the Nth clock is released is given by

$$T_i = N\Delta T_i + T_{0i}, \tag{15}$$

so that we may write

$$(\Delta T_i/a)\rho = T_i - T_{0i}. \tag{16}$$

From Eqs. (7a) and (7b) we obtain

$$dT/dR = \epsilon \frac{(1-2M/R_i)^{1/2}}{(1-2M/R)(2M/R-2M/R_i)^{1/2}}, \tag{17}$$

from which, by integrating from (R_i, T_i) to (R, T), one obtains

$$T_i = T - \epsilon(1-2M/R_i)^{1/2} \times \int_{R_i}^{R} \frac{dy}{(1-2M/y)(2M/y-2M/R_i)^{1/2}}. \tag{18}$$

Upon substituting Eq. (18) into Eq. (16), we obtain the transformation between the Schwarzschild coordinates (R, T) and our operationally defined space coordinate ρ:

$$\frac{\Delta T_i}{a}\rho = T - T_{0i} - \epsilon(1-2M/R_i)^{1/2} \times \int_{R_i}^{R} \frac{dy}{(1-2M/y)(2M/y-2M/R_i)^{1/2}}. \tag{19}$$

The integrals in the transformations in Eqs. (9) and (19) will involve terms in $\ln(1-2M/R)$ with all the attendant difficulties when $R \leq 2M$. However, upon eliminating T, we obtain the following combination of Eqs. (9) and (19), which does not diverge anywhere:

$$\tau - (1-2M/R_i)^{1/2}\frac{\Delta T_i}{a}\rho = (1-2M/R_i)^{1/2}T_{0i} + \epsilon\int_{R_i}^{R} \frac{dy}{(2M/y-2M/R_i)^{1/2}}. \tag{20}$$

This combination, which can be integrated, gives R as an implicit function of ρ and τ.

When the differential of Eq. (20),

$$(2M/R-2M/R_i)^{1/2}d\tau - \epsilon dR = (1-2M/R_i)^{1/2}(2M/R-2M/R_i)^{1/2}(\Delta T_i/a)d\rho, \tag{21}$$

is substituted into Eq. (10), we obtain the metric form in terms of the coordinates (ρ, τ),

$$ds^2 = (\Delta T_i/a)^2(2M/R-2M/R_i)d\rho^2 - d\tau^2 + R^2 d\Omega^2, \tag{22}$$

which is diagonal and analytic for all $R > 0$.

If we set $\Delta T_i = a$ and let the dropping radius $R_i \to \infty$, the metric form of Eq. (22) reduces to

$$ds^2 = (2M/R)d\rho^2 - d\tau^2 + R^2 d\Omega^2, \tag{23}$$

which was obtained by Lemaître[4] in 1933. However, the condition $R_i \to \infty$ means that a falling clock would use up an infinite amount of its time in making the journey to regions of finite R, and this infinite journey time should be taken into account in the transformations of Eqs. (9), (19), and (20). The difficulties associated with an infinite journey time can be avoided if R_i is kept finite.

When calculated with the metric form of Eq. (22), the proper distance $L = \int_{x^4=\text{const}} ds$ between two events simultaneous relative to τ is

$$L = (\Delta T_i/a)\int_{R_1}^{R_2} (2M/R-2M/R_i)^{1/2} d\rho. \tag{24}$$

Setting $d\tau = 0$ in Eq. (21) and substituting the remainder into Eq. (24), we obtain

$$L = -\epsilon \int_{R_1}^{R_2} \frac{dR}{(1-2M/R_i)^{1/2}} = -\epsilon \frac{R_2 - R_1}{(1-2M/R_i)^{1/2}}, \tag{25}$$

which (up to an immaterial + or − sign) is identical to the expression for proper distance given in Eq. (13). This equivalence is to be expected since the τ's in the nondiagonal metric form in Eqs. (10) and the diagonal metric form in Eq. (22) are identical, so that it makes no difference which of these metric forms is used to calculate $\int ds$ subject to the condition $d\tau = 0$.

V. CONCLUSION

We have shown that when, in the spherically symmetric case, the time coordinate is given by the times shown on appropriately synchronized falling standard clocks, we obtain a simple physical picture of the Schwarzschild radial coordinate as a measure of proper distance between simultaneous events, where proper distance is defined as

$$L = \int_{x^4 = \text{const}} ds . \tag{26}$$

Further, when a spatial coordinate is then introduced not for mathematical reasons but solely because it is naturally related to the falling clocks, the former nondiagonal metric becomes diagonal, and does so at no cost of simplicity. The line integral in Eq. (26) does not lie along a spacelike geodesic connecting the two τ-simultaneous events. However, the radial line of integration is a geodesic in the subspace defined by $\tau = \text{constant}$.

Equation (26) is not the only way proper distance can be defined. Landau and Lifshitz,[5] for example, starting from a space-time described by the general metric

$$ds^2 = g_{ij}dx^i dx^j + 2g_{i4}dx^i dx^4 + g_{44}(dx^4)^2 \quad (i,j = 1,2,3), \tag{27}$$

define infinitesimal proper distance dL by means of a "radar" method in the following manner: Let an observer with fixed coordinates x_0^i reflect a light signal from an event infinitesimally close to his world line. If the proper time measured on a clock at the observer's location between the emission and reception of the radar signal is $d\tau_0$, the infinitesimal proper distance between the observer and the event from which the light signal is reflected is defined to be

$$dL = \tfrac{1}{2} d\tau_0 . \tag{28}$$

With this definition Landau and Lifshitz show

$$dL = [(g_{ij} - g_{i4}g_{j4}/g_{44})dx^i dx^j]^{1/2}, \tag{29}$$

where dx^i is the spatial coordinate difference between the event and the observers world line.

It is seen that for diagonal metrics Landau and Lifshitz's definition of infinitesimal proper distance agrees with the infinitesimal form of the definition given in Eq. (26). If the metric is not diagonal, however, Eqs. (26) and (29) yield different results. A physical reason for this disagreement is obtained by recalling the discussion of Sec. III. The time coordinate τ in the nondiagonal metric in Eq. (10) is measured by clocks moving relative to the space coordinate R, whereas the definition of Landau and Lifshitz would require measuring time τ_0 with clocks that are fixed at $R = \text{const}$ in Eq. (10). But clocks fixed at $R = \text{const}$ in the metric in Eq. (10) are no different from clocks fixed at $R = \text{const}$ in the metric in Eq. (1), so one will eventually obtain the same result for proper distance given in Eq. (6) if the definition of Landau and Lifshitz is used with the metric of Eq. (10). Moreover, one cannot have physical clocks fixed at $R = \text{const} < 2M$, which is why the falling geodesic clocks were introduced in the first place in Sec. III.

Note added in proof. E. Newman and J. N. Goldberg, Phys. Rev. 114, 1391 (1959), in defining distance in terms of the geodesic deviation of null rays, have shown that in a Schwarzschild field the distance is proportional to the radial coordinate R. We thank Professor J. Stachel for bringing this to our attention.

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