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Kinematical conditions in the construction of spacetime

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We adopt the point of view that a solution of Einstein's equations is an evolution of given initial Cauchy data. Implementing the evolution equations necessarily requires a determination, not directly dictated by the field equations, of the kinematics of the observers in terms of which the evolution is represented. In this paper we study the observers' kinematics (velocities and accelerations) in terms of the geometry of their congruences of world lines relative to families of time slicings of spacetime, which contrasts with the more usual approach of imposing particular "gauge" or "coordinate conditions." The types of conditions we suggest are adapted to the exact Einstein equations for general strong-field, dynamic spacetimes that have to be calculated numerically. Typically, the equations are three-dimensionally covariant, elliptic, and linear in the kinematical functions (the lapse function and shift vector) that they determine. The gravitational field enters in nonlinear form through the presence of curvature in the equations. We present a flat-space model of such elliptic equations (e.g. for maximal slicing) which suggests that this curvature leads to an exponential decrease in the proper time between time slices at late times. We show how the use of maximal slicing with minimal-distortion observers generalizes the notion of a stationary rest frame to dynamical asymptotically flat spacetimes. In cosmological spacetimes the use of minimum-distortion observers is shown to differentiate between those universes which contain only kinematic time dependence (e.g. open Kasner universe) and those in which dynamical degrees of freedom are present (e.g. mixmaster universe). We examine many examples and construct new coordinate systems in both asymptotically flat and cosmological solutions to illustrate these properties.

I. INTRODUCTION

A systematic approach to solving the Einstein field equations has been developed in a practical form over the last decade.[1] The program is based on treating general relativity in terms of its Cauchy problem.[2] Initial data for the gravitational field and matter variables are chosen on a spacelike hypersurface that represents the starting moment of the physical situation one wishes to study. The four elliptic equations of constraint are solved numerically. Then these data are propagated into the future by numerically integrating the hyperbolic equations of evolution. The latter step involves first choosing the velocity and acceleration vectors of the observers who are describing the evolving spacetime. That is, one must prescribe the "kinematics" (i.e., spacetime gauge) of the construction. After the spacetime has been built, any information of interest, such as gravitational radiation, event horizons, particle trajectories, etc., can be extracted numerically.

In this paper we shall study spacetime kinematics from the viewpoint of spacetime geometry rather than the more usual approach of "coordinate conditions."[3] The geometric approach leads to new ideas that are theoretically appealing and useful in calculations. Our presentation of new kinematical conditions involves necessarily also a brief unified review of the initial-value and evolution equations that shows where the kinematical conditions fit into the program.

The generality of the methods is such that no special symmetries need be imposed, in principle, on the spacetime. (However, only axisymmetric cases have been calculated in detail so far.[1]) Moreover, we work with the exact theory and do not employ any approximation schemes. This means one can construct general solutions of the Einstein equations that possess good spacelike initial data. Cosmological problems can also be treated by these techniques, but here we shall mainly focus attention on asymptotically flat spacetimes. In particular, any solution of interest in astrophysics, such as collapsing stellar cores, formation of black holes, colliding black holes or stars, etc., can be generated by this algorithm. This could be of great importance, for example, in planning gravitational-wave detection experiments.[4] The most effective sources of gravitational waves should be those with strong internal fields and high-speed, large-amplitude internal motions. No known approximation techniques apply to such problems.[5] Hence, one must consider a systematic approach to the full theory that is adapted to numerical calculations.

Because we are considering a solution of the Einstein equations as an evolution of given Cauchy initial data, we regard the gravitational field as the time history of two spatial tensor fields[6] (γ_{ab}, K_{ab}) that represent the intrinsic geometry and extrinsic curvature of a spacelike hypersurface. One point of view[7] regards the evolution as involving the choice of time slices, one after another, which push forward into the future. Another point of view that is equivalent to the first imagines a family of Eulerian observers,[8] each with a four-velocity $\hat{n}^a$ orthogonal to the time slices. These observers move into the future and record the relative proper distance configuration (γ_{ab}) and the relative shearing and expansion in local proper time (K_{ab}) of the neighboring observers.

An essential difficulty in general relativity is that there is no *a priori* preferred family of time slices for an unknown gravitational field. The spacetime metric in general admits no time-translation-invariant hypersurfaces as it does in Minkowski spacetime. Even in the "time-independent" Schwarzschild and Kerr spacetimes, of course, the time-translation invariance is not global in the sense that the symmetry vector is not everywhere timelike. Therefore, one must necessarily construct a system of reference in order to be able to calculate the gravitational field. Since there are no preferred observers, one must simply choose some set of world lines orthogonal to a family of spacelike hypersurfaces and define the Eulerian observers in terms of these.[8] Obviously, any other choice of Eulerian observers or time slices is just as good in principle. Therefore, one has to sort our kinematics and dynamics as much as possible even though they are intertwined in the time history of (γ_{ab}, K_{ab}).

Time slicing is the fundamental kinematical choice because it determines which sequence of data (γ_{ab}, K_{ab}) represents spacetime. We want the slicing to cover enough of the full maximally extended spacetime for the relevant physics to be imprinted in (γ_{ab}, K_{ab}). The slicing must stay nonsingular throughout the calculation, even though a portion of a slice may be inside a black hole while another part is in nearly flat spacetime. This requirement severely restricts the allowed slicing. Given such a slicing, one has the history of the gravitational field in terms of the related Eulerian observers.

However, the Eulerian observers may not represent the optimal choice for sorting out kinematics and dynamics. Given a sequence of time slices and arbitrary three-dimensional coordinates on the initial slice, one can imagine a non-normal congruence of curves threading the slices (nowhere tangent to any slice) with each such curve acquiring its three "labels" in the initial slice.[9] Now we have in addition to the Eulerian observers a set of "coordinate observers" for the same family of time slices. The relation of the two is specified at any point on a slice by a velocity shift. For instance, one could use such a shift to make the spatial coordinates constant along the world lines of a fluid source, in which case the new observers would be of the "Lagrangian"[8] type. Alternatively, one might wish to adopt a set of trajectories that minimizes certain coordinate effects in the representation of (γ_{ab}, K_{ab}). In any event, the idea is to determine velocities rather than coordinates to determine the geometrical properties of the observer congruence. This leaves the maximum amount of useful coordinate covariance at one's disposal.

Typically we are led to four elliptic equations to be solved on each slice to yield the instantaneous kinematics. The solution tells how far to advance along the normals of the present slice before stopping to construct the next slice and how far to move parallel to the slice if a velocity shift is required. These elliptic equations are three-dimensionally covariant and contain the curvature of the three-space. This is a novel and important feature, as most previous studies of kinematics (in terms of coordinate conditions)[3] have been motivated by the behavior of weak fields and have not, therefore, involved curvature. By way of contrast, our conditions are meant precisely for use in highly curved dynamic spacetimes. Both the elliptic character of the equations and the presence of curvature suggest a kind of "feedback" effect that, in a sense, allows the kinematics to be adjusted by the dynamical state of the field at a given moment.

In Sec. II we review the basic notions of geometry and kinematics that motivate our study. Time slicings are the subject of Sec. III, which contains descriptions of geodesic, maximal, and hyperboloid slicings as particular cases. We present new results on the behavior of maximal slices in regions of strong curvature. This is followed by a description of various shift-vector conditions in Sec. IV. Included here are the minimal-distortion and related shift vector criteria.

The final section contains a number of examples of our kinematical conditions in familiar spacetimes. We first show the intimate relation of Killing vector fields (if they exist) to the maximal-minimal gauge. An examination of the stationary rest frame allows us to generalize this notion to arbitrary dynamical asymptotically flat spacetimes. We show how several well-known black-hole coordinate systems can be geometrically unified. Turning to homogeneous cosmologies we see that the use of a minimal-distortion shift vector sorts out dynamical degrees of freedom from coordinate modes.

II. KINEMATICS AND DYNAMICS

We shall regard solving Einstein's equations as constructing the evolution of a Cauchy initial-data set. The algorithm that guides our work breaks into four parts: (1) We find Cauchy data that satisfy the constraint equations and represent the initial state of the physical system we wish to study. (2) We erect the observers' velocity and acceleration four-vectors. (3) We evolve the data along the observers' trajectories to the next time slice. (4) We sort out the gravitational physics from its coordinate representation. In this section, we briefly outline these steps and set up the notation we shall need.

A. Initial data

Separating the four initial-value equations from the ten Einstein equations

$$G_{ab} = T_{ab} \tag{2.1}$$

is a process that is well known.[10] We recall here only the salient points. Spacetime, which is characterized by a metric g_{ab} with signature $(-+++)$, is sliced into a family of spacelike hypersurfaces $\mathcal{T}$. These slices are labeled by a monotonically increasing function τ, such that $\tau = $ constant on each slice. The congruence of timelike curves that meets these slices orthogonally has a unit timelike tangent vector field $\hat{n}^a$, with

$$g_{ab}\hat{n}^a\hat{n}^b = -1 . \tag{2.2}$$

The vector field $\hat{n}^a$ represents the four-velocities of the Eulerian observers. On each slice $\mathcal{T}$, these observers are momentarily at rest. They measure their local proper distances in $\mathcal{T}$ by the three-metric γ_{ab} induced by g_{ab} on $\mathcal{T}$ and given by the "projection operator"[11]

$$\gamma_{ab} = g_{ab} + \hat{n}_a\hat{n}_b , \quad \perp^a_b = \delta^a_b + \hat{n}^a\hat{n}_b . \tag{2.3}$$

The description of the embedding of $\mathcal{T}$ in spacetime requires also the extrinsic curvature (second fundamental tensor) K_{ab}, which is conveniently defined by[9]

$$K_{ab} = -\tfrac{1}{2}\mathcal{L}_{\hat{n}}\gamma_{ab} , \tag{2.4}$$

where $\mathcal{L}_{\hat{n}}$ is the Lie derivative along $\hat{n}^a$. The tensors γ_{ab} and K_{ab} are called "spatial" tensors because they depend on the choice of slicing $\mathcal{T}$ and satisfy

$$\hat{n}^a\gamma_{ab} = \hat{n}^aK_{ab} = 0 . \tag{2.5}$$

The pair (γ_{ab}, K_{ab}) form the gravitational part of the complete Cauchy data for a solution of Einstein's equations.[10] As such, they must satisfy the initial-value equations on $\mathcal{T}$:

$$D_b(K^{ab} - \gamma^{ab}\,\mathrm{tr}K) = j^a , \tag{2.6}$$

$$\mathcal{R} - K_{ab}K^{ab} + (\mathrm{tr}K)^2 = 2\rho . \tag{2.7}$$

Here D_a is the covariant derivative operator induced on $\mathcal{T}(D_a\gamma_{bc} = 0)$, $\mathrm{tr}K = \gamma^{ab}K_{ab} = g^{ab}K_{ab}$, and $\mathcal{R}$ is the scalar curvature of γ_{ab}. These four equations are the four Einstein equations

$$\hat{n}^c\gamma^{ab}G_{bc} = \hat{n}^c\gamma^{ab}T_{bc} , \tag{2.8}$$

$$\hat{n}^c\hat{n}^dG_{cd} = \hat{n}^c\hat{n}^dT_{cd} . \tag{2.9}$$

The external sources of the gravitational field are characterized by a stress-energy tensor T_{ab}. It produces a stress density S_{ab}, a momentum density j^a, and an energy density ρ given by

$$S_{ab} = \gamma^c_a\gamma^d_bT_{cd} , \tag{2.10}$$

$$j^a = -\gamma^{ab}T_{bc}\hat{n}^c , \tag{2.11}$$

$$\rho = T_{cd}\hat{n}^c\hat{n}^d , \tag{2.12}$$

as determined by the Eulerian observers of $\mathcal{T}$.

A constructive algorithm exists for obtaining representative initial data for a gravitating system. From physical or mathematical considerations, one makes a first guess at "bare" initial data (γ'_{ab}, K'_{ab}) and $(\rho', j^{a\prime})$. Then the initial-value equations are turned into a set of four quasilinear elliptic equations determining four constrained pieces or "potentials" (ϕ, W^a). These equations are solved, either analytically or numerically, and the bare data are then dressed by (ϕ, W^a) so that the resulting data (γ_{ab}, K_{ab}) and (ρ, j^a) satisfy (2.6) and (2.7). For details, see Ó Murchadha and York.[12]

B. Eulerian and coordinate observers

The remaining Einstein equations deal with the evolution of (γ_{ab}, K_{ab}) away from $\mathcal{T}$ along a vector field t^a. This vector field is tangent to the congruence of world lines of "coordinate observers," so called because their world lines are assumed to be permanently labeled by the spatial coordinate values x^i $(i = 1, 2, 3)$ that they acquire in the initial slice. In general t^a is not coincident with $\hat{n}^a$. Because $\hat{n}^a$ is a unit vector orthogonal to the slices $\mathcal{T}$, there is a scalar function[13] $(-\alpha)$ relating $\hat{n}_a$ and $\nabla_a\tau$, that is,

$$\hat{n}_a = -\alpha\nabla_a\tau . \tag{2.13}$$

The role of the scalar "lapse function" α is to specify the orthogonal proper-time interval $\alpha\delta\tau$

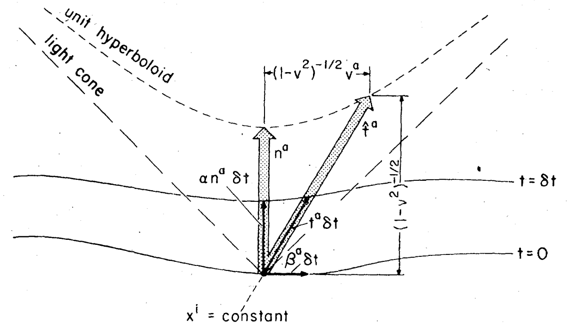

FIG. 1. The local kinematics of the gravitational field are described by the lapse function α and the shift vector β^a. Given a slice $\tau = 0$, we can erect unit normal vectors $\hat{n}^a$ at each point on the slice. To define the slice labeled by $\tau = \delta\tau$, we advance a proper distance $\alpha\delta\tau$ along each $\hat{n}^a$. To reach the point in the slice $\tau = \delta\tau$, which we wish to have the same spatial coordinates x^i $(i = 1, 2, 3)$ as the base point of $\hat{n}^a$, we move along a vector $t^a = \alpha\hat{n}^a + \beta^a$. This is equivalent to a velocity boost between $\hat{n}^a$ and $\hat{t}^a$ with three-velocity $v^a = \alpha^{-1}\beta^a$.

between the slices $\mathcal{T}(\tau)$ and $\mathcal{T}(\tau + \delta\tau)$ (see Fig. 1). The kinematic freedom available in α allows a portion of a slice in a strong-field region to advance at a different rate from another portion in a weaker-field region, a feature that turns out to be of crucial importance in applications.

The Eulerian observers by definition are at rest in $\mathcal{T}$ and have no (spatial) rotation,[14]

$$\hat{n}_{[a}\nabla_b \hat{n}_{c]} = 0\,, \qquad (2.14)$$

because $\hat{n}^a$ is hypersurface-orthogonal [cf. (2.13)]. However, these observers in general have a nonvanishing acceleration vector given by

$$a^a = \gamma^{ab} D_b(\ln\alpha), \quad a^a\hat{n}_a = 0\,, \qquad (2.15)$$

which shows that the lapse function is the "acceleration potential" for Eulerian observers. These observers in general will need to accelerate in order to counterbalance the tendency of gravity to focus timelike trajectories and cause a "collapse" of the slices $\mathcal{T}$ ("coordinate singularity").

The above relations characterize the "time" τ in terms of a family of spacelike hypersurfaces .[15] The complementary or dual aspect of time is described by the four-vector $t^a = (\partial/\partial\tau)^a$ along which the data on $\mathcal{T}$ are to be evolved. The use of t^a (coordinate observers) corresponds in hydrodynamics to the use of mixed Euler-Lagrange trajectories because in general t^a lies along neither the Eulerian trajectories nor the matter trajectories. In this case, besides α, we need the "shift vector" β^a (Ref. 13) to relate the two (see Fig. 1):[9]

$$t^a = \alpha\hat{n}^a + \beta^a, \quad \beta^a\hat{n}_a = 0\,. \qquad (2.16)$$

The basic requirement on t^a is that it be a tangent field of a congruence of curves that "threads" the slices τ = constant, in the sense that t^a should be nowhere tangent to $\mathcal{T}$. Hence we require that $\alpha > 0$. On the other hand, it is not necessary to require that $\alpha^2 > \beta^2$ $(\beta^2 = \gamma_{ab}\beta^a\beta^b = g_{ab}\beta^a\beta^b)$; that is, t^a need not necessarily be timelike. One here recognizes the important fact that the relation of Eulerian and coordinate observers is specified in addition to α by a relative "velocity" β^a that has nothing whatever to do with spatial coordinate conditions *per se*. Therefore, the points in an initial $\mathcal{T}$ can be labeled by any convenient coordinate system while the kinematics of evolution involves only the relationships among families of observer trajectories (spacetime geometry).

Before proceeding with details of the kinematics when there is a nonzero shift vector, one may observe that the evolution can always be carried out along the nonunit normal vector $T^a = \alpha\hat{n}^a$ that is always timelike for $\alpha > 0$. If t^a and T^a are smooth vector fields and $\beta^a \neq 0$, then these two vectors merely generate different diffeomorphisms between fixed slices $\mathcal{T}(\tau)$ and $\mathcal{T}(\tau + \delta\tau)$. The apparently more complicated choice of t^a rather than T^a is often preferable because, as we discuss in Sec. IV, an appropriate choice of $\beta^a \neq 0$ can simplify the description of the spacetimes calculated in a Cauchy evolution.

For purposes of illustration, we assume that t^a is timelike, as in Fig. 1. Because t^a has a "velocity" or shift β^a relative to the Eulerian observers, we may imagine the transformation from $\hat{n}^a$ to t^a "frames" as occurring in two steps that demonstrate how the kinematical choices inherent in special relativity (β^a or the v^a defined below) and general relativity (α = lapse function = acceleration potential = "gravitational red-shift factor") are both being used. First, we boost the four-velocity $\hat{n}^a$ to the four-velocity $\hat{t}^a$ $(g_{ab}\hat{t}^a\hat{t}^b = -1)$ of observers whose world lines coincide with those of t^a. However, note that $\hat{t}^a$'s clocks do not keep the same time τ as t^a's clocks (see below). Also, we do not bother to boost the arbitrary three-basis tangent to $\mathcal{T}$ because it is already the one we want to use. The boost of $\hat{n}^a$ to $\hat{t}^a$ is defined by the physical three-velocity

$$v^a = \alpha^{-1}\beta^a \qquad (2.17)$$

of $\hat{t}^a$ relative to $\hat{n}^a$. From (2.16) we find

$$\hat{t}^a = (\alpha^2 - \beta^2)^{-1/2} t^a = \alpha^{-1}(1 - v^2)^{-1/2}(\alpha\hat{n}^a + \alpha v^a)$$
$$= \Gamma(\hat{n}^a + v^a), \quad \Gamma = (1 - v^2)^{-1/2}, \qquad (2.18)$$

which is the usual special-relativistic result. The second step recognizes that the boosted observers

$[\hat{t}^a = (\partial/\partial\tau_p)^a]$ have clocks that keep $\hat{t}^a$'s local proper time τ_p, not the time τ. The relative rate of τ clocks and τ_p clocks is given by

$$t^a = \alpha\Gamma^{-1}\hat{t}^a \quad \text{or} \quad \frac{\partial}{\partial\tau} = \alpha\Gamma^{-1}\frac{\partial}{\partial\tau_p}, \qquad (2.19)$$

which follows from (2.18). This step shows how the lapse function α is used to adjust clock rates in an appropriate way. Clearly, the most important aspect of the adjustment process is the manner in which α varies from point to point [the relative "tilting" of $\mathcal{T}(\tau)$ and $\mathcal{T}(\tau+\delta\tau)$ in Kuchař's[16] terminology]. This variation of α on $\mathcal{T}$ is equivalent to the acceleration of the Eulerian observers (2.15). Finally, we note that in general the $\hat{t}^a$ observers will possess "local spatial rotation,"

$$\hat{t}_{[a}\nabla_b\hat{t}_{c]} \neq 0\,, \qquad (2.20)$$

and therefore these observers cannot define a time slicing of spacetime.

In summary, the Eulerian and coordinate observers are related by a scalar and a three-vector (α, β^a) that define the kinematics of the evolution. It is especially important that the specification of initial data on a slice proceeds entirely independent of the kinematics of the trajectories that carry the data away from the slice.

C. Evolution equations

The dynamical Einstein equations are concerned with the evolution of the spatial metric γ_{ab}. They can be written in either second-order form[17] for $\pounds_t\pounds_t\gamma_{ab}$ or in first-order form[18] for $\pounds_t\gamma_{ab}$ and $\pounds_t K_{ab}$. The latter are preferable for our purposes for two reasons. Firstly, the initial-value equations are very much more difficult to treat if we try to use $(\gamma_{ab}, \pounds_t\gamma_{ab})$ as initial data[19] instead of (γ_{ab}, K_{ab}). Moreover, the preferred latter form of the constraints [Eqs. (2.6) and (2.7)] contains no reference to α and β^a. Secondly, the dynamical equations $\pounds_t\pounds_t\gamma_{ab}$ contain both (α, β^a) as well as $(\pounds_t\alpha, \pounds_t\beta^a)$,[17] in contrast to the equations for $\pounds_t\gamma_{ab}$ and $\pounds_t K_{ab}$ that contain in essence only (α, β^a) and their derivatives within a given slice. Thus, the behavior of α and β^a away from a given slice is irrelevant in the first-order formalism. This means that the kinematical conditions that are imposed to fix α and β^a need only depend on the known instantaneous state of the field at a fixed time.

In the presence of external sources, the first-order equations of motion are

$$\pounds_t\gamma_{ab} = -2\alpha K_{ab} + \pounds_\beta\gamma_{ab}\,, \qquad (2.21)$$

$$\pounds_t K_{ab} = -D_a D_b\alpha + \alpha[\mathcal{R}_{ab} + (\mathrm{tr}K)K_{ab} - 2K_{ac}K^c_b - (S_{ab} - \tfrac{1}{2}\gamma_{ab}\,\mathrm{tr}S) - \tfrac{1}{2}\rho\gamma_{ab}] + \pounds_\beta K_{ab}\,, \qquad (2.22)$$

where $\mathcal{R}_{ab}$ is the Ricci curvature tensor of γ_{ab}. The principal kinematical terms in (2.21) and (2.22) are

$$D_a D_b\alpha = \alpha(D_a a_b + a_a a_b) \qquad (2.23)$$

and

$$\pounds_\beta\gamma_{ab} = \nabla_a\beta_b + \nabla_b\beta_a + 2\hat{n}_a\hat{n}_b\beta^c a_c\,, \qquad (2.24)$$

$$\pounds_\beta K_{ab} = \beta^c\nabla_c K_{ab} + K_{ac}\nabla_b\beta^c + K_{cb}\nabla_a\beta^c\,. \qquad (2.25)$$

Equation (2.21) is actually an identity following from (2.4) and (2.16).

Two equations that will prove useful in the sequel are obtained from the traces of (2.21) and (2.22):

$$\pounds_t[\ln(\det\gamma)^{1/2}] = -\alpha\,\mathrm{tr}K + D_a\beta^a\,, \qquad (2.26)$$

$$\pounds_t(\mathrm{tr}K) = -\Delta\alpha + \alpha[K_{ab}K^{ab} + \tfrac{1}{2}(\rho + \mathrm{tr}S)] + \pounds_\beta\mathrm{tr}K\,. \qquad (2.27)$$

Here $\det\gamma$ is defined[20] by $\det\gamma = \alpha^{-2}\det(-g_{ab})$ and $\Delta = \gamma^{ab}D_a D_b$ is the three-dimensional covariant scalar Laplacian operator. Observe that (2.26) is a geometrical identity while (2.27) is a consequence of Einstein's equations.

Assuming that the constraints (2.6) and (2.7) are satisfied, on the right-hand side of (2.22) we could replace ρ using (2.7) to obtain $\pounds_t K_{ab}$ in terms of $(\gamma_{ab}, K_{ab}, S_{ab}, \alpha, \beta^a)$. We see from this step that to evolve a set of initial data that is compatible with the constraints we need a specification of α and β^a besides the gravitational data (γ_{ab}, K_{ab}) and the initial stress tensor S_{ab}. The stress tensor S_{ab} will be known in terms of the same source initial data that was required in constructing ρ and j^a. Because of the Einstein equations (2.1) and the Bianchi identity, we have $\nabla_b T^{ab} = 0$. Thus, in principle we do not need to use separate evolution equations for ρ and j^a. The equation $\nabla_b T^{ab} = 0$ can easily be rewritten to give $\pounds_t\rho$ and $\pounds_t j^a$ in terms of $(\gamma_{ab}, K_{ab}, \rho, j^a, \alpha, \beta^a)$ and their spatial derivatives. The evolution of S_{ab} would be obtained from any remaining independent equations of motion of the source or its equation of state. However, in practice there are a number of other ways to evolve matter fields. We refer the reader to the work of Wilson[21] for how this is done.

D. Gravitational degrees of freedom

An unavoidable consequence of the kinematical freedom present in general relativity is that the history $\{\gamma_{ab}(\tau), K_{ab}(\tau)\}$ of the gravitational field has a mixture of time dependence that results partly from the choice of observers and partly from "real" gravitational dynamics.[22] The only situation in which one can definitely separate these two aspects is when there exists in the spacetime an exact Killing vector $\xi^a(\pounds_\xi g_{ab} = \nabla_a \xi_b + \nabla_b \xi_a = 0)$ that is timelike sufficiently far (in spacelike directions) from the central strong-field region. In this case one can arrange that $t^a = \xi^a$ and there is no dynamics. On the other hand, if one selects a $t^a \neq \xi^a$ in such a case, there will appear in $\{\gamma_{ab}, K_{ab}\}$ a "fictitious" time dependence. One of the major goals in studying kinematical constructions by geometric methods is to avoid such problems as far as possible.

Our approach attempts a separation of kinematical and dynamical effects in a three-dimensionally covariant manner on a family of "good" slices (Sec. III). We ask what geometric object constructed from the spatial metrics γ_{ab} of the slices $\mathcal{T}$ can be regarded as containing the dynamical degrees of freedom.[23] The simplest answer that we are aware of is the conformal three-geometry[24] of $\mathcal{T}$, represented by the "conformal metric" $\tilde{\gamma}_{ab} = (\det\gamma)^{-1/3}\gamma_{ab}$. The conformal factor $(\det\gamma)^{1/3}$, by which $\tilde{\gamma}_{ab}$ is multiplied to produce γ_{ab}, contains the overall scale of the metric and, in an asymptotically flat spacetime, contains the information on the total energy of the system.[25] This conformal factor is not part of the freely specifiable data in the initial-value problem, but is found by using one of the constraint equations.[26] [The factor $(\det\gamma)^{1/3}$ is essentially equivalent to ϕ^4, where ϕ is the gravitational scalar potential referred to in Sec. II A.]

One expects the conformal spatial metric to convey information on the anisotropy of the gravitational field. In asymptotically flat spacetimes, such anisotropy may be regarded as representing gravitational radiation.[27] (In weak fields the conformal three-geometry is determined by h_{ab}^{TT}, the usual transverse and traceless wave variables.)[28] In compact three-geometries, the anisotropy is referred to as gravitational degrees of freedom.[27] (For example, in the "mixmaster universe,"[29] the anisotropy variables β_+ and β_- determine the conformal three-geometry of the "Ω-time" slices.) In both cases, the underlying dynamical object is the conformal three-geometry induced on a family of slices. Hence, in separating kinematics and dynamics, we focus attention on the time rate of change of the conformal three-geometry, which is represented by $\pounds_t\tilde{\gamma}_{ab}$. Clearly $\pounds_t\tilde{\gamma}_{ab}$ will change if we use different t^a observers *even if the slicing is fixed.* This means that our choice of shift vectors β^a is the handle by which we can reduce kinematical effects in the evolution of Cauchy data on a given family of slices. This process is treated in Sec. IV.

III. TIME SLICES

In dynamic spacetimes there are no timelike symmetries to suggest a preferred choice of slices or the lapse function α. We know only that if $\alpha \to 1$ and $\beta^a \to 0$ sufficiently rapidly at great distances from the strong-field region of an asymptotically flat spacetime, then t^a reduces to a constant time translation at spatial infinity. Here we have $t^a \to \hat{n}^a$, which is appropriate in the case that the curved spacetime Eulerian observers are to be identified asymptotically with the standard observers of Minkowski spacetime. In spacetimes with compact time slices, the above reasoning does not apply. Here the time slicing must be constructed wholly from the spacetime dynamics itself without appealing to an external standard reference system such as "observers at infinity." We will now examine the properties of several natural time-slicing conditions.

A. Geodesic slicing

The simplest procedure would be to start with a given slice and to set $\alpha = 1$ everywhere. This implies that the Eulerian observers are freely falling [Eq. (2.15) yields $a_a = 0$] and that the slices are geodesically parallel. It is now well known that this condition fails in general to give a family of slices that adequately cover the spacetime regions of interest.[30] What happens is that the Eulerian geodesics tend to focus to a caustic, owing basically to the attractive nature of the gravitational interaction. This can be seen from the Einstein equation (2.27)

$$\pounds_{\hat{n}}(\mathrm{tr}K) = K_{ab}K^{ab} + \tfrac{1}{2}(\rho + \mathrm{tr}S) \tag{3.1}$$

if $\alpha = 1$ and $\beta^a = 0$. The first term on the right-hand side is always non-negative and so is the second if the strong energy condition[31] is satisfied. The convergence $\mathrm{tr}K = -\nabla_a\hat{n}^a = -\pounds_{\hat{n}}\ln(\det\gamma)^{1/2}$ of the Eulerian geodesics thus tends to increase without limit, resulting in a coordinate singularity ($\det\gamma \to 0$). This happens on a free-fall time scale ($\tau \sim M$ for black holes).

As a specific example, let us briefly consider the evolution of the time-symmetric slice $\tau_{\mathrm{Sch}} = 0 = v_{\mathrm{Kruskal}}$ of the extended Kruskal-Schwarzschild[32] spacetime. This is the prototype of all black-hole spacetimes because it contains an Einstein-Rosen bridge and an event horizon. At late times any nonrotating uncharged collapse forming a black hole should settle down to a Schwarzschild black hole. The choice of time slices to the future of the initial slice is completely determined by giving α. If we choose $\alpha = 1$, then the region of the spacetime covered by these slices is shown in Fig. 2(a). Since the free-fall time to the singularity is πM from the center point of the $\tau = 0$ slice, the slices will asymptotically go only to $\tau_{\mathrm{last}} = \pi M$. Note that virtually none of the spacetime exterior to the

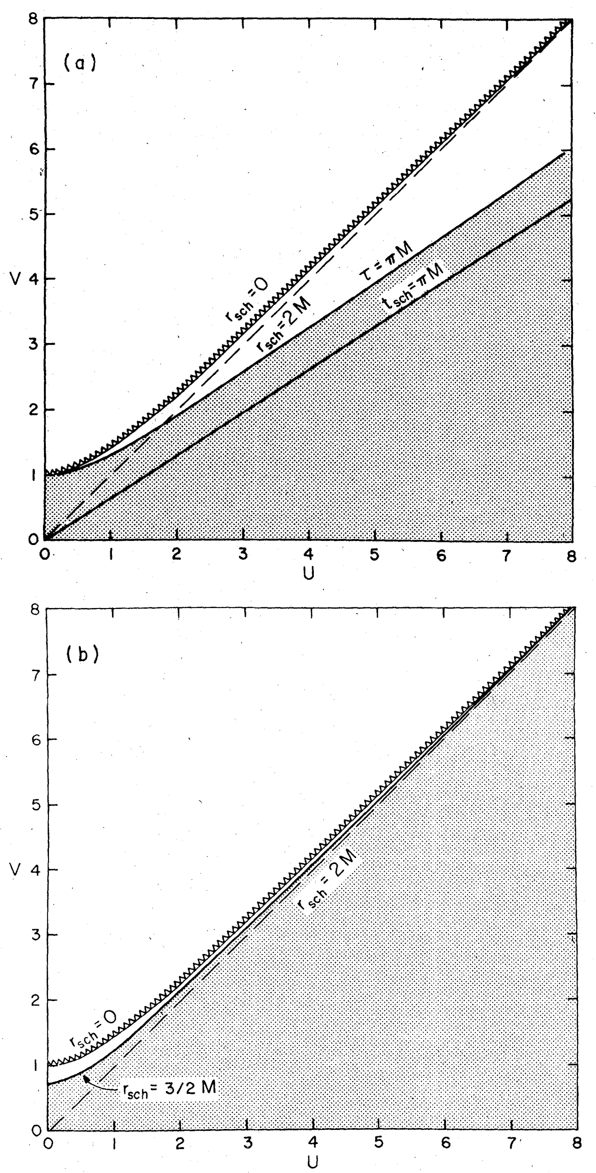

FIG. 2. (a) The shaded region represents the development of the initial slice $v = 0$ in Schwarzschild-Kruskal spacetime by geodesic slices ($\alpha = 1$). Note that the slice $\tau = \pi M$ hits the singularity (Ref. 6) and covers very little of the exterior of the black hole, i.e., it is asymptotic to $t_{\rm Sch} = \pi M$. This is why one needs a more sophisticated choice of α. (b) Here the development of the same initial slice is by maximal slices [Eqs. (3.4)]. The slicing remains nonsingular and the entire exterior of the black hole is covered. The "limit surface" which the maximal slices wrap up around (Ref. 37) is $r_{\rm Sch} = \frac{3}{2}M$.

black hole is covered. This indicates that if we had perturbed the metric and radiation had been generated near the horizon, it would not have had time to propagate to the wave zone ($r \gtrsim 20M$) before the geodesic slices became singular in the central region. (For more details see Sec. V.)

Borrowing the terminology of Choquet-Bruhat and Geroch,[33] we call any such partial foliation of a spacetime a *development*. Each slicing condition on α will yield a development of the initial-data set. Our goal is to construct a development which will contain as much of the exterior of the black-hole spacetime as possible.

B. Maximal slicing

The unsuitable features of geodesic slicing were recognized long ago by Lichnerowicz,[34] who suggested a possible remedy. He pointed out that α can be chosen in such a way that tr$K$ remains zero if it vanishes on an initial slice. As a result, the Eulerian observers have zero convergence and their local volume element $(\det\gamma)^{1/2}$ is time independent [Eq. (2.26)]. Since Eulerian observers are by definition irrotational (hypersurface orthogonal), we may think of them as forming an incompressible, irrotational, shearing "test" fluid in the spacetime. The fluid remains incompressible because the Eulerian observers are accelerating ($D_a\alpha \neq 0$) to balance the focusing effect of gravity. From the dual hypersurface point of view, the demand tr$K = 0$ implies that the volume of any region Ω of the hypersurface $\mathcal{T}$ is maximized[35] relative to any other spacelike hypersurface which coincides with $\mathcal{T}$ outside of Ω.

If we define time slices by tr$K = \mathcal{L}_t(\mathrm{tr}K) = 0$, then from (2.27) we see that the required equation[34] for α on each time slice is

$$\Delta\alpha - [K_{ab}K^{ab} + \tfrac{1}{2}(\rho + \mathrm{tr}S)]\alpha = 0, \qquad (3.2)$$

a linear elliptic equation which must be supplemented by appropriate boundary conditions.[36] If the constraint equation (2.7) is satisfied then (3.2) can be rewritten as

$$\Delta\alpha - [\mathcal{R} - \tfrac{3}{2}(\rho - \tfrac{1}{3}\mathrm{tr}S)]\alpha = 0. \qquad (3.3)$$

Because the coefficient of α is non-negative in (3.2), by the same argument as in (3.1), the same must be true of the coefficient of α in (3.3). Let us therefore confine our attention to the vacuum case ($\rho = \mathrm{tr}S = 0$) where the equation becomes

$$\Delta\alpha - \mathcal{R}\alpha = 0, \quad \mathcal{R} \geq 0. \qquad (3.4)$$

Returning to our example of Schwarzschild-Kruskal spacetime, let us ask what the development of the $\tau = 0$ initial data is if α satisfies (3.4) on each slice. If $\alpha = 0$ at the throat ($r_{\rm Sch} = 2M$ at $\tau = 0$), then the answer is the usual Schwarzschild time coordinate slicing[32] which covers the entire exterior of the black hole (up to $\tau = +\infty$), but none of the interior. Because α is antisymmetric

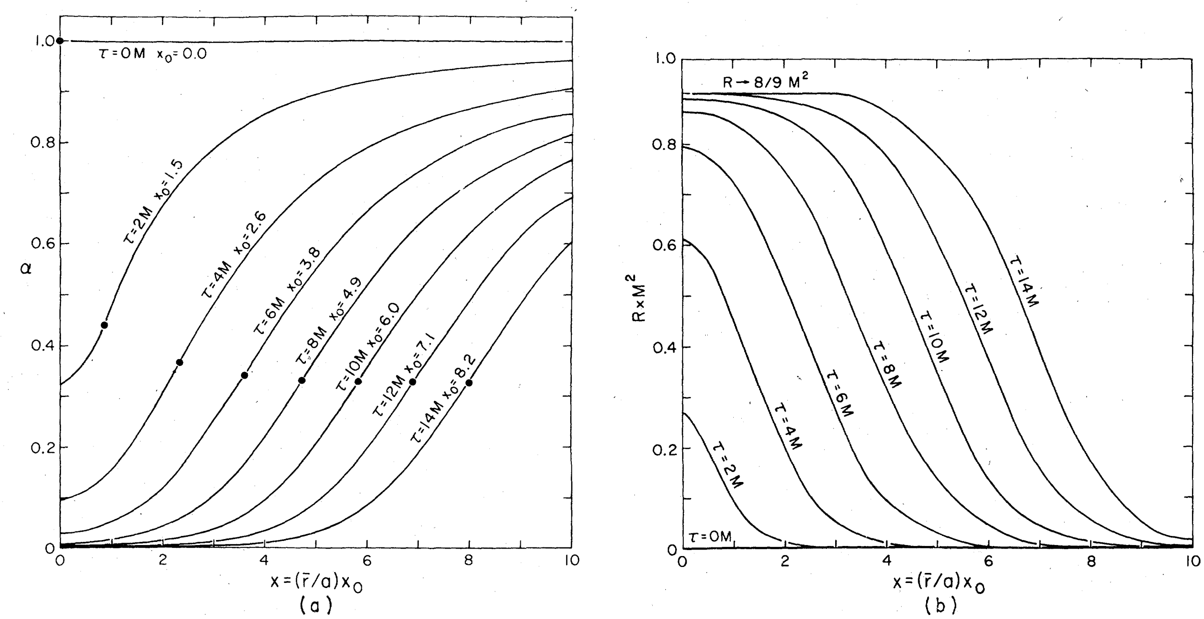

FIG. 3. (a) The spherically symmetric lapse function α is plotted versus the dimensionless radius $x \equiv (\bar{r}/a)x_0$ where $\bar{r}$ is the proper radial distance from the throat and x_0 and a are as defined by Eq. (3.7). These plots of $\alpha(x)$ are given for a series of time slices of the symmetric maximal slices of Schwarzschild-Kruskal spacetime studied by Estabrook *et al.* (Ref. 37). Notice the rapid collapse of the lapse near the throat at late times ($\tau \gtrsim 10M$). Time slices are labeled by the strength parameter x_0 and by proper time at a large finite distance (where we set $\alpha = 1$). The curves rise more rapidly than in Fig. 4 because this distance is not infinite. The location of the event horizon $r_{\rm Sch} = 2M$ is denoted by a dot on each slice (see Eppley, Ref. 1). (b) For the same time slices as in (a), we plot the Ricci scalar $\mathcal{R}(x)$ of the three-metric of the slice. At $\tau = 0$, $\mathcal{R} = 0$ from the constraint equations. It grows in the strong-field region ($x \cong 0$) as time increases. This is what forces the lapses to zero in (a). At late times the central value of $\mathcal{R}$ goes to $8/9M^2$, the value of $\mathcal{R}$ for the hypercylinder $r_{\rm Sch} = 3M/2$. The "effective radius" a [Eq. (3.7)] is found to grow linearly in time and be approximately the proper radial distance from the throat to the horizon in the time slice.

across the throat, time advances "downward" on the left side of the Kruskal diagram. The case where α is symmetric across the throat ($\alpha = 1$ at $\tau = 0$) was investigated by Estabrook *et al.* (EWCDST)[37] and Reinhart[38] in 1973. This development is shown in Fig. 2(b). Note that not only does it contain the entire future of the exterior of the black hole, it also contains a large portion of the interior spacetime.

Now in order for the proper time $\int_0^\infty \alpha\, d\tau \to \infty$ along an observer's world line far from the black hold and yet have $\int_0^\infty \alpha\, d\tau < \pi M$ in the central region, the lapse function α must go to zero very rapidly in this strong-field region while $\alpha \to 1$ at spatial infinity. Figures 3(a) and 3(b) show the results of the EWCDST investigation as later refined by Smarr and Eppley.[39] We see that α drops to zero as the three-scalar curvature $\mathcal{R}$ rises. This behavior was *qualitatively* explained by EWCDST using Eq. (3.4) and the Hopf maximum-minimum principle for elliptic equations. Since this early research, more spacetimes have been constructed[40] (colliding black holes, collapsing stars, and strong imploding gravity waves) with the same qualitative results as shown in Figs. 2(b) and 3(a). This behavior of the maximal time slicing has been termed "singularity avoidance."[41]

Despite the facts of the above discussion, "avoidance of spacelike singularities" is *not* a correct characterization of the properties of maximal foliations. In the Reisner-Norström solution, α goes to zero inside the outer horizon,[42] but the singularities avoided in this case are timelike. In a self-similar dust collapse studied by Eardley and Smarr,[43] there is a spacelike singularity that is *not* avoided by the maximal slicing [Eq. (3.3)]. Both of these examples are described in Sec. V.

What we do know is that maximal slicing prevents the focusing of the world lines of the Eulerian observers. In order to get an idea of how this is related to the behavior of curvature and matter density on the slices, we now turn to the study of a simple example.

C. Simplified solution of the maximal equation

The solution of (3.4) depends on γ_{ab} both because Δ is the covariant Laplacian and because $\mathcal{R}(\gamma_{ab})$ is the coefficient of α. To try to understand which is the dominant factor, we shall follow Wheeler[44] in

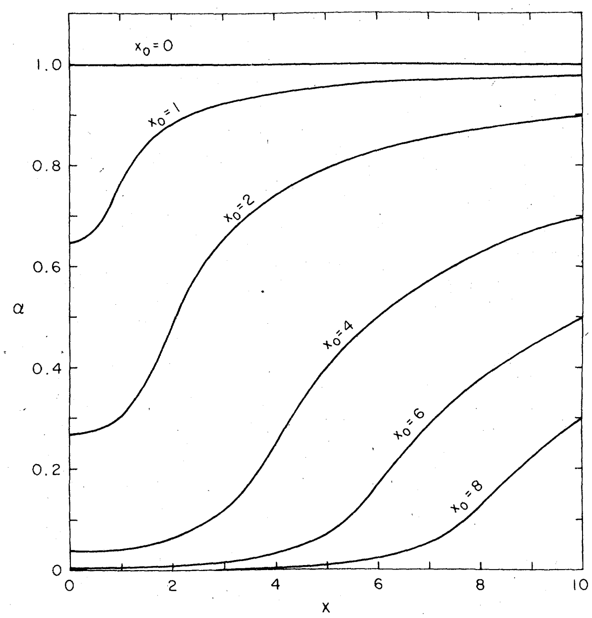

FIG. 4. Plotted here are the analytic solutions [Eq. (3.5)] of our flat-space model equation for maximal slicing. The solutions are labeled by the "strength" x_0 of the curvature. Note the striking similarity between these curves and the lapse function resulting from the maximal slicing of a black hole [Fig. 3(a)].

a method he used on a similar problem. We model (3.4) by assuming that γ_{ab} is a *flat* metric in spherical polar coordinates and that $\mathcal{R}$ is spherically symmetric and constant ($\mathcal{R}_0$) in a region of radius a about the origin and zero outside. Of course, the last step breaks the correct relation between scalar curvature and metric, but a perturbative treatment of the model (adding small curvature terms to the metric) shows that the results are still qualitatively correct. Moreover, as we shall see, the resemblance to the actual maximal slicing of Schwarzschild spacetime turns out to be remarkable in several respects.

The solution of (3.4) in this case is

$$\alpha = (\cosh x_0)^{-1} \frac{\sinh x}{x}, \quad 0 \leq x \leq x_0 \tag{3.5a}$$

$$\alpha = 1 + \frac{\tanh x_0 - x_0}{x}, \quad x \geq x_0 \tag{3.5b}$$

$$x \equiv r\sqrt{\mathcal{R}_0}, \quad x_0 \equiv a\sqrt{\mathcal{R}_0}, \tag{3.5c}$$

where α and $\partial\alpha/\partial r$ are matched at $r=a$. (See Fig. 4.) This is a one-parameter family of solutions. The natural parameter turns out to be x_0, the dimensionless "strength" of the scalar curvature hill. If $x_0 = 0$, corresponding to zero scalar curvature, then $\alpha = 1$ everywhere. As x_0 increases from zero, the lapse function begins to fall in the region where $\mathcal{R}_0 \neq 0$ and rise back to unity as $\alpha \to 1 + O(1/r)$ as $r \to \infty$. The minimum value of α always occurs at the origin with value

$$\left.\begin{aligned} \alpha_{\min} &= (\cosh x_0)^{-1} \\ D_a\alpha &= 0 \end{aligned}\right\} r = 0\,. \tag{3.6}$$

Thus, for large x_0 the minimum value of the lapse goes as $\alpha_{\min} \sim e^{-x_0}$.

The qualitative behavior of α in our "flat space with scalar curvature" model, Fig. 4, is very similar to what happens in the full dynamical curved spacetimes that have been evolved by maximal slicing [e.g. Fig. 3(a)]. This indicates that the solutions of (3.4) or (3.3) in the actual slicings will be determined primarily by the behavior of the "strength" of the scalar curvature on these slices.

There are now two questions that naturally arise: (1) What is a reasonable curved-space generalization of the "strength parameter" $x_0 = a\sqrt{\mathcal{R}_0}$? (2) How does $\alpha_{\min}$ depend on x_0 and on time? Knowing the latter will tell us, for example, how long it takes for $\alpha_{\min}$ to reach zero and halt the evolution in the strong-field region. Will this be less than the time from the initial slice to any singularity?

To generalize the strength parameter $x_0 = a\sqrt{\mathcal{R}_0}$ in the spherically symmetric case, we write down the simple proper line integral

$$x_0 = \int_0^\infty \mathcal{R}^{1/2}\gamma_{rr}{}^{1/2}dr \equiv a[\mathcal{R}(r=0)]^{1/2}. \tag{3.7}$$

Since $\mathcal{R} \geq 0$, $\mathcal{R} = O(r^{-4})$, and $\gamma_{rr} = 1 + O(r^{-1})$, x_0 in (3.7) is always real and finite. [In nonspherical problems, (3.7) would have to be replaced by an average of such integrals over different directions.] In the nonvacuum case, in place of (3.7) we have

$$x_0 = \int_0^\infty [K_{ab}K^{ab} + \tfrac{1}{2}(\rho + \mathrm{tr}S)]^{1/2}\gamma_{rr}{}^{1/2}dr\,. \tag{3.8}$$

Let us return to the numerical results for maximal slicing of the Schwarzschild-Kruskal spacetime.[45] In Fig. 3(a) we have labeled each α graph by x_0 [calculated using (3.7)] as well as by τ. In Fig. 5 we plot $\alpha_{\min}$ versus x_0. Note that at late times $\alpha_{\min} \sim e^{-0.97x_0}$ in almost exact agreement with our flat-space example. This indicates the "collapse of the lapse" is due to general properties of elliptic equations and is not dependent on the details of strong gravitational fields. Because Figs. 3(a) and 3(b) are the result of a Cauchy evolution, x_0 is a function of time. In Fig. 6 we plot $x_0(\tau)$. For reasons we do not yet fully understand, it is almost exactly linear: $x_0 = (\tau/1.77M) + 0.34$. In-

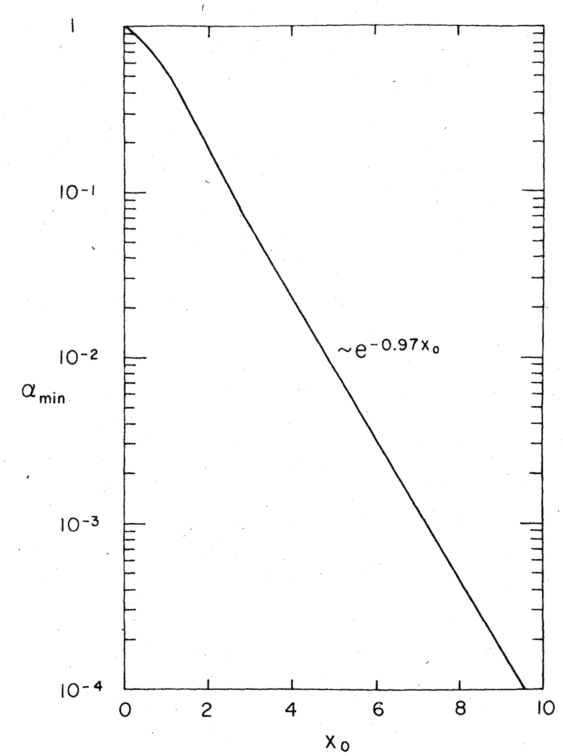

FIG. 5. From Fig. 3(a), we have plot α_{min} versus the strength parameter x_0 defined by 3.7. At late times we find $\alpha_{min} \sim e^{-0.97x_0}$, in almost exact agreement with our flat-space model, Fig. 4.

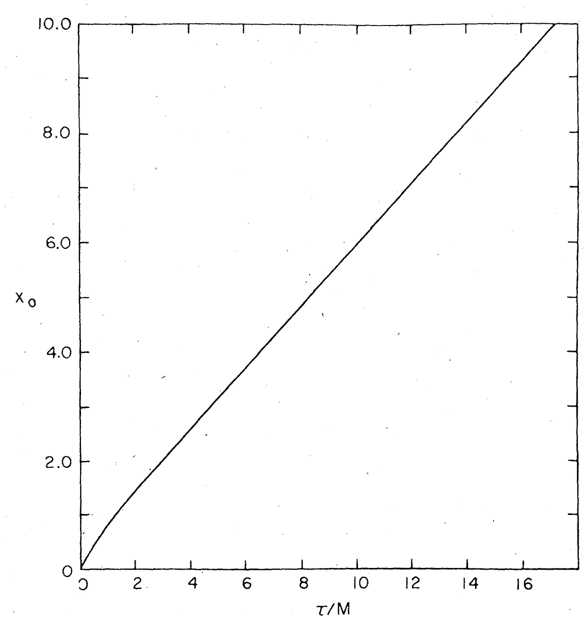

FIG. 6. The value x_0 from Fig. 3(a) is plotted here versus time. It is remarkable that from $\tau = 0$, the growth of x_0 is almost exactly linear with an inverse slope of $\tau_e \simeq 1.8M$. This together with Fig. 5 implies $\alpha_{min} \sim e^{-\tau/\tau_e}$ at late times for maximal slicing of Kruskal-Schwarzschild spacetime.

serting this relation into that found in Fig. 5 for $\alpha_{min}(x_0)$, one finds that at late times

$$\alpha_{min} \sim e^{-\tau/\tau_e}, \quad \tau_e \simeq 1.82M. \tag{3.9}$$

This exponential decay of the lapse induced by maximal slicing is a new result and gives a much clearer insight into the so-called "singularity avoidance."

It suggests that maximal slicing halts the evolution before any singularities are reached *if (3.7) grows rapidly enough*, as opposed to the unweighted curvature alone, for example. This is not too surprising since the maximal equation is elliptic and hence α_{min} will "feel" a global, rather than a local, measure of the strength of the curvature. In certain extreme cases of the spherical collapse of self-similar dust studied by Eardley and Smarr,[43] the generalized parameter x_0 does not grow fast enough for the maximal slices to halt (i.e., for $\alpha \to 0$ in the central region) before reaching a singularity. There, the main support of the integrand of (3.7) has insufficient linear measure (a "spike" in ρ at the origin).

However, in almost all "reasonable" cases studied, including colliding black holes and collapsing stars,[40] the maximal slicing does halt the evolution in the strong-field region before the singularity is reached. Again, as an example, consider the maximal slicing of Schwarzschild-Kruskal spacetime with $D_a\alpha = 0$ on the throat (to be thought of as the analog of $r = 0$ in our model). The throat will free-fall collapse to the singularity in $\tau_{ff} = \pi M$. The e-folding time for maximal slicing is $\tau_e = 1.80M$ and the total elapsed proper time until the maximal slicing halts the collapse at $r_{Sch} = 1.5M$ is $\tau_{max} = \int_0^\infty \alpha_{min} dt$ or

$$\tau_{max} = \left(\frac{\pi}{3} + \frac{\sqrt{3}}{2}\right) \simeq 1.91M, \tag{3.10}$$

which is well short of τ_{ff}. We could have estimated τ_{max} from our model using (3.6) as

$$\begin{aligned} \tau_{max} &= \int_0^\infty (\cosh x_0)^{-1} dt \\ &= \int_0^\infty [\cosh(t/\tau_e)]^{-1} dt \\ &= \frac{\pi}{2}\tau_e \simeq 2.87M < \tau_{ff}. \end{aligned} \tag{3.11}$$

In this case the model predicts that the last maximal slice occurs before the singularity is reached, just as in the actual calculation.

The last conclusion can be seen to be essentially a consequence of the maximum-minimum principle for the elliptic equation (3.4). At the interior positive minimum $\alpha = \alpha_{\min}$, we have of course that $D_a\alpha_{\min} = 0$. Therefore, the observer at this place has acceleration $a_a = D_a \ln\alpha = 0$ and is in free fall. However, his immediate neighbors are accelerated outward to prevent convergence of the world lines near him. Hence, $\int_0^\infty \alpha_{\min} dt < \tau_{ff}$.

D. Hyperboloid slicing

Up to now we have assumed that our time slices would be like the planes of Minkowski spacetime as we move far enough from strong-field regions. However, there is another set of potentially useful slicings that we shall mention. These are the hyperboloid slicings with $\text{tr}K = K_0 = \text{constant}$.[46] In Minkowski spacetime the $\text{tr}K = K_0$ slices are just the "mass" hyperboloids which are described in Sec. V. These slices are asymptotically null and approach null infinity $\mathscr{I}^\pm$ rather than spacelike infinity. This suggests that they might be well adapted to studying gravitational radiation from a bounded region.

Such slices have been shown to exist in several asymptotically flat spacetimes. Brill[46] studied their properties in Schwarzschild-Kruskal spacetime and Eardley and Smarr[43] showed that they could be also used for star collapse with a Schwarzschild exterior. Whether strictly $\text{tr}K = \text{constant}$ slices exist in radiating spacetimes is apparently not certain. Goddard[46] has conjectured that the difficulty in finding "good cuts of $\mathscr{I}^+$" may doom the existence of such slices in general for asymptotically flat spacetime.

No such problems exist for $\text{tr}K = \text{constant}$ slicings in compact cosmologies, where $\text{tr}K$ is a natural time coordinate.[27] All homogeneous cosmologies[47] have these slices (with $\text{tr}K$ a different constant for each slice), and they have proved to be useful in studying singularities in generic inhomogeneous cosmologies.[48] Moreover, theorems demonstrating the existence of such slices have been given without restrictions to homogeneity.[49]

E. Other slicings

We wish to emphasize that general slicings of any spacetime, compact or infinite, can be obtained constructively by choosing any initial value for $\text{tr}K$ and then fixing its velocity $\pounds_t(\text{tr}K)$. One obtains in this way an elliptic equation for α that generates a family of slices by solving

$$\Delta\alpha - [K_{ab}K^{ab} + \tfrac{1}{2}(\rho + \text{tr}S)]\alpha = \pounds_\beta \text{tr}K - \pounds_t \text{tr}K. \tag{3.12}$$

On any slice, everything in this equation is known except the shift β^a (and, of course, α). Therefore, in general the equations for α and β^a (next section) have to be solved simultaneously because they are coupled. This points out one of the advantages of $\text{tr}K = 0$ or $\text{tr}K = \text{constant}$ slicings. In these cases β^a does not appear in (3.12) and the equations for α and β^a are uncoupled.[50] This uncoupling (first find α, then find β^a) is analogous to the fact that the initial-value equations (for the quantities φ and W^a mentioned in Sec. II A) uncouple whenever $\text{tr}K = \text{constant}$.

In summary, the simplest cases are those in which α can be found independently of β^a. These are, for asymptotically flat spaces, (1) maximal slicing: $\text{tr}K = 0$, $\pounds_t(\text{tr}K) = 0$, (2) hyperboloid slicing: $\text{tr}K = \text{constant}$ ($D_a\text{tr}K = 0$) and $\pounds_t(\text{tr}K) = 0$. For compact spaces we have $\text{tr}K = \text{constant}$ ($D_a\text{tr}K = 0$) and $\pounds_t(\text{tr}K) = \text{constant} > 0$.

IV. SHIFT VECTORS

A. Curve congruences and shift vectors

We assume that we now have a choice of an initial slice and a prescription for determining the lapse function α. This will lead to a development of the initial data that is foliated by a family of spacelike slices $\mathscr{T}$ labeled by a scalar function τ, the time variable.

The existence of a development described in terms of a particular foliation leaves only the choice of a congruence of curves, threading the leaves of the foliation, to be designated as the "time congruence." Whatever this congruence is chosen to be, we shall parametrize each of its curves by the value of τ it acquires in intersecting the slices . That is, the *proper-time* clocks carried by observers moving tangent to the time congruence are to be adjusted in such a way that they measure the time τ. We assume that, at great distances from the strong-field region, the adjustment required becomes arbitrarily small, so that τ is the proper time of observers "at infinity." As described in Sec. II B, this adjustment process requires a knowledge of both α and the relative three-velocity of observers moving tangent to the time congruence with respect to the Eulerian observers. (Alternatively, the latter is given by the hyperbolic "tilt angles" of the time congruence relative to $\hat{n}^a$.) Thus, if the proper time from τ to $\tau + \delta\tau$ measured *along the time congruence* is $\delta\tau_p$, then the adjustment factor is defined by $\delta\tau_p = (\alpha^2 - \beta^2)^{1/2}\delta\tau$. Hence, we require a specification of β^a.

We now have two independent curve congruences of interest, the time congruence and the normal congruence. The latter may be regarded as being defined automatically by the foliation, as one needs to know only α and $\hat{n}^a$. The curves of the normal congruence can also be parametrized by τ. The adjustment factor analogous to the above is simply α. The use of the normal congruence thus parametrized in calculating the development would restrict one to time-orthogonal coordinate frames. This is a restriction that should be avoided in general. For example, it is clearly inappropriate for describing rotating sources.

On the initial slice, we may suppose that there is an arbitrary coordinate system (or overlapping systems) x^i with basis $(\partial/\partial x^i)^a$, $i=1,2,3$. Each world line of the time congruence is permanently labeled by the coordinate values x^i that it acquires in the initial slice. Therefore, the basis $(\partial/\partial x^i)^a$ of the initial slice is carried along the time congruence and remains tangent to the leaves of the foliation. The time leg of the frames of the "coordinate observers" is given by $t^a = \alpha\hat{n}^a + \beta^a$. With a geometrical specification of the folitation and the shift vector, one has an unambiguous definition of the coordinate frame

$$\left\{ t^a, \left(\frac{\partial}{\partial x^i}\right)^a \right\} = \left\{ \frac{\partial}{\partial \tau}, \frac{\partial}{\partial x^i} \right\}$$

in terms of the geometry of curve congruences.

B. Shift vectors and matter flow

In addition to the two congruences described above, there is a third set of independent world lines describing the history of any sources that may be present. For definiteness, let us suppose that the source is a fluid. Then there is a unit timelike four-velocity field $\hat{u}^a$ tangent to the fluid world lines. The relation of $\hat{u}^a$ and $\hat{n}^a$ is similar to that between $\hat{t}^a$ and $\hat{n}^a$:

$$\hat{u}^a = (1 - w^2)^{-1/2}(\hat{n}^a + w^a), \qquad \hat{u}^a\hat{u}_a = -1, \quad \hat{n}^a w_a = 0, \tag{4.1}$$

where w^a is the three-velocity of the fluid relative to the Eulerian observers $\hat{n}^a$. To describe the fluid's motion on the family of slices $\mathcal{T}$ relative to the coordinate frames, we can first define its three-velocity $\tilde{w}^a$ using τ time rather than Eulerian proper time ("clocks adjusted by α"). This gives $\tilde{w}^a = \alpha w^a$, which is analogous to $\beta^a = \alpha v^a$ in the discussion in Sec. II B. It follows immediately that the three-velocity of the fluid relative to the coordinate frames

$$\left\{ \frac{\partial}{\partial \tau}, \frac{\partial}{\partial x^i} \right\}$$

is given by $\tilde{w}^a - \beta^a$.

The above discussion suggests one possible physically motivated way of choosing β^a. We can correlate the fluid motion and the coordinate observers' frames by demanding a definite relationship between β^a and $\tilde{w}^a$. For example, the well-known comoving[51] condition is $\beta^a = \tilde{w}^a$ ($v^a = w^a$). In this case, a given element of the fluid has fixed spatial coordinates x^i. (Of course, even when $\beta^a = \tilde{w}^a$, the time τ does not coincide with the local proper time defined in the rest frame of the fluid element.)

Tying β^a to $\tilde{w}^a$ can simplify the description of the sources of a gravitational field in some cases. However, this is not always true. If a bounded fluid source is spatially rotating relative to spatial infinity, then setting $\beta^a = \tilde{w}^a$ in the fluid, with $\beta^a \to 0$ at infinity, would cause the spatial coordinates in the fluid to "wind-up" in a complicated way in the evolution.

In such cases, and in general, it may be more important to use β^a to simplify the description of the *gravitational field* that *results* from the evolution of a given source. This is especially important also whenever a horizon or gravitational radiation can form. In any case, the tying of β^a to $\tilde{w}^a$ cannot be used in a vacuum spacetime region.

C. Shift vectors and geometry of the gravitational field

Elsewhere,[54] we have presented a three-covariant method of choosing the shift vector as part of a "radiation gauge" for general relativity. It was shown that this "minimal-distortion" shift vector arises from a simple variational principle and that it provides a natural three-covariant strong-field generalization of the radiation gauges of ADM and Dirac, which were motivated by consideration of weak fields. Here, we shall describe the geometric foundations of this and similar prescriptions. In Sec. V, we show by examples that the minimal-distortion shift vector also simplifies the description of the field even when no radiation is present (e.g., in the Kerr metric).

We begin by recalling the well-known decomposition[52] of the covariant derivative of any timelike unit vector $\hat{Z}^a$. (Similar decompositions exist for null and spacelike vectors as well.) One writes

$$\nabla^a \hat{Z}^b = \omega^{ab} + \sigma^{ab} + \tfrac{1}{3} h^{ab}\theta - \hat{Z}^a \zeta^b, \tag{4.2}$$

where the tensors

$$h^{ab} = g^{ab} + \hat{Z}^a\hat{Z}^b, \tag{4.3a}$$

$$\omega^{ab} = h^a_c h^b_d \nabla^{[c}\hat{Z}^{d]}, \tag{4.3b}$$

$$\sigma^{ab} = h^a_c h^b_d(\nabla^{(c}\hat{Z}^{d)} - \tfrac{1}{3}h^{cd}\nabla_e\hat{Z}^e), \tag{4.3c}$$

$$\theta = \nabla_e \hat{Z}^e, \tag{4.3d}$$

$$\zeta^b = \hat{Z}^c\nabla_c\hat{Z}^b, \tag{4.3e}$$

are, respectively, the projection tensor (local "three-metric" orthogonal to $\hat{Z}^a$), twist ($\omega^{ab} = -\omega^{ba}$), shear ($\sigma^{ab} = \sigma^{ba}$, $h^{ab}\sigma_{ab} = 0$), expansion, and acceleration of $\hat{Z}^a$. All these tensors are orthogonal to $\hat{Z}^a$:

$$h^{ab}\hat{Z}_b = \omega^{ab}\hat{Z}_b = \sigma^{ab}\hat{Z}_b = \zeta^b\hat{Z}_b = 0. \quad (4.4)$$

If we take $\hat{Z}^a = \hat{n}^a$, then we find $h_{ab} = \gamma_{ab}$ (three-metric of $\mathcal{T}$), $\omega^{ab} = 0$ ($\hat{n}^a$ is "surface-forming"), $\zeta^a = a^a$ (acceleration of Eulerian observers), and $\sigma^{ab} + \frac{1}{3}\theta h^{ab} = -K^{ab}$, K^{ab} = the extrinsic curvature of $\mathcal{T}$. Likewise, the fluid motion ($\hat{Z}^a = \hat{u}^a$) can be analyzed in terms of its twist, shear, expansion, and acceleration as determined in the local rest frame of the fluid.

Here we wish to describe, in terms similar to the above, a strain tensor Θ_{ab} for the time congruence t^a. This quantity will be analogous to the strain tensor

$$\theta_{ab} = \sigma_{ab} + \tfrac{1}{3}\theta\gamma_{ab} = \gamma^c_a\gamma^d_b\nabla_{(c}\hat{n}_{d)}, \quad (4.5)$$

defined in the ordinary way[53] (using $\hat{n}^a$) for the normal congruence. Note that θ_{ab} characterizes a strain induced in the three-geometry of $\mathcal{T}$, with unit normal $\hat{n}^a$, as we pass from τ = constant to a nearby *parallel* slice:

$$2\theta_{ab} = -2K_{ab} = \pounds_{\hat{n}}\gamma_{ab}. \quad (4.6)$$

Therefore, the strain induced in the three-geometry of $\mathcal{T}$ in passing along the normal congruence from τ to $\tau + \delta\tau$ (slices *not* parallel) is simply

$$\pounds_{\alpha\hat{n}}\gamma_{ab} = -2\alpha K_{ab} = 2\perp^c_a\perp^d_b\nabla_{(c}\alpha\hat{n}_{d)} \equiv 2\perp\nabla_{(a}\alpha\hat{n}_{b)}\ . \quad (4.7)$$

Because $t^a = \alpha\hat{n}^a + \beta^a$, $\beta^a\hat{n}_a = 0$, the strain tensor Θ_{ab} that we seek will be the sum of (4.7) and the additional purely intrinsic deformation $\pounds_\beta\gamma_{ab} = D_a\beta_b + D_b\beta_a$ familiar from ordinary elasticity theory.[53] (Here it is convenient to think of β^a as describing an "active" motion of the points of τ = constant.) Therefore, one can define the strain tensor Θ_{ab} associated with the time congruence by

$$2\Theta_{ab} = \perp\pounds_t\gamma_{ab} = -2\alpha K_{ab} + \perp\pounds_\beta\gamma_{ab} = 2\perp\nabla_{(a}\alpha\hat{n}_{b)}\ . \quad (4.8)$$

This strain tensor compares the intrinsic metric on τ to that on $\tau + \delta\tau$ by (1) transferring ("dragging") the former along the time congruence, (2) taking a projection of the thus deformed three-metric into $\tau + \delta\tau$ (not orthogonally to t^a), (3) taking the difference between the deformed metric and that "already" on $\tau + \delta\tau$, and (4) pulling the difference back to τ.

One may conveniently visualize the above process in terms of small deformations of thin shells[53] in Euclidean space. (See Fig. 7.) One may think of an "already curved" shell (γ_{ab} not flat, $K_{ab} \neq 0$) that is given a *specified* normal deflection $\delta\tau(\alpha\hat{n}^a)$ and an arbitrary tangential deformation $\delta\tau\beta^a$. A natural question to ask is: What tangential deformation $\delta\tau\beta^a$ will in some sense minimize the resulting strain tensor Θ_{ab}?

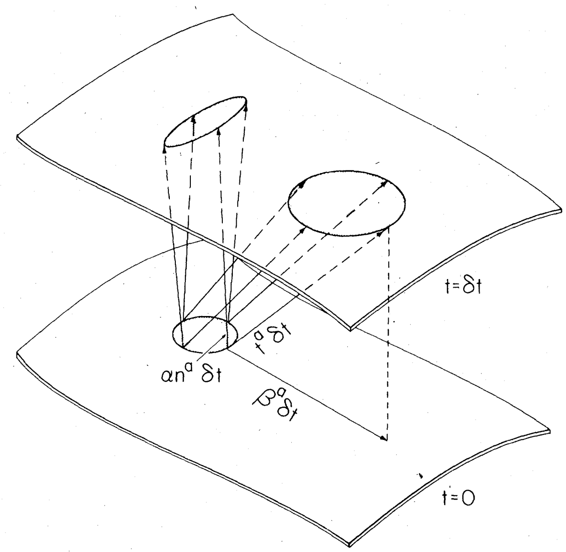

FIG. 7. This schematic diagram illustrates the use of the minimal-distortion shift vector to reduce coordinate shear. If a small sphere (here one spatial dimension is suppressed) is transported along the normal $\hat{n}^a$ to the next slice $\tau = \delta\tau$, it will be sheared into an ellipsoid. If the slicing is maximal, the volume will be preserved to first order. On the other hand, if a shift vector is also used, then some of this coordinate shear can be removed, although with the possible introduction of some change in volume.

D. Minimizing changes in the three-metric and in the conformal three-metric

It is in principle possible to use any shift vector, as the effect of the shift can be viewed as that of a three-dimensional coordinate transformation in the slice. However, one can choose the shift in such a way as to minimize[54] the strain discussed above in a global sense. On each slice, one forms the non-negative "square" of the strain $\Theta_{ab}\Theta^{ab}$ in terms of (4.8) and integrates this over a slice. Variation with respect to β^a yields a "minimal strain" shift vector satisfying

$$D^b\Theta_{ab} = 0 \text{ or } D^b(D_b\beta_a + D_a\beta_b) = D^b(2\alpha K_{ab}). \quad (4.9)$$

This linear elliptic equation for β^a has unique solutions for appropriate boundary conditions.[36] The equation is three-covariant and involves the curvature of the slice in an essential way, as one sees from

$$D^b(D_a\beta_b + D_b\beta_a) \equiv \Delta\beta_a + D_a(D_b\beta^b) + \mathcal{R}_{ab}\beta^b, \quad \Delta \equiv D^cD_c.$$

This criterion involves the *velocity* of the metric because $\Theta_{ab} = \frac{1}{2}\perp\pounds_t\gamma_{ab}$. If we interpret the result in terms of definite components γ_{ij} of the metric in a basis $\partial/\partial x^i$, then the integrand of the variational principle is $\frac{1}{4}\gamma^{ik}\gamma^{jl}\dot{\gamma}_{ij}\dot{\gamma}_{kl}$. Hence, we are minimizing time changes of the metric component functions in an "average" sense.

A related method of choosing the shift is perhaps more fundamental. We can define a "shear" tensor associated with Θ_{ab} by taking its trace-free part:

$$\Sigma_{ab}=\Theta_{ab}-\tfrac{1}{3}\gamma_{ab}\,\mathrm{tr}\Theta.$$

To distinguish Σ_{ab} from the usual shear tensor σ_{ab} discussed earlier, we call Σ_{ab} the *distortion*. As for any shear-type tensor, it can be viewed as describing the change of *shape* of a small figure during the deformation process. (See Fig. 7.) It has five independent components that contain the relative stretching of two of the axes of the figure relative to the third one ("pure stretching") and the changes of the angles between the three pairs of axes. The distortion tensor says nothing about the change of volume [i.e., $(\det\gamma)^{1/2}$] of the figure.

The interest in Σ_{ab} stems from the fact that it is essentially the velocity of the conformal three-metric $\tilde{\gamma}_{ab}=(\det\gamma)^{-1/3}\gamma_{ab}$. Thus,

$$\begin{aligned}\Sigma_{ab}&=\tfrac{1}{2}(\det\gamma)^{1/3}\perp\mathcal{L}_t\tilde{\gamma}_{ab}\\&=\tfrac{1}{2}\perp(\mathcal{L}_t\gamma_{ab}-\tfrac{1}{3}\gamma_{ab}\,\mathrm{tr}\,\mathcal{L}_t\gamma)\\&=-\alpha(K_{ab}-\tfrac{1}{3}\gamma_{ab}\,\mathrm{tr}K)+\tfrac{1}{2}(L\beta)_{ab}\ ,\\(L\beta)_{ab}&\equiv D_a\beta_b+D_b\beta_a-\tfrac{2}{3}\gamma_{ab}D_c\beta^c\ .\end{aligned}\tag{4.10}$$

We mentioned earlier that it is convenient to think of gravitational dynamical degrees of freedom as being characterized by $\tilde{\gamma}_{ab}$. Because the number of degrees of freedom of a field at each point on a slice is the number of independent *velocity* components of the metric on the slice, we would expect an appropriate choice of β^a to effectively fix three of the five components of Σ_{ab} at each point. Moreover, to minimize "coordinate effects" in the representation of the evolution of $\tilde{\gamma}_{ab}$ on the given foliation, we see that the optimum choice of β^a would be to minimize the integral of $\Sigma_{ab}\Sigma^{ab}$ over a slice. This process yields the *minimal-distortion* shift vector as the unique solution of

$$D^b\Sigma_{ab}=0$$

or

$$D^b(L\beta)_{ab}=D^b[2\alpha(K_{ab}-\tfrac{1}{3}\gamma_{ab}\,\mathrm{tr}K)]\ .\tag{4.11}$$

Notice that $(K_{ab}-\frac{1}{3}\gamma_{ab}\,\mathrm{tr}K)=-\sigma_{ab}$. Therefore one may regard this choice of β^a as the one that (globally) most nearly compensates the shear $\alpha\sigma_{ab}$ over which one has no control if there has been prescribed a definite foliation of a development. This criterion for β^a also contains the curvature[55]

$$D^b(L\beta)_{ab}=\Delta\beta_a+\tfrac{1}{3}D_a(D_b\beta^b)+\mathcal{R}_{ab}\beta^b\ .$$

In component language, we have used as integrand in the variational principle

$$\Sigma_{ab}\Sigma^{ab}=\tfrac{1}{4}(\det\gamma)^{2/3}\gamma^{ik}\gamma^{jl}\dot{\tilde{\gamma}}_{ij}\dot{\tilde{\gamma}}_{kl}\ .\tag{4.12}$$

The minimal distortion equation has a three-dimensional component form equivalent to $D^j\dot{\tilde{\gamma}}_{ij}=0$.

E. Discussion of minimal-distortion condition

There are several features of the minimal-distortion condition that are worth mentioning explicitly. Elsewhere,[36] we have pointed out that in a wave zone, where the field is relatively weak, this condition generalizes and includes the "*TT* gauges" of ADM and Dirac.[3] In this case, the β^a's are "small." However, they are not small in general. Deep in a "near zone," even in the absence of radiation, the β^a's are "large" and using them simplifies significantly the description of the structure of strong fields. This is illustrated in Sec. V. It has also been demonstrated for black-hole examples by Eppley[56] and by Duncan.[56]

A further feature is the close relationship of minimal-distortion shift vectors to the "isotropic" type coordinates that are used almost exclusively in parametrized-post-Newtonian (PPN) approximations[57] to general relativity (and other metric theories of gravity). The simplest example of isotropic coordinates is in the Schwarzschild metric, where the spatial metric has its explicitly conformally flat form $\gamma_{ij}=(1+m/2r)^4 f_{ij}$, where f_{ij} is a flat metric and $r=(x^2+y^2+z^2)^{1/2}$ if $f_{ij}=\delta_{ij}$. In general, three-metrics are not conformally flat, but isotropic-type coordinates can be defined as those that satisfy,[36] in Cartesian component language, $\tilde{\gamma}_{ij,j}=O(r^{-3})$. The significant point is that if we choose shifts β^i that vanish at infinity, the minimal-distortion shift always preserves the condition above for isotropic-type coordinates.[36] [Here we also assume the customary condition $K_{ij}=O(r^{-2})$ that guarantees finite linear momentum.] Therefore, repeated solution of the minimal-distortion equation generates in the evolution of Cauchy data a time-dependent three-dimensional coordinate transformation from quasi-isotropic coordinates to quasi-isotropic coordinates.

This discussion and the examples in Sec. V, demonstrate that the minimal-distortion shift vector is a natural choice from the points of view of (1) the initial-value problem (conformal treatment), (2) anisotropic cosmologies, (3) description of gravity in the wave zone, (4) description of strong-field structure in Schwarzschild and Kerr metrics, (5) exact treatment of dynamics of geometry, and (6) PPN approximation schemes.

V. EXAMPLES

A. Killing equations

We turn finally to a detailed investigation of a number of familiar spacetimes. These will include Minkowski spacetime, stationary spacetimes (including Kerr black holes), and homogeneous cosmologies. Our purpose is to use the general framework developed in previous sections to elucidate the close relationship between many coordinate systems which heretofore have been considered only in relationship to the specific spacetime at hand. This will illustrate some of the practical difficulties one encounters in constructing global coordinate systems. Further, the insight we gain from applying our general ideas to analytic models, where we can solve the equations in closed form, will be crucial for the numerical applications in dynamical asymptotically flat spacetimes or inhomogeneous cosmologies.

Since most of the model spacetimes we will consider are analytic exact solutions of the field equations, we pause briefly to consider a fundamental difference between the present approach to solving Einstein's equations and the more traditional one. Typically exact solutions have been discovered by making use of preferred vector fields in the sought-for spacetime, e.g. Killing vectors, degenerate principal null directions, etc. The coordinate system in which the spacetime line element is exhibited is tightly built around these vector fields. However, this technique is useful only for a very restricted set of spacetimes containing such vector fields.

To solve for physically interesting radiating spacetimes one must have a qualitatively different approach, such as the Cauchy evolution discussed above. There one poses initial data and evolves these data into the future along some congruence of curves t^a specified by a choice of the functions α and β^a. It is from this point of view that we examine some well-known spacetimes.

Our first task is to consider how a Killing vector ξ^a appears in the present approach. Assume the slicing (α) is given and we have yet to choose a shift vector (β^a). If we decompose ξ^a along the normal congruence $\hat{n}^a$ to the slicing as

$$\xi^a = \mu \hat{n}^a + \omega^a , \tag{5.1}$$

where $\omega^a \hat{n}_a = 0$, then we can write

$$\xi^a = \mu \alpha^{-1} t^a , \tag{5.2}$$

where t^a is given by Eq. (2.16). We project[58] the Killing equation

$$\nabla_a \xi_b + \nabla_b \xi_a = 0 \tag{5.3}$$

into the surface, yielding

$$D_a \beta_b + D_b \beta_a - 2\alpha K_{ab} = -\beta_a D_b \ln(\alpha^{-1}\mu) - \beta_b D_a \ln(\alpha^{-1}\mu) . \tag{5.4}$$

We can then evaluate the distortion tensor Σ_{ab} (4.13) by removing the trace from this equation:

$$2\Sigma_{ab} = -\beta_a D_b \ln(\alpha^{-1}\mu) - \beta_b D_a \ln(\alpha^{-1}\mu) + \tfrac{2}{3}\gamma_{ab}\beta^c D_c \ln(\alpha^{-1}\mu) . \tag{5.5}$$

Two cases present themselves. If the slicing is carried into itself by the ξ^a isometry, then $\xi^a = f(t)\, t^a$ [with $f(t)=1$ usually] and $D_a \ln(\alpha^{-1}\mu)=0$. This allows us to use ξ^a as our coordinate congruence. In this case the Killing shift vector *is* a minimal-distortion shift vector since from (4.13)

$$2\Sigma_{ab} = (L\beta)_{ab} - 2\alpha\sigma_{ab} = 0 \tag{5.6}$$

and the rate of change of the conformal three-geometry is zero. The other case is one in which the slicing is chosen in such a way that it is not carried into itself by the isometry. If one wishes to choose the coordinate congruence to lie along the Killing congruence, then $\xi^a = f(x) t^a$ and $\partial_a \ln(\alpha^{-1}\mu) \neq 0$. The shift vector so obtained is *not* a minimal-distortion shift vector since the divergence of (5.5) contains the $\alpha^{-1}\mu$ terms. Furthermore, $\Sigma_{ab} \neq 0$ and the distortion can be minimized further by solving the full minimal-distortion equation (4.11). However, in general the distortion cannot be reduced to zero unless the more restrictive condition, (5.6), holds everywhere on the slice. Since these equations overdetermine β^a, one cannot in general find a zero-distortion coordinate system. In the examples discussed below, we will encounter both of these cases.

In applications of the minimal-distortion condition one must keep in mind the fact that the shift vector one finds will not be unique unless appropriate boundary conditions are imposed. In asymptotically flat spacetimes there are conditions at spatial infinity and on any possible inner boundaries.[28] Asymptotically $\beta^a_{\rm min} \to 0$ if at great distances the slices are like Minkowski hyperplanes (e.g., tr$K=0$). On the other hand, for asymptotically boosted slices, such as tr$K$ $=$ constant $\neq 0$, a more natural choice in keeping with the interpretation of β^a as a velocity is to require $|v^a| = |\beta^a|\alpha^{-1} \to 1$ as we shall see below. Also, we note on such asymptotically boosted slices $\alpha = O(r)$.

In cosmological spacetimes with closed slices, the closure condition itself provides the only boundary condition needed. Here $\beta^a_{\rm min}$ is always unique up to any possible conformal Killing vectors. The latter would only give an overall rescaling of the three-metric. For open cosmologies, there may not be any natural asymptotic condition, in which case $\beta^a_{\min}$ may be nonunique. Here $\beta^a_{\min}$ can be selected on the basis of the resulting simplicity of the metric. All of these features are illustrated below.

B. Minkowski spacetime

Usually one does not use a time slicing for Minkowski spacetime other than the standard planes which are normal to the time Killing vector:

$$\xi^a = \partial/\partial t \ ; \qquad (5.7)$$

t = Minkowski time coordinate. Since no curvature is present, these slices are at once geodesic ($\alpha = 1$), time symmetric ($K_{ij} = 0$), maximal ($\mathrm{tr}K = 0$), and flat ($\mathcal{R}_{ij} = 0$). The slices are trivially mapped into each other by ξ^a, so the minimal-distortion coordinate congruence lies along the Killing trajectories with $\beta^a_{\min} = 0$.

Another useful slicing is the hyperboloid,[46] $\mathrm{tr}K$ = constant. The time slices are given by

$$(t-\tau)^2 - r^2 = a^2 \ , \qquad (5.8)$$

where τ = constant on each slice and a is the radius of curvature of the slice:

$$\mathrm{tr}K = -3/a \ . \qquad (5.9)$$

The slices are no longer geodesic, time symmetric, or flat because we find

$$\alpha = (1 + r^2/a^2)^{1/2} \ , \qquad (5.10)$$

$$K_{ij} = \tfrac{1}{3}\gamma_{ij}\,\mathrm{tr}K \ , \qquad (5.11)$$

$$\mathcal{R} = -6/a^2 \ . \qquad (5.12)$$

The slices are carried into themselves by ξ^a and so the minimal-distortion congruences lies along ξ^a, requiring a nonzero β^a:

$$\beta^a = -\left(\frac{r}{a}\right)\left(1 + \frac{r^2}{a^2}\right)^{1/2}\frac{\partial}{\partial r} \ . \qquad (5.13)$$

With this shift vector the three-metric on the slice can be written in two forms familiar from cosmology:

$$dl^2 = (1 + r^2/a^2)^{-1}\,dr^2 + r^2 d\Omega^2 \ , \qquad (5.14)$$

or, in terms of relabeling $r = a\sinh\eta$, conformal to a unit hyperboloid:

$$dl^2 = a^2(d\eta^2 + \sinh^2\eta\, d\Omega^2) \ . \qquad (5.15)$$

Here [using (5.10), (5.13), and (5.14)] one sees $|v^a| = |\beta^a|\alpha^{-1} \to 1$ as $r \to \infty$, so this $\beta^a_{\min}$ satisfies the natural asymptotic conditions mentioned above. Note that, as expected, the three-metric is time independent, i.e., there are no gravitational degrees of freedom present. As pointed out in Sec. III, the hyperboloid slices intersect null infinity and not spacelike infinity.

We see that already for these simple slices it is important to use the minimal-distortion congruence. If, for instance, we had the same hyperboloid slicing, but chose the shift vector equal to zero, the spacetime line element would read

$$ds^2 = -[1 + \mathrm{csch}^2(\sigma - \tau/a)]\,d\tau^2 + a^2\,\mathrm{csch}^2(\sigma - \tau/a)(d\sigma^2 + d\Omega^2) \ , \qquad (5.16)$$

where σ is a radial coordinate constant along the normal congruence. Since the normal trajectories are moving through a static spacetime:

$$r = a\,\mathrm{csch}(\sigma - \tau/a) \ , \qquad (5.17)$$

the lapse and three-metric appear to be time dependent. Furthermore, the radial coordinate σ has range

$$\tau/a \le \sigma < \infty \ , \qquad (5.18)$$

which depends on τ. Notice that $\beta^a = 0$, in this example, is also a minimal-distortion shift vector (the *conformal* three-metric is time independent). However, this shift vector does *not* satisfy the natural asymptotic condition for boosted slices. This vividly demonstrates that even when no gravitational field is present, and even though a "natural" smooth time slicing is used, the time dependence of the three-metric is highly dependent on the choice of shift vector. The optimum choice is the minimal-distortion vector (5.13) that satisfies the natural asymptotic condition.

C. Stationary spacetimes

Let us move now to the case where a gravitational field is present (with or without matter), but with a Killing trajectory still present. For example, consider the usual stationary axisymmetric line element[59]

$$ds^2 = -e^{2\nu}dt^2 + e^{2\mu}(d\rho^2 + dz^2) + \rho^2 B^2 e^{-2\nu}(d\phi - \omega dt)^2 \ , \qquad (5.19)$$

where ν, μ, B, and ω are functions of ρ and z, but not t or ϕ. Here the time slices are chosen to be orthogonal to the twist-free part of the time Killing vector ξ^a. In fact, it is easy to show that

$$\xi^a = \alpha\hat{n}^a - \omega^a , \qquad (5.20)$$

where ω^a is the vorticity vector formed from ξ^a. The normals $\hat{n}^a$ to the slices are often referred to as "Bardeen locally nonrotating" observers. From our point of view they are the Eulerian observers for this particular slicing (the normals to any slicing are "nonrotating" by definition).

The lapse function is given by

$$\alpha = e^{\nu} \tag{5.21}$$

and the slices are maximal,[60] therefore we require $\beta^a_{\min} \to 0$ at spatial infinity. If $\omega^a \neq 0$, the slices are "momentarily stationary"[60] and if $\omega^a = 0$, they are time symmetric. The only case in which they are geodesic is if $\nu = \omega^2 = 0$, which is Minkowski spacetime, in which case the time planes are recovered. The slices are obviously carried into themselves by ξ^a, and thus $\beta^a = -\omega^a$ is a minimal-distortion vector field. As in Minkowski spacetime, the first-order form of minimal distortion (5.6) is satisfied. We note that here β^a minimizes the distortion caused by differential rotation shear, not radial shear as in the Minkowski space hyperboloid slicing. Again, if a different shift vector were used, the three-metric would appear to be time dependent.

Stationary spacetimes may contain (rotating) masses or vacuum black holes. In the former case $\alpha > 0$ everywhere. However, in the latter case the topology of the three-slice is $S^2 \times R$ instead of R^3. As a result, there is a "throat" on which an "inner boundary condition" is required. In the standard stationary coordinate systems (including Schwarzchild and Boyer-Lindquist systems), the boundary conditions on the throat are seen to be

$$\alpha = 0 \,, \tag{5.22}$$

$$[(L\beta)_{ij} + 2\alpha\sigma_{ij}]\hat{e}^j = 0 \,, \tag{5.23}$$

where $\hat{e}^i$ is an outward normal to the two-surface of the throat in the slice. The latter is a Neumann condition, so $\beta^a_{\min} \neq 0$ on the throat. It can be shown in general from the variational principle for $\beta^a_{\min}$ (Ref. 28) that the Neumann condition (5.23) gives an absolute minimum for Σ_{ab}. In the static case there is no dragging of inertial frames and $\sigma_{ij} = 0$. Therefore in this case, $\beta^a_{\min} = 0$ everywhere satisfies (5.23).

Our analysis of the stationary line element in terms of maximal slicing with minimal-distortion shift vector instead of in terms of Killing vectors allows us to generalize to a spacetime of no symmetries. That is, we believe the natural generalization of the notion of a stationary rest frame to a dynamic spacetime is obtained by first choosing initial data for which the total three-momentum vanishes[28] and then by evolving Einstein's equations using a lapse and shift satisfying (3.2), (4.11), and (5.22), (5.23). The t^a trajectories are then, in a sense, "close" to Killing-type trajectories.

D. Black holes

It is important to realize that the standard coordinates for stationary black holes are by no means the only useful ones. Let us consider, for simplicity, Schwarzschild-Kruskal spacetime.[32] Here the Killing congruence ξ^a is hypersurface orthogonal, but it is timelike only outside of the event horizon. The static slicing orthogonal to ξ^a therefore only covers the spacetime outside of the black hole.

One way to explore the dynamic region inside the black hole is to use geodesic slices. Lemaître[61] first used this idea to show that the surface $r = 2M$ was not singular. His system is based on a normal congruence of marginally bound radial geodesics. If σ = constant along each geodesic, then the line element is

$$ds^2 = -d\tau^2 + (4M/3)^{2/3}(\sigma - \tau)^{-2/3}d\sigma^2 + (9M/2)^{2/3}(\sigma - \tau)^{4/3}d\Omega^2 . \tag{5.24}$$

As in the hyperboloid example in Minkowski spacetime, one sees here that demanding a normal coordinate congruence ($\beta^r = 0$) causes an apparent time dependence in the three-metric. By choosing instead a minimal-distortion shift vector, we find that the line element becomes

$$ds^2 = -(1 - 2M/r)d\tau^2 + 2(2M/r)^{1/2}dr d\tau + dr^2 + r^2 d\Omega^2 , \tag{5.25}$$

where r is the standard Schwarzschild curvature r. Again, because ξ^a maps the time slices into themselves, the coordinate congruence lies along the ξ^a trajectories. Note, however, that the falloff in the slices of various quantities is different from the falloff usually assumed. Here

$$\beta^r \sim r^{-1/2} , \tag{5.26}$$

$$K_{ij} \sim r^{-3/2} . \tag{5.27}$$

This is why the three-slices can be flat and not contain the mass. In addition, each slice intersects the Schwarzschild singularity at $r = 0$.

The Lemaître system is useful as an example of the simplification induced by the minimal-distortion shift vector. However, as a model for more complicated nonspherical spacetimes, we would like a nonsingular evolution. In Sec. III, we discussed various developments of the time-symmetric slice $v_{\text{Kruskal}} = 0$ in the Kruskal diagram. There we saw the crucial dependence of the slicing on the choice of lapse function. The use of a shift vector in such black-hole spacetimes[56] where the three-space is not topologically R^3 is beginning to be investigated.

For the geodesic slicing, the coordinates adapted to the normal congruence are termed Novikov coordinates.[62] This coordinate system is a special case of the Bondi-Tolman metrics.[63] Such coordinates are natural for studies of dust ball collapse matched[64] to a Schwarzschild exterior. There one can ask for the shift vector which sends the coordinate observers along the ξ^a trajectories and consider the problems of matching the lapse and shift across the surface of the star. For a discussion see Smarr and Welty.[65] For the maximal slicing, the shift vector which matches the t^a trajectories with those of ξ^a was found by Estabrook *et al.*[37] Note, however, that neither the geodesic nor the maximal slicing is carried into itself by the ξ^a trajectories, since the slicing advances symmetrically across the throat, and therefore these shift vectors are *not* minimal-distortion shift vectors.

This leads us to a subtle point involving boundary conditions. There are two types of "inner boundaries" which may occur. One is when there is a throat present, as in the Schwarzschild solution, and the other is when a central massive body is present. In the latter case, for instance, one might want to choose the shift vector inside the matter by the requirement that the coordinate congruence be comoving with the matter, and in the vacuum exterior choose a minimal-distoriton shift vector. In either case, one must choose an inner boundary condition on β^a.

As discussed above, for the shear to be an absolute minimum, one must use the Neumann-type boundary condition on β^a, i.e., (6.6). This in general requires that $\beta^a \neq 0$ on the boundary. However, if this occurs then the coordinate congruence must *cross* the boundary. In the case of a symmetric slicing of a throat spacetime, this means the isometry between the upper and lower sheets will be broken. In the case of a star, such an absolute minimum means either the surface of the star cannot lie along a coordinate line, or if comoving coordinates inside are demanded, then the shift vector is discontinuous across the surface. Another possibility, of course, is to choose $\beta^a = 0$ on the boundary. In this case, the boundary will lie along a coordinate line, but the distortion will not be an absolute minimum. In particular, distortion will pile up near the boundary. An example of such a minimal-distortion shift vector with a maximal slicing has been given by Eppley.[1]

Finally, we make some remarks about whether maximal slicing always leads to nonsingular developments. In Sec. III we presented a model which indicated that maximal slicing causes the lapse function to drop to zero exponentially in time as a nonlocal strength of curvature and matter terms increases. Two questions arise: (1) Is this drop *always* fast enough so as to produce a nonsingular development? (2) In the cases where it is, does this mean the slices are "avoiding a spacelike singularity to the future"?

The answer to both of these questions, as mentioned earlier, is no. A counterexample to the first claim was produced by Eardley and Smarr.[43] They investigated the maximal slicings of a class of Bondi-Tolman solutions representing dust ball collapses. In most cases the maximal slicing behaves qualitatively as described in Sec. III. But, for a sharp enough rise in the density ρ toward the center of the ball, resulting in a very inhomologous collapse, the maximal slices hit the spacelike singularity in a finite time. In terms of our model calculation, this is because with such a sharp "spike" in the central density, the weighted strength in Eq. (3.8) gives too slow a time rise to x_0 and therefore too slow an exponential falloff for α [Eq. (3.9)]. However, it seems likely that more reasonable equations of state will not allow this to occur.

To see how question (2) can have a negative answer, we turn to the charged version of Schwarzschild spacetime: Reissner-Nordström. By solving for the maximal slicing of the analytically extended spacetime, Duncan[42] finds the $r=$ constant surfaces which are the analogs of the $r=\frac{3}{2}M$ limiting maximal surface in Kruskal-Schwarzschild spacetime. Here there are two surfaces $r=$ constant which have $\mathrm{tr}K=0$:

$$r_{1,2}=\tfrac{3}{4}M\left[1\pm\left(1-\frac{8}{9}\frac{Q^2}{M^2}\right)^{1/2}\right]. \tag{5.28}$$

One of these, r_1, lies outside of the Cauchy horizon:

$$r_- = M[1-(1-Q^2/M^2/^{1/2}] \tag{5.29}$$

for $0<Q/M<1$. The Cauchy evolution by maximal slices from the time-symmetric initial slice wraps up around r_1. Now even if the maximal slices filled the entire interior of the Cauchy horizon ($r>r_-$) they could not hit a singularity. Thus, they must be "avoiding" something else. In this case it is the Cauchy horizon r_- itself, since it is a null hypersurface with zero volume. This example again indicates that maximal slices try to avoid global regions of small volume.

All of these remarks may be carried over to dynamical spacetimes containing no Killing trajectories. Geodesic slicings seem to work well if curvatures are very small[66] (weak gravitational waves). For strong fields, caustics develop in geodesic slicing and a better method, such as maximal slices, must be used. This has been done[40] for star collapse, strong gravitational waves, and colliding black holes. The general properties shown by our flat-space example seem to carry over, even in highly nonspherical problems. Presumably, this is because at late times any nonrotating black hole settles down to a spherical one. The use of nonzero shift vectors for these spacetimes is only beginning to be explored. However, it is clear that the choice of zero shift leads to a time evolution of the three-metric dominated by coordinate shear.[1] The use of hyperboloid slicing is also beginning to be explored.[43]

E. Homogeneous cosmologies

Just as we study the kinematics of stationary black-hole spacetimes to prepare us for asymptotically flat dynamical spacetimes, so the homogeneous cosmologies[47] are the models from which we gain insight for the inhomogeneous case. In the black-hole spacetimes, the stationarity forced the "standard time slices" to be maximal ($\mathrm{tr}K=0$); in the cosmological spacetimes the choice of surfaces of homogeneity as time slices implies the slicing $\mathrm{tr}K=f(\tau)$. However, the homogeneity also implies the normals $\hat{n}^a$ are geodesic ($\alpha=1$). As was suspected for a long time and now has been rigorously proved, *families* of maximal slices do not exist in general for closed cosmologies.[67]

The use of a shift vector is uncommon in the study of cosmologies, except in the case of tilted cosmologies,[68] where the shift vector is chosen to maintain comoving coordinates when the matter flow lines are not orthogonal to the surfaces of homogeneity. We show below that the minimal-distortion shift vector can be useful for simplifying the three-metric of homogeneous cosmologies. Furthermore, as in asymptotically flat spacetimes, we *define* gravitational degrees of freedom to be present when $\Sigma_{ab}(\beta^c_{\min})\neq 0$. This contrasts with the usual investigations of homogeneous cosmologies when $\beta^a=0$ is the preferred gauge. The minimal-distortion shift equation becomes

$$D^a(L\beta)_{ab}=2J_b\,. \tag{5.30}$$

If the cosmology is tilted, then this full equation must be solved with a matter current J^a. For the untilted case (5.30) simplifies to the source-free case

$$D^a(L\beta)_{ab}=0. \tag{5.31}$$

Here again the *boundary conditions play the crucial role*. If the universe is open we may have

$$(L\beta)_{ab}=\mathrm{const} \tag{5.32}$$

and a nontrivial β^a. (In particular, the constant may be $2\alpha\sigma_{ab}$.) If it is closed then

$$(L\beta)_{ab}=0 \tag{5.33}$$

and the only nontrivial solutions are conformal Killing vectors.[55] If no such symmetries exist, then we have

$$\beta_a=0. \tag{5.34}$$

For an application to an open universe let us study vacuum Kasner spacetime, a Bianchi type-I homogeneous cosmology.[47] The line element in the usual form with zero shift is

$$ds^2=-d\tau^2+\prod_{i=1}^{3}\tau^{2p_i}(dx^i)^2, \tag{5.35}$$

where p_i are three constants. The slicing is geodesic but also satisfies $D_a\,\mathrm{tr}K=0$, with $\mathrm{tr}K(\tau)$ being given by

$$\mathrm{tr}K=-\tau^{-1}=\mathrm{tr}K(\tau). \tag{5.36}$$

Note that in these coordinates the three-metric is time dependent, even though there exists a homothetic Killing vector[69] (self-similarity) in the Kasner universe given by

$$\xi^a=\tfrac{1}{2}c\tau\left[\frac{\partial}{\partial\tau}+\sum_{i=1}^{3}(1-p_i)\tau^{-1}x^i\frac{\partial}{\partial x^i}\right], \tag{5.37}$$

$$\nabla_a\xi_b+\nabla_b\xi_a=cg_{ab}\,. \tag{5.38}$$

Since the slices are carried into themselves by ξ^a, it seems likely that, by an extension of our discussion in Sec. V A, a minimal-distortion shift vector will result if we choose t^a to lie along the ξ^a trajectories:

$$\xi^a=\tfrac{1}{2}c\tau t^a=f(\tau)t^a. \tag{5.39}$$

To verify this conjecture, we find new coordinates y^i which are constant along t^a:

$$t^a(y^i)=0,\qquad y^i=\tau^{p_i-1}x^i. \tag{5.40}$$

In these new coordinates the line element becomes

$$ds^2=-\left\{1-\sum_i[(p_i-1)y^i]^2\right\}d\tau^2-2\tau\sum_i(p_i-1)y^i dy^i d\tau+\tau^2\sum_i(dy^i)^2. \tag{5.41}$$

We see that the three-metric is manifestly conformally flat and thus *the time rate of change of the conformal three-metric vanishes*. Even though $\sigma_{ab}\neq 0$, we easily verify that $\Sigma_{ab}=0$ [Eq. (4.10)] and therefore gravitational degrees of freedom are entirely absent. The apparently dynamical behavior in (5.35) is purely kinematical and is eliminated if we use a minimal-distorition shift vector:

$$(L\beta)_{ab}=2\alpha\sigma_{ab}=\mathrm{constant}. \tag{5.42}$$

However, as noted above, in open cosmologies this shift vector is not unique, since one can add to it any solution of (5.33). In particular, one can add $-\partial/\partial y^i$ to β^i in (5.41). Along the new t^a trajectories we introduce new constant coordinates $z^i=\tau y^i$ and the line element becomes

$$ds^2=-\left[1-\sum_i (p_i z^i \tau^{-1})^2\right]d\tau^2 - 2\sum_i p_i z^i \tau^{-1} dz^i d\tau + \sum_i (dz^i)^2. \qquad (5.43)$$

The three-slices are in fact flat, not just conformally flat. It is interesting that this feature of the flat three-spaces, which we also saw in the Lemaître coordinates of Schwarzschild, is only manifest in the minimal-distortion coordinates.[73]

We turn now from the Kasner universe to the Kantowski-Sachs-Thorne[70] (KST) universe. This is a very interesting example since it has features representative of both black holes and cosmologies. Among the homogeneous cosmologies, the KST is the only one[47] not belonging to a Bianchi class, by virtue of its three-surfaces having topology $S_2\times R$. As was noted by its discoverers, it can be mapped isometrically onto the Kruskal-Schwarzschild[32] spacetime for $r<2M$. There the "natural" time slices are the $r_{\rm Sch}$ = constant hypercylinders ($S_2\times R$). These are homogeneous surfaces (independent to $t_{\rm Sch}$) and therefore are constant tr$K$ slices

$$\operatorname{tr}K=\frac{2}{\tau^2}\left(\frac{2M}{\tau}-1\right)^{-1/2}(\tau-\tfrac{3}{2}M) = \operatorname{tr}K(\tau), \qquad (5.44)$$

where we are using $\tau=r_{\rm Sch}$ as a time coordinate to emphasize the interchange of roles of time and space variables for $r_{\rm Sch}<2M$. Note that the KST cosmology has a "big bang" at $\tau=2M$, expands until tr$K=0$ at $\tau=r_{\rm Sch}=3M/2$, and recontracts to a singularity at $\tau=0$.

As in the Kasner universe, the use of a minimal-distortion shift vector removed the time dependence for the three-metric, which in normal coordinates reads

$$dl^2=\left(\frac{2M}{\tau}-1\right)^{-1} dt_{\rm Sch}^2+\tau^2 d\Omega^2. \qquad (5.45)$$

With a new radial coordinate

$$\rho=r_{\rm Sch}^{-3/2}(2M-r_{\rm Sch})^{1/2}t_{\rm Sch} \qquad (5.46)$$

defined to be constant along the minimal-distortion congruence, the four-metric decomposes as

$$\alpha=\left(\frac{\tau}{2M-\tau}\right)^{1/2}, \quad \beta^a=\frac{\tau(3M-\tau)}{2M-\tau}\,\rho\frac{\partial}{\partial\rho}, \qquad (5.47)$$

$$dl^2=\tau^2(d\rho^2+d\Omega^2). \qquad (5.48)$$

Note that, as in the Kasner universe (5.41), the lapse and three-metric are independent of the spatial variables because of the homogeneity, but the shift vector depends linearly on distance. This is necessary in order for the minimal-distortion shift vector to "undo" the coordinate shear in the extrinsic curvature. Our calculation verifies in a new way the fact that KST possesses no dynamical degrees of freedom.[71]

Let us return to the interpretation of KST as the $r_{\rm Sch}$ = constant slices of Kruskal-Schwarzschild spacetime for $r_{\rm Sch}<2M$. Consider now the tr$K=K_0$ = constant slicing of the full manifold. What has been found by Eardley and Smarr[43] and Brill *et al.*[46] is that such a slicing "wraps up" around an $r_{\rm Sch}=r_0$ constant slice inside the horizon ($r=2M$) where tr$K(\tau=r_0)=K_0$ using Eq. (8.44). Thus a "limit slice" exists for hyperboloid slices just as it does for maximal slices. This is very important since it means the hyperboloid slices combine the advantages of null infinity with the strong-field behavior of maximal slices.

As a final cosmological example, let us consider the mixmaster universe,[72] which is a closed Bianchi type-IX homogeneous universe.[47] Here, of course, the gravitational degrees of freedom *are* excited.

Following our discussion at the beginning of this section, the closed nature of the time slices (topology S_3) implies that the minimal-distortion shift vector vanishes modulo conformal Killing vectors (which provide only an overall rescaling of the coordinates). Thus, the anisotropic shear in the normal coordinates is *entirely* caused by gravitational shear.[27] Thus, there are no superfluous coordinate effects.

One could extend our analysis to the other Bianchi types and thus complete this study of homogeneous cosmologies. It would be very instructive if one could find a cosmology which is, in a sense, a Kasner universe crossed with a mixmaster universe, i.e., a cosmology in which the normal coordinates induce *both* kinematic and dynamic modes into the three-metric. The minimal-distortion shift would then remove just the kinematic terms. Candidates among the homogeneous cosmologies would require the three-metric of the time slices of homogeneity *not* to be conformally flat. For other reasons such a classification has been recently carried out by Spero and Szafron.[74] Using their results and notation we would claim that there are *no* dynamical modes excited (slices are conformally flat) in any Bianchi type-I, V, VI$_{-1}$($a-n_2=-n_3$), VII$_h$($n_2=n_3$), VII$_0$($n_1=n_2$), IX($n_1=n_2=n_3$), or Kantowski-Sachs-Thorne cosmology. Which ones among the others actually have $\Sigma_{ab}(\beta^c_{\rm min})\neq 0$ remains to be calculated. Also of interest would be to perform this analysis on inhomogeneous cosmologies, such as the Gowdy universe.

VI. CONCLUSIONS

We have discussed how the choice of a coordinate system in which to represent a spacetime metric can be analyzed geometrically. One considers a foliation or slicing of the spacetime to define the time coordinate. The three spatial coordinates are then constant along a congruence of curves which thread these slices. This decomposition defines two sets of "observer" world lines: those normal to the slices, termed "Eulerian," and those along the coordinate lines. The kinematics of these observers is then governed by the acceleration of the Eulerian observers and the three-velocity of the coordinate observers. These quantities are codified in the lapse function and shift vector of Arnowitt, Deser, and Misner.

The choice of slicing is crucial in order to obtain a nonsingular development of an initial-data slice. We illustrated this by comparing geodesic and maximal slicing of Kruskal-Schwarzschild spacetime. The use of the maximal slicing requires solving a curved-space elliptic equation on each slice. We presented a model flat-space example which seems to capture most of the features of the time development of the maximal lapse function. This model indicates that at late times the lapse drops exponentially to zero in strong-field regions.

With the slicing determined, the shift vector may be used to sort out the time dependence of the three-metric. In particular, the use of a certain "minimal-distortion" shift vector is seen to be of great kinematic value. Its properties were discussed in general and then in explicit examples. As models for more complicated spacetimes we investigated Minkowski spacetime, black-hole spacetimes, and homogeneous cosmologies. These familiar spacetimes seen from the present viewpoint show the way in which maximal or hyperboloid slicing and minimal-distortion shift vectors unify a wide range of coordinate systems. More importantly, these methods are *directly* applicable to spacetimes possessing no symmetries, which can be found from the numerical evolution of Einstein's equations.

Note added in proof. After this work was completed, Robert T. Jantzen communicated his results on the use of shift vectors in Bianchi cosmologies [in Relativistic Cosmology and Bianchi Universes, edited by R. Ruffini (to be published)]. He finds, by group-theoretical methods, that the natural shift vectors indeed do satisfy the minimal-distortion conditions that we have proposed.

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[1]L. Smarr, A. Čadež, B. DeWitt, and K. Eppley, Phys. Rev. D 14, 2443 (1976); L. Smarr, Ann. N.Y. Acad. Sci. (to be published); L. Smarr, Ph.D dissertation, Univ. of Texas, 1975 (unpublished); K. Eppley, Ph.D. dissertation, Princeton Univ., 1975 (unpublished).

[2]A. Lichnerowicz, J. Math. Pure Appl. 23, 37 (1944); Y. Choquet-Bruhat, in *Gravitation*, edited by L. Witten (Wiley, New York, 1962). See for a historical review B. S. DeWitt, Phys. Rev. 160, 1113 (1967).

[3]P. A. M. Dirac, Phys. Rev. 114, 924 (1959); R. Arnowitt, S. Deser, and C. W. Misner, in *Gravitation*, edited by L. Witten (Wiley, New York, 1962); J. L. Anderson, Rev. Mod. Phys. 36, 929 (1964); L. Smarr and J. W. York, Phys. Rev. D 17, 1945 (1978).

[4]For a review, see e.g., K. S. Thorne, Report No. OAP-462, 1976 (unpublished).

[5]For an important exception see P. D. D'Eath, Phys. Rev. D. (to be published).

[6]We use $a, b, c, \cdots$ to represent general spacetime tensor indices and $i, j, k, \cdots = 1, 2, 3$, when referring to a particular basis of spatial coordinates on a slice. Our conventions are such that spacelike vectors have positive norms, and curvature is defined as in C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973). Units are such that $8\pi G = c = 1$.

[7]A. Lichnerowicz, (Ref. 2); R. Arnowitt, S. Deser, and C. W. Misner (Ref. 3); J. A. Wheeler, in *Relativity, Groups, and Topology*, edited by C. DeWitt and B. DeWitt (Gordon and Breach, New York, 1964).

[8]We borrow this terminology from hydrodynamics. Our Eulerian observers are at rest in the chosen time slices. A similar point of view was expressed by F. A. E. Pirani, *Les Theories Relativistes de la Gravitation* (CNRS, Paris, 1962), pp. 85–91. See also A. E. Fischer and J. E. Marsden, J. Math. Phys. 13, 546 (1972).

[9]The geometric approach that replaces "coordinates" by "slices and congruences" in the present context owes much to the work of J. Stachel. See J. Stachel,

Acta. Phys. Polonica 35, 689 (1969); J. Stachel, Ph.D. dissertation, Stevens Institute of Technology, 1962 (unpublished).

[10]For the original work see Ref. 2. For a modern review see Ref. 6.

[11]J. A. Schouten, *Ricci Calculus* (Springer, New York, 1954); C. Cattaneo, Ann. Math. Ser. 4, 48, 46 (1959); J. Ehlers, P. Jordan, W. Kundt, and R. Sachs, Akad. Wiss. Lit. Mainz Abh. Math.-Natur. Kl, 11 (Mainz 4), 793 (1961); Ref. 9.

[12]N. Ó Murchadha and J. W. York, Phys. Rev. D 10, 428 (1974).

[13]We follow Dirac (Ref. 3), Arnowitt, Deser, and Misner (ADM, Ref. 2), Wheeler (Ref. 7), DeWitt (Ref. 2), and Fischer and Marsden (Ref. 8) in choosing $\alpha \equiv (g^{00})^{-1/2}$ and $\beta_i \equiv g_{0i}$ as our fundamental gauge variables. This is in contrast to the following authors who choose $(g_{00})^{-1/2} = (\alpha^2 - \beta_k \beta^k)^{-1/2}$ and $g_{0i}(g_{00})^{-1/2}$ as their gauge variables: C. Cattaneo (Ref. 11); H. D. Wahlquist and F.B. Estabrook, NASA Technical Report No. 32-868 (unpublished); V. D. Zakharov, *Gravitational Waves in Einstein's Theory* (Wiley, New York, 1972), pp. 115–135. These latter treatments are based on the time *congruence* as fundamental instead of the time *slices*. Their variables are suited to describing the local three-metric orthogonal to the time congruence t^a. However, such metrics cannot be fitted together globally to form the geometry of any slice. Hence, their variables are not the natural ones when we wish to describe the global "instantaneous state" of the gravitational field as required in a Cauchy problem. While this approach can be useful for some problems, it is not suited to imposing geometric coordinate conditions. We note that often in the literature (e.g., ADM above), the lapse function α is denoted by N and the shift vector β_i by N_i.

[14]The square brackets imply antisymmetrization over the indices enclosed.

[15]Equivalently, this family is called a "foliation" of spacetime of codimension one with spacelike leaves. Such a foliation is defined by a global one-form $\hat{n}_a$ such that $\hat{n}_a \wedge (d\hat{n})_{bc} = 0$, d = exterior derivative, $\wedge$= exterior product, and $g^{ab}\hat{n}_a \hat{n}_b = -1$. These conditions are equivalent to (2.14) and (2.2), respectively.

[16]K. Kuchař, J. Math. Phys. 17, 777 (1976).

[17]Y. Choquet-Bruhat (Ref. 2); J. Stachel (Ref. 9).

[18]R. Arnowitt, S. Deser, and C. W. Misner (Ref. 3); A. E. Fischer and J. E. Marsden (Ref. 8). If further variables representing first spatial derivatives of γ_{ab} are introduced, the Einstein equations can be treated as a completely first-order quasilinear hyperbolic system. See S. G. Hahn and R. W. Lindquist, Ann. Phys. (N.Y.) 29, 304 (1964); A. E. Fischer and J. E. Marsden, Commun. Math. Phys. 28, 1 (1972). Both of these papers impose particular coordinate conditions: the first geodesic normal and the second spacetime harmonic. For a discussion of this approach with general coordinates (i.e., α, β^i) see L. Smarr, Ref. 1.

[19]Using $(\gamma_{ab}, \mathcal{L}_t \gamma_{ab})$ as initial data corresponds to the "thin-sandwich" approach to the constraint problem. (See MTW, Ref. 6, Chap. 21.) Here α and β^a become the *unknowns* of the initial-value equations and are not available for use in sorting out the kinematics. Moreover, the constraint equations are quite complicated when written in terms of $(\gamma_{ab}, \mathcal{L}_t \gamma_{ab}, \alpha, \beta^a)$ and no useful results have been obtained in these terms, to our knowledge.

[20]Note that γ_{ab} is a "degenerate" spacetime tensor $(\gamma_{ab}\hat{n}^b = 0)$. We have chosen to define $\det(\gamma_{ab})$ geometrically using "volume elements" as $\alpha^{-2}\det(-g_{ab})$ in order to avoid the explicit introduction of a three-dimensional coordinate basis at this stage, i.e. (volume element on $\mathcal{T}$) = (α^{-1}) × (volume element of spacetime). This is permissible since a slicing is assumed to be given (the spacetime scalar α is known). Then $\mathcal{L}_t \det(\gamma_{ab})$ is well defined via $\mathcal{L}_t \alpha$ and $\mathcal{L}_t \det(-g_{ab})$. This leads to the usual coordinate formulas when $t^a = (\partial/\partial\tau)^a$, e.g., Eq. (2.26).

[21]J. R. Wilson, Astrophys. J. 173, 431 (1972); J. R. Wilson, Ann. N. Y. Acad. Sci. 262, 123 (1975); J. R. Wilson, in Proceedings of Enrico Fermi International School of Physics, Varenna, Italy, 1975 (unpublished); L. Smarr and J. R. Wilson (papers in preparation).

[22]For earlier attempts to separate kinematics from dynamics, see e.g. C. W. Misner, *Conférence Internationale les Théories Relativistes de la Gravitation*, edited by L. Infeld (Gauthier-Villars, Paris, 1964); F. B. Estabrook and H. D. Wahlquist (Ref. 13) and J. Math. Phys. 5, 1629 (1964); V. D. Zakharov (Ref. 13).

[23]Of course, one can characterize many of the physical properties of the evolving spacetime in terms of invariants of a spacetime Riemann tensor (see L. Smarr, Ref. 18, for examples). These quantities are independent of the choice of an initial slice and of α and β^a. However, one must still have a method of selecting t^a suitably in order that the resulting development covers a portion of the maximal development sufficient for computation of the invariants at late times.

[24]P. A. M. Dirac (Ref. 3); J. W. York, Phys. Rev. Lett. 26, 1656 (1971); 28, 1082 (1972); J. Math. Phys. 14, 456 (1973); N. Ó Murchadha and J. W. York (Ref. 12).

[25]D. R. Brill, Ann. Phys. (N.Y.) 7, 466 (1959); H. Araki, *ibid.* 7, 456 (1959); R. Arnowitt, S. Deser, and C. W. Misner, Phys. Rev. 110, 1100 (1960); 122, 997 (1961); D. R. Brill and S. Deser, Ann. Phys. (N.Y.) 50, 548 (1968); N. Ó Murchadha and J. W. York, Phys. Rev. D 10, 2345 (1974); Gen. Relativ. Gravit. 7, 257 (1976).

[26]A. Lichnerowicz (Ref. 2); Y. Choquet-Bruhat (Ref. 2); Commun. Math. Phys. 21, 211 (1971); Gen. Relativ. Gravit. 5, 47 (1974); D. R. Brill (Ref. 25); N. Ó Murchadha and J. W. York, Ref. 12 and J. Math. Phys. 14, 1551 (1973).

[27]J. W. York, Phys. Rev. Lett. 28, 1082 (1972).

[28]L. Smarr and J. W. York (Ref. 3).

[29]C. W. Misner, Phys. Rev. 186, 1319 (1969); 186, 1328 (1969).

[30]For discussion see, e.g., S. W. Hawking and G. F. R. Ellis, *The Large Structure of Space-Time* (Cambridge Univ. Press, New York, 1973), pp. 96–102.

[31]See Ref. 30, p. 95. In our notation, this implies $\rho + \mathrm{tr}\, S \geq 0$.

[32]See, e.g., C. W. Misner, K. S. Thorne, J. A. Wheeler (Ref. 6).

[33]Y. Choquet-Bruhat and R. Geroch, Commun. Math. Phys. 14, 329 (1969).

[34]A. Lichnerowicz (Ref. 2). For an extensive discussion of maximal slices and bibliography see Y. Choquet-Bruhat, A. E. Fischer, and J. E. Marsden, in Proceedings of Enrico Fermi International School of

Physics, Varenna, Italy, 1976 (unpublished).
[35]Lichnerowicz referred to such surfaces as "minimal" which would be true if the three-surface were embedded in a four-metric of signature (++++). For a proof of the maximization of volume see e.g. Y. Choquet-Bruhat, Ann. Sc. Norm. Super. Pisa, Serie IV, 3, 361 (1976).
[36]Appropriate asymptotic and "inner boundary" conditions for both lapse and shift equations are discussed in L. Smarr and J. W. York (Ref. 3).
[37]F. Estabrook, H. Wahlquist, S. Christensen, B. DeWitt, L. Smarr, and E. Tsiang, Phys. Rev. D 7, 2814 (1973).
[38]B. Reinhart, J. Math. Phys. 14, 719 (1973).
[39]See Eppley (Ref. 1). We thank Kenneth Eppley for providing some numerical results on the lapse function which we use in this paper.
[40]See L. Smarr, Ann. N. Y. Acad. Sci. (to be published) for a review of this work. Also, K. Eppley, J. Wilson, P. Chrzanowski, and J. Belli, work in progress.
[41]See, e.g., Ref. 37 and Ref. 40.
[42]M. Duncan, report, University of Texas, 1977 (unpublished).
[43]D. M. Eardley and L. Smarr, report, 1977 (unpublished).
[44]J. A. Wheeler (Ref. 7) used this flat-space technique on the elliptic equation resulting from D. R. Brill's study (Ref. 25) of the initial data for time-symmetric gravitational waves.
[45]Unpublished results from the work reported on in Ref. 37 and Ref. 39.
[46]For the use of functions of K_{ij} to define time slices see K. Kuchař, J. Math. Phys. 11, 3322 (1970); Phys. Rev. D 4, 955 (1971); For the use of $\mathrm{tr}(K)=\mathrm{constant}$ surfaces see J. W. York (Ref. 27); Y. Choquet-Bruhat (Ref. 35); D. R. Brill, J. Cavalle, and J. Isenberg report, 1976 (unpublished); A. J. Goddard, D. Phil. thesis, Oxford Univ., 1975 (unpublished), Gen. Relativ. Gravit. 8, 525 (1977); D.M. Eardley and L. Smarr (Ref. 43).
[47]For a review of this subject see M. P. Ryan, Jr. and L. C. Shepley, *Homogeneous Relativistic Cosmologies* (Princeton Univ. Press, Princeton, New Jersey, 1975).
[48]G. Barker, D. M. Eardley and L. Smarr (paper in preparation).
[49]Y. Choquet-Bruhat, C. R. Acad. Sci. (Paris) 280, 169 (1975); M. Cantor, A. Fischer, J. Marsden, N. Ó. Murchadha, and J. W. York, Commun. Math. Phys. 49, 187 (1976).
[50]J. W. York (Ref. 27); N. Ó Murchadha and J. W. York, J. Math. Phys. 14, 1551 (1973).
[51]This is the general relativistic generalization of the use of Lagrangian coordinates in hydrodynamics. P. Chrzanowski and J. Belli have used this condition in conjunction with maximal slicing in sperical star collapse (unpublished work).
[52]See, e.g., J. Ehlers, P. Jordan, W. Kundt, and R. Sachs (Ref. 11).
[53]If a flat background metric is available, the strain is the difference between the perturbed metric and the flat one. In our general approach no background is introduced. Therefore, we call the difference between two nearby nonflat metrics the strain. This is in accord with the terminology of elasticity theory. For an elementary discussion see, e.g., I. S. Sokolnikoff, *Tensor Analysis* (Wiley, New York, 1958) or L. Landau and E. M. Lifshitz, *Theory of Elasticity* (Addison-Wesley, Reading, Mass., 1959).
[54]J. W. York and L. Smarr, Bull. Am. Phys. Soc. 20, 544 (1975). For a discussion in component language, see L. Smarr and J. W. York (Ref. 3).
[55]This operator is discussed in detail in J. W. York, J. Math. Phys. 14, 456 (1973); Ann. Inst. Henri Poincaré 21, 319 (1974); M. Cantor, Composit. Math. (to be published).
[56]K. R. Eppley (Ref. 1) and report, 1977 (unpublished); M. Duncan (Ref. 42).
[57]An extensive discussion of the PPN approximation scheme is given in Ref. 6.
[58]B. K. Berger, J. Math. Phys. 17, 1268 (1976).
[59]See, e.g., J. M. Bardeen, in *Black Holes*, edited by C. DeWitt and B. S. DeWitt (Gordon and Breach, New York, 1973).
[60]J. M. Bardeen, Astrophys. J. 162, 71 (1970); R. D. Greene, E. L. Schucking, and C. V. Vishveshwara, J. Math. Phys. 16, 153 (1975).
[61]G. Lemaître, Ann. Soc. Sci. Bruxelles I A53, 51 (1933).
[62]I. D. Novikov, Sov. Astron.—AJ 7, 587 (1963). For a discussion see Ref. 6, p. 826.
[63]R. C. Tolman, Proc. Natl. Acad. Sci. USA 20, 169 (1934); H. Bondi, Mon. Not. R. Astron. Soc. 107, 410 (1947).
[64]D. L. Beckedorff, A. B. senior thesis, Princeton Univ., 1962 (unpublished); P. Szekeres, Nuovo Cimento 17B, 187 (1973); M. P. Korkina, Sov. Astron.—AJ 19, 185 (1975).
[65]D. E. Welty, A. B. senior thesis, Princeton Univ., 1976 (unpublished); L. Smarr, and D. E. Welty (paper in preparation).
[66]K. R. Eppley, L. Smarr, and S. Teukolsky (paper in preparation).
[67]F. J. Flaherty and D. Brill, Commun. Math. Phys. 50, 157 (1976); Y. Choquet-Bruhat (Ref. 35).
[68]A. R. King and G. F. R. Ellis, Commun. Math. Phys. 31, 209 (1973).
[69]See, e.g., D. M. Eardley, Commun. Math. Phys. 37, 287 (1974).
[70]R. Kantowski and R. K. Sachs, J. Math. Phys. 7, 443 (1966); K. S. Thorne, Ph.D. thesis, Princeton Univ., 1965 (unpublished).
[71]For earlier proofs see B. K. Berger, D. M. Chitre, V. E. Moncrief, and Y. Nutku, Phys. Rev. D 5, 2467 (1972); F. Lund, *ibid.* 8, 3247 (1973); W. G. Unruh, *ibid.* 14, 870 (1976).
[72]C. W. Misner, Phys. Rev. Lett. 22, 1971 (1969).
[73]Note that in these preferred slices, but with our shift vector, the three-metric is static $\gamma_{ij}=\delta_{ij}$ and the extrinsic curvature is purely longitudinal $K_{ij}=D_{(i}\beta_{j)}=P_i\tau^{-1}\delta_{ij}$ (no summation).
[74]A. Spero and D. A. Szafron, report, 1977 (unpublished). We thank L. P. Hughston, R. A. Matzner, and L. C. Shepley for conversations on this topic.