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Crichton ambiguities with infinitely many partial waves

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We construct families of spinless two-particle unitary cross sections that possess a nontrivial discrete phase-shift ambiguity, with in general an infinite number of nonvanishing partial waves. A numerical investigation reveals that some of the previously known finite Crichton ambiguities are merely special cases of the newly constructed examples.

I. INTRODUCTION

In 1966, Crichton[1] constructed two different elastically unitary scattering amplitudes, each with exactly the same modulus. Crichton's amplitudes contained only S, P, and D waves, but they provided an example of a nontrivial ambiguity in the determination of a scattering amplitude from a differential cross section. It was subsequently shown[2] that Crichton's phase-shift ambiguity was only one point in a continuous family of SPD examples that produced the same cross section. Next, twofold ambiguities were constructed with nonvanishing S, P, D, and F waves[3] and with S, P, D, F, and G waves.[4] Meanwhile, it had been shown that a twofold ambiguity was the maximal possible uncertainty in the case of an entire, nonpolynomial amplitude.[5] Very recently, Berends and Van Reisen[6] have constructed Crichton ambiguities of a special kind in which an arbitrary but finite number of partial waves is nonzero.

In all the above works, a doubt about the relevance of the Crichton ambiguities to actual elastic-region phase-shift analyses can legitimately be raised in view of the fact that physical partial-wave amplitudes should fall off exponentially as the angular momentum tends to infinity. Even when the partial-wave series is cut off sharply, as in unsophisticated partial-wave analyses, the highest partial waves that are retained should be small; and this feature is not present in the known Crichton examples. Thus, one may wonder whether there is a Crichton ambiguity at all when the partial-wave series is infinite and has a finite ellipse of convergence. We shall show in this paper that actually the ambiguity still exists when one allows an exponential tail, and that some of the known finite Crichton ambiguities are but special cases in which this tail fortuitously vanishes.

Gersten[7] observed that if $F(z)$ is a polynomial amplitude of order L in z, the cosine of the scattering angle, then one may write

$$F(z)=F(1)\prod_{i=1}^{L}\frac{z-z_i}{1-z_i}, \tag{1.1}$$

where the z_i are the zeros of the amplitude. New amplitudes may be obtained by complex conjugating some or all of the z_i, leaving the modulus of (1.1) invariant for real z. Thus all of the alternative amplitudes would correspond to the same differential cross section. However, if the energy is such that no inelastic channels are open, the amplitude must satisfy the equality conditions of elastic unitarity. In general these equalities are not preserved under complex conjugation of some or all of the z_i.

We follow Berends and Van Reisen[6] in limiting our search for an alternative unitary amplitude to the case that only one of the zeros, say z_1, is complex conjugated. Thus we intend to construct a new amplitude, $F'(z)$, such that

$$F'(z)=\frac{z-z_1^*}{1-z_1^*}\frac{1-z_1}{z-z_1}F(z), \tag{1.2}$$

in which $F(z)$ has a zero at $z=z_1$, and $F'(z)$ has a zero at $z=z_1^*$. However, we shall drop the artificial and unnecessary assumption that only a finite number of partial waves are nonvanishing. If we write in fact

$$F(z)=\frac{1}{2i}\frac{z-z_1}{1-z_1}\sum_{l=0}^{\infty}(2l+1)(\gamma_l-1)P_l(z), \tag{1.3}$$

then $F'(z)$ is obtained by replacing z_1 by z_1^*. The elastic unitarity condition for $F(z)$ may be written

$$|S_l|=1, \tag{1.4}$$

for $l=0,1,2,\ldots$, where

$$S_l=1+2iF_l, \tag{1.5}$$

F_l being the usual partial-wave amplitude of $F(z)$. By using the recurrence relations for the Legendre polynomials we find

$$S_l = \frac{\beta_l - z_1\gamma_l}{1 - z_1}, \tag{1.6}$$

where

$$\beta_l = \frac{(l+1)\gamma_{l+1} + l\gamma_{l-1}}{2l+1}. \tag{1.7}$$

Since we wish F' to be different from F, we require $\mathrm{Im}z_1 \neq 0$, and this means that if both amplitudes are to be unitary, then for all l,

$$|\beta_l|^2 - 2\,\mathrm{Re}z_1\,\mathrm{Re}(\beta_l^*\gamma_l) + |z_1|^2|\gamma_l|^2 = |1 - z_1|^2 \tag{1.8}$$

and

$$\mathrm{Im}(\beta_l^*\gamma_l) = 0. \tag{1.9}$$

We shall regard (1.8) as the main recurrence relation and (1.9) as a subsidiary condition. In fact, since we require $\gamma_l \to 1$ as $l \to \infty$, as a necessary condition for the convergence of the series (1.3) in some neighborhood of the physical region, than we can show that (1.9) is equivalent to

$$\mathrm{Im}(\gamma_l^*\gamma_{l-1}) = 0. \tag{1.10}$$

We may use the subsidiary condition (1.10) to divide the various solutions of the nonlinear recurrence relation (1.8) into classes. Observe first that, if all the γ_l are real, then $F'(z) = F^*(z)$. This is the well-known trivial ambiguity which is of no further interest. If γ_l is complex, then (1.10) says that γ_{l-1} must have the same phase, modulo π, unless it vanishes, and hence that γ_{l-2} must also have the same phase, unless it is zero, and so on. Thus we may divide the possible solutions into groups depending on which of the γ_l's are zero. Since the γ_l must be asymptotically real and we want to exclude the trivial ambiguity from consideration, it follows that at least one γ_l must be complex, and hence that at least one other γ_l must vanish. A particular family would consist of all amplitudes for which, say,

$$\gamma_{n_1} = \gamma_{n_2} = \cdots \gamma_{n_\mu} = 0.$$

Then the γ_l, for $0 \leq l < n_1$, would have to be relatively real (i.e., they all have the same phase, modulo π). For $n_1 < l < n_2$, the common phase could be different from that in the first interval, and so on, and for $l > n_\mu$, the γ_l's would all have to be real. We define the "class L" amplitudes as all those solutions of our equations for which $\gamma_{L-1} = 0$ and $\gamma_l \neq 0$ for $l \geq L$. Clearly L must be finite since convergence of the partial-wave series implies $\gamma_l \to 1$ as $l \to \infty$. The family to which we alluded above belongs to class $n_\mu + 1$, and the special case of a class L amplitude for which $\gamma_l = 1$ for $l \geq L$ is an Lth-order polynomial.

In Sec. II we shall consider class 2 amplitudes and show that some of the Crichton *SPD* solutions are close to amplitudes with an infinite number of partial waves. This is interesting, since it shows that the twofold ambiguities are much more common than one might have thought on the basis of the known polynomial examples. Continuous paths between the *SPD* ambiguity and some *SPDF* cases have already been plotted[8] by means of a computer program. The polynomial amplitude occurs as a special case when $\mathrm{Re}z_1$ takes on a particular value. We consider in Sec. II also the special case that arises when $\mathrm{Re}z_1 = 1$: Here the partial-wave series is unending, but the ellipse of convergence is infinite, so we have an entire amplitude of the sort considered by Itzykson and Martin.[5]

In Sec. III we discuss in more detail the numerical problems involved in the construction of phase-shift ambiguities with an infinite number of partial waves. We find ambiguities for $\mathrm{Re}z_1 > 1$, a region where phase-shift ambiguities for polynomial amplitudes do not exist, as well as for $\mathrm{Re}z_1 < 1$, and we show that polynomial solutions occur in the latter region when $\mathrm{Re}z_1$ takes on certain specific values. In Sec. IV we present detailed numerical results, in the form of tables of the phase shifts, in which the transition from amplitudes with finite ellipses of convergence to polynomials and to order-zero entire functions can be clearly seen.

We have solved completely the problem of constructing Crichton-type ambiguities associated with the complex conjugation of one zero of the scattering amplitude. However, the subject is far from exhausted since one could consider the complex conjugation of any finite or even an infinite number of zeros. Indeed, one can consider transformations that leave the cross section invariant but which do not amount to complex conjugation of zeros on the first Riemann sheet of the z plane. Despite the simplicity of our approach we have demonstrated that Crichton-type ambiguites are much more common than one might have thought from a consideration of the polynomial cases alone. In this paper, we have constructed examples in which the dispersive part only has poles inside the large Lehmann ellipse; but in fact it has already been shown by an implicit method[8] that Crichton-type ambiguities can be constructed in which the first singularity of the dispersive part is a branch point. Such a case corresponds in fact to a transformation more general than the complex conjugation of physical sheet zeros.

It is clear that certain classes of differential cross sections do not admit of Crichton-type ambiguities (in particular those for which the contraction mapping proof of Martin is applicable). One surely expects that many cross sections outside the contractive region will also not possess two different unitary amplitudes. However, the question is still open as to just how common the Crichton-type ambiguities are. Are most unitary amplitudes close to amplitudes that belong to Crichton pairs, or is this almost never the case? We do not yet know.

II. CLASS 2 AMPLITUDES

In this section, we shall be exclusively interested in the simple class of solutions of the subsidary condition (1.10) that is characterized by

$$\gamma_1 = 0\,, \tag{2.1}$$

so that γ_0 may be complex, but γ_l must be real for $l \geq 2$. Our task is to solve the recurrence relation (1.8), subject to these conditions, and subject also to the asymptotic requirement

$$\gamma_l \to 1\,, \tag{2.2}$$

as $l \to \infty$. It is convenient to introduce the notation

$$\gamma_l = 1 - \epsilon_l\,, \tag{2.3}$$

and in terms of ϵ_l, the relations (1.8) become

$$\eta_l^2 - 2\eta_l + 2\,\mathrm{Re}\, z_1(\eta_l + \epsilon_l - \eta_l \epsilon_l) + |z_1|^2(\epsilon_l^2 - 2\epsilon_l) = 0\,, \tag{2.4}$$

for $l \geq 2$, where

$$\eta_l = 1 - \beta_l = \frac{(l+1)\epsilon_{l+1} + l\epsilon_{l-1}}{2l+1}\,, \tag{2.5}$$

with the understanding that $\epsilon_1 = 1$ in accordance with (2.1). For $l=0$ and $l=1$, we find from (1.8) that

$$|\gamma_0| = \left|\frac{1-z_1}{z_1}\right| \tag{2.6}$$

and

$$\mathrm{Re}\gamma_0 = \frac{1}{4\gamma_2}\left(9|1-z_1|^2 - \left|\frac{1-z_1}{z_1}\right|^2 - 4\gamma_2^2\right). \tag{2.7}$$

One can first imagine trying a naive iteration of the system (2.4), (2.6), and (2.7). Choose a complex z_1 and a real γ_2 such that (2.6) and (2.7) can be solved for γ_0, i.e., such that $(\mathrm{Re}\gamma_0)^2 \leq |\gamma_0|^2$. Then solve (2.4) for η_l in terms of ϵ_l for $l = 2, 3, \ldots$. For a given l, ϵ_l and ϵ_{l-1} are already known, and so (2.5) allows ϵ_{l+1} to be calculated from η_l. Evidently, this iteration can be deemed successful only if all the discriminants of the quadratic equations for the η_l are non-negative and if

$$\epsilon_l \to 0\,, \tag{2.8}$$

as $l \to \infty$. If this asymptotic condition is indeed observed, then for l large enough we may approximate (2.4) by retaining only the terms linear in ϵ_l and η_l:

$$\frac{(l+1)\epsilon_{l+1} + l\epsilon_{l-1}}{2l+1} = u\epsilon_l\,, \tag{2.9}$$

where

$$u = \mathrm{Re}z_1 + \frac{(\mathrm{Im}z_1)^2}{\mathrm{Re}z_1 - 1}\,. \tag{2.10}$$

The general solution of (2.9) is a linear superposition of the Legendre functions of the first and second kinds with index l and argument u. The requirement (2.8) can be met by choosing z_1 such that $|u| > 1$, which implies $|z_1| > 1$, and by picking out the solution

$$\epsilon_l = CQ_l(u)\,, \tag{2.11}$$

where C is a real constant. The corresponding partial-wave series for $F(z)$ and $F'(z)$ would converge in the interior of the ellipse $E(u)$, defined by

$$E(u) = [z:\ |z + (z^2-1)^{1/2}| = |u| + (u^2-1)^{1/2}]. \tag{2.12}$$

In fact if $|u| \leq 1$, there is no solution of (2.9) that leads to an exponentially convergent partial-wave series (except for the trivial case of a polynomial), so we shall always consider $|u| > 1$.

The major difficulty with such an ascending iteration is that in general the asymptotic recurrence relation (2.9) would be ruined by an admixture of $P_l(u)$, which is inconsistent with (2.8), and indeed which invalidates the passage from (2.4) to (2.9). Evidently there is something like an eigenvalue condition operative, albeit a nonlinear one: We cannot choose z_1 and γ_2 freely if we want to be sure to hit the asymptotic form (2.11). We must somehow make γ_2 an implicit function of z_1; but in fact even if we knew how to do this, it would still be unadvisable to iterate the system (2.4) in the ascending mode. The reason is that rounding errors would accumulate and these would eventually allow a leakage of the unwanted solution, $P_l(u)$. Evidently it is much better to start with (2.11) for $l = L$ and $l = L+1$, for some large L and some C, and then to iterate the exact recurrence relation (2.4) in the descending mode down to $l = 2$. This yields finally η_2 and hence ϵ_1 or equivalently $\gamma_1 = 1 - \epsilon_1$. For a general choice of the asymptotic normalization C, the condition (2.1) would be violated, and so we must now construe γ_1 as an implicit function of C and determine the latter by the condition

$$\gamma_1(C) = 0\,. \tag{2.13}$$

Having done this, we would finally determine γ_0 from (2.6) and (2.7).

This scheme can only be fully implemented with the help of a computer, but before we embark upon such a numerical analysis, we shall show that the above proposal makes sense when $|\epsilon_l| \ll 1$ for all $l \geq 2$. In this case $|\eta_l| \ll 1$ for all $l \geq 3$, but

$$\eta_2 = \frac{3\epsilon_3 + 2\epsilon_1}{5} \approx \tfrac{2}{5}, \qquad (2.14)$$

since $\epsilon_1 = 1$ and ϵ_3 is negligible. Hence the linearization (2.9) of the nonlinear recursion relation (2.4) is valid down to $l=3$, at which value it yields

$$\epsilon_2 = C Q_2(u). \qquad (2.15)$$

On the other hand, ϵ_2 can be obtained from (2.4) for $l=2$, if we inject the approximate value $\frac{2}{5}$ for η_2. We linearize with respect to ϵ_2 and obtain

$$\epsilon_2 = \tfrac{2}{5}\frac{4 - 5\,\mathrm{Re}z_1}{3\,\mathrm{Re}z_1 - 5|z_1|^2}, \qquad (2.16)$$

which is only consistent with the above approximations if indeed the right-hand side is much smaller in absolute value than unity. If this is the case, then we can read off C by comparing (2.15) and (2.16), and then indeed $|\epsilon_l| \ll 1$ for all $l \geq 2$ since we assume z_1 to have been chosen so that $|u| > 1$.

A limiting case of the above approximation occurs if we set $\mathrm{Re}z_1 = \frac{4}{5}$ so that C and all ϵ_l for $l \geq 2$ vanish. The amplitude is then a second-order polynomial, namely the Crichton ansatz, which we write

$$F(z) = \tfrac{15}{2} e^{i\delta_2} \sin\delta_2 (z+\alpha)(z+\beta), \qquad (2.17)$$

where δ_2 is the D wave phase shift, where

$$\alpha = -z_1 = -\tfrac{4}{5} + \tfrac{1}{5} i \cot\delta_2, \qquad (2.18)$$

and where β satisfies Eqs. (20) and (21) of Ref. 2 which may be shown to be equivalent to (2.6) and (2.7) in the case $\gamma_2 = 1$. It should be noted that for the polynomial solution neither $|u|$ nor $|z_1|$ need be greater than unity, but only if $|u| > 1$ can we find solutions close to the Crichton amplitude that are analytic in the ellipse (2.12).

It should be noted that if $|u| > 1$ and if l is large enough, then the asymptotic form (2.11) should be a good approximation to the exact solution. However, if $\mathrm{Re}z_1$ is not close to $\frac{4}{5}$, then ϵ_l is not small for some low values of l, and one must solve the quadratic equations (2.4) instead of the linearized ones. For each l one has a choice between two solutions, and in practice not all choices lead to real solutions for all the ϵ_l down to ϵ_1. The condition (2.13) must finally be used to determine C by a method of successive approximations, as we shall see in Sec. III.

To conclude this section, we shall examine the exponential tail more closely. The amplitude (1.3) can be written

$$F(z) = \frac{1}{2i}\frac{z - z_1}{1 - z_1}[G(z) - \epsilon_0], \qquad (2.19)$$

where ϵ_0 is complex but $G(z)$ is real since γ_l is real for $l \geq 1$. The alternative amplitude, $F'(z)$, is obtained from this by replacing z_1 by z_1^* and hence,

$$-F'(z^*) = \frac{1}{2i}\frac{z - z_1}{1 - z_1}[G(z) - \epsilon_0^*]. \qquad (2.20)$$

From this it follows that

$$F(z) = -F'^*(z^*) - \frac{z - z_1}{1 - z_1}\mathrm{Im}\epsilon_0, \qquad (2.21)$$

and thus $F(z)$ and $-F'^*(z^*)$ differ only in their S and P waves. Hence, $\delta_l = -\delta_l'$ for $l \geq 2$ and so the two tails are identical (up to the trivial ambiguity). It is easy to generalize this proof to class L in which $\gamma_{L-1} = 0$, and γ_l is real for $l \geq L$, and we find $\delta_l = -\delta_l'$ for $l \geq L$. This is a result of possible physical significance, for often one has a model for the high-l tail, and our results show that this is not sufficient in general to remove the twofold ambiguity since our amplitudes have the same tails and differ only in the low partial waves.

Finally, one can see from (2.10)–(2.12) that as $\mathrm{Re}z_1 \to 1$, the ellipse of convergence of the partial-wave series tends to infinity, although the amplitude will not in general be a polynomial. We have here examples of twofold ambiguities with entire functions of the sort studied by Itzykson and Martin.[5] One can see directly that when $\mathrm{Re}z_1 = 1$, the solutions of (2.4) are

$$\eta_l = \epsilon_l \pm \mathrm{Im}z_1[\epsilon_l(2 - \epsilon_l)]^{1/2}. \qquad (2.22)$$

For sufficiently large l, $|\epsilon_l| \ll 1$, and so

$$\epsilon_l \approx \epsilon_{l-1}{}^2/t, \qquad (2.23)$$

where

$$t = 8(\mathrm{Im}\,z_1)^2. \qquad (2.24)$$

This approximate recurrence relation can be solved for $l \geq L$, where $L \gg 1$ and $|\epsilon_L| \ll t$, to give

$$\epsilon_l = t(\epsilon_L/t)^{2^{l-L}}, \qquad (2.25)$$

which tends to zero faster than any exponential, so that the corresponding amplitudes are entire functions. Indeed, since the difference between $F(z)$ and $-F'^*(z^*)$ is a polynomial, we know from the results of Itzykson and Martin[5] that the order of our functions is zero.

III. PROPERTIES OF THE DESCENDING ITERATION

In this section we shall discuss in more detail the numerical procedure outlined in Sec. II. We shall analyze the various conditions that a class n amplitude must satisfy. In particular, we shall show that z_1, the zero of the amplitude which is complex-conjugated, must lie in a bounded region of the complex z_1 plane. We also investigate the analytic properties of class n amplitudes.

In the actual calculation we do not linearize (2.4) but solve it for $\epsilon_{l-1}=1-\gamma_{l-1}$:

$$\epsilon_{l-1}=\frac{2l+1}{l}\epsilon_l \operatorname{Re}z_1-\frac{l+1}{l}\epsilon_{l+1}+\frac{2l+1}{l}(1-\operatorname{Re}z_1)\times\left[1-\tau_l\left(1+\frac{\operatorname{Im}^2 z_1}{(\operatorname{Re}z_1-1)^2}\epsilon_l(2-\epsilon_l)\right)^{1/2}\right], \tag{3.1}$$

where $\tau_l=\pm 1$. In principle we could choose either value of τ_l for every value of l. However, if we want ϵ_l to decrease exponentially for increasing l, it is clear that we must have $\tau_l=+1$ for all l greater than some L. In the special case of an Lth-order polynomial, $\epsilon_l=0$ for $l\geq L$ but $\epsilon_{L-1}\neq 0$. This implies $\tau_L=-1$, and then (3.1) yields

$$\epsilon_{L-1}=\frac{2(2L+1)}{L}(1-\operatorname{Re}z_1). \tag{3.2}$$

If we iterate (3.1) down to $l=2$ (for class 2 amplitudes) and find that $\epsilon_1=1$ and that we can solve (2.6) and (2.7) for γ_0, then we obtain indeed a polynomial ambiguity of degree L.

To construct a phase-shift ambiguity with an infinite number of partial waves, we must first of all choose a finite number of l values for which $\tau_l=-1$. We then choose z_1 such that $|z_1|>1$. Next we take a sufficiently large value of l, say M ($M\gg L$, where L is the highest l value for which $\tau_l=-1$), and some value of C, say C_1, and set

$$\epsilon_{M+1}=C_1Q_{M+1}(u),\quad \epsilon_M=C_1Q_M(u), \tag{3.3}$$

where u is given by (2.10). Then we use (3.1) to iterate down to $l=1$ (class 2 amplitudes). This is possible only if during the iteration the condition

$$1+\frac{\operatorname{Im}^2 z_1}{(\operatorname{Re}z_1-1)^2}\epsilon_l(2-\epsilon_l)\geq 0 \tag{3.4}$$

is satisfied for $l\geq 2$. Let us assume for the moment that this is the case. We have then obtained a value of ϵ_1, say $\epsilon_1^{(1)}$. We repeat this procedure with a second value of C, say C_2, and find $\epsilon_1^{(2)}$. Once we have two values of ϵ_1, we make a linear extrapolation to find an improved value which we then iterate:

$$C_{n+1}=\frac{\epsilon_1^{(n)}C_{n-1}-\epsilon_1^{(n-1)}C_n+C_n-C_{n-1}}{\epsilon_1^{(n)}-\epsilon_1^{(n-1)}}. \tag{3.5}$$

We have found that this simple procedure is usually sufficient to find a value of C such that

$$\epsilon_1(C)=1, \tag{3.6}$$

except, of course, whenever (3.6) is incompatible with (3.4). Once we have C such that (3.6) is satisfied, we use $\gamma_2=1-\epsilon_2$, corresponding to this value of C, to calculate γ_0 from (2.6) and (2.7). It is easily seen that γ_2 must satisfy

$$\frac{3}{2}\left|\frac{1-z_1}{z_1}\right|(|z_1|-\tfrac{1}{3})\leq|\gamma_2|\leq\frac{3}{2}\left|\frac{1-z_1}{z_1}\right|(|z_1|+\tfrac{1}{3}). \tag{3.7}$$

Once again this will not always be the case. The problem is then to choose z_1 such that (3.4), (3.6), and (3.7) are all satisfied, and this is essentially a numerical problem. Fortunately, we can restrict the possible values of z_1 considerably, as we shall now show.

First of all we shall prove, for amplitudes of class n, that $|\operatorname{Im}z_1|\leq 1$ is a necessary condition for the existence of an ambiguity of the type considered in this paper. For a class n amplitude, we have $\gamma_{n-1}=0$ and γ_l real for $l\geq n$ but γ_l complex for $l<n-1$. Condition (3.4) implies

$$|\gamma_l|\leq\frac{|1-z_1|}{|\operatorname{Im}z_1|}\ \text{ for } l\geq n. \tag{3.8}$$

We write

$$\gamma_{n-2}=e^{i\phi}\mu_{n-2},\quad \gamma_{n-3}=e^{i\phi}\mu_{n-3},$$

where μ_{n-2} and μ_{n-3} are real. (They are equal to plus or minus the modulus of the corresponding γ.) Thus,

$$S_{n-2}=\frac{e^{i\phi}}{1-z_1}\left(\frac{n-2}{2n-3}\mu_{n-3}-z_1\mu_{n-2}\right), \tag{3.9}$$

and from $|S_{n-2}|=1$ we obtain the condition,

$$|\mu_{n-2}|\leq\frac{|1-z_1|}{|\operatorname{Im}z_1|}. \tag{3.10}$$

Now we consider the unitarity condition for $l=n-1$. We have

$$S_{n-1}=\frac{1}{1-z_1}\left(\frac{n}{2n-1}\gamma_n+\frac{n-1}{2n-1}e^{i\phi}\mu_{n-2}\right), \tag{3.11}$$

and from $|S_{n-1}|=1$ we find, using (3.8) and (3.10),

$$|1-z_1|\leq\frac{n}{2n-1}|\gamma_n|+\frac{n-1}{2n-1}|\mu_{n-2}|\leq\frac{|1-z_1|}{|\operatorname{Im}z_1|},$$

which implies

$$|\operatorname{Im}z_1|\leq 1. \tag{3.12}$$

For class 2 amplitudes we can obtain a slightly better bound by combining (3.7) and (3.8). These two conditions imply we must require

$$\frac{|1-z_1|}{|\operatorname{Im}z_1|}\geq\frac{3}{2}\left|\frac{1-z_1}{z_1}\right|(|z_1|-\tfrac{1}{3}). \tag{3.13}$$

Condition (3.13) gives an upper bound for $|\operatorname{Im}z_1|$ which equals 1 for $\operatorname{Re}z_1=0$ and approaches $\frac{2}{3}$ from above as $|\mathrm{Re}\, z_1| \to \infty$.

We shall now show that $|\mathrm{Re} z_1|$ cannot be arbitrarily large. To do this we use (1.8), which we write in the form

$$\beta_l = \frac{(l+1)\gamma_{l+1} + l\gamma_{l-1}}{2l+1}$$
$$= \gamma_l \,\mathrm{Re} z_1 \pm (|1 - z_1|^2 - \gamma_l^2 \mathrm{Im}^2 z_1)^{1/2}. \qquad (3.14)$$

For simplicity in exposition, we shall first consider class 2 amplitudes and then generalize the result to arbitrary class. Since there is certainly a bound of the type

$$|\gamma_l| \leq B_1, \quad l \geq 1 \qquad (3.15)$$

where B_1 is independent of l, we know that also

$$|\beta_l| = |\gamma_l \,\mathrm{Re} z_1 \pm (|1 - z_1|^2 - \gamma_l^2 \mathrm{Im}^2 z_1)^{1/2}|$$
$$\leq B_1, \quad l \geq 2. \qquad (3.16)$$

Let us start with (3.8):

$$B_1 = \frac{|1 - z_1|}{|\mathrm{Im} z_1|}. \qquad (3.17)$$

It is easily seen that for $|\mathrm{Re} z_1| > 1$ we obtain from (3.16) a better bound for $|\gamma_l|$. In fact,

$$|\gamma_l| \leq B_2$$
$$= \frac{|1 - z_1|}{|z_1|^2 |\mathrm{Im} z_1|}[|\mathrm{Re} z_1| + |\mathrm{Im} z_1|(|z_1|^2 - 1)^{1/2}], \qquad (3.18)$$

which is valid for $l \geq 1$. Note that the right-hand side of (3.18) is already finite in the limit $|\mathrm{Re} z_1| \to \infty$. The bound (3.18) can be improved by repeating the argument which led us from (3.16) to (3.18). This iterative process can be continued as long as $B_{k+1} < B_k$, which implies

$$B_k > \frac{|1 - z_1|}{(|z_1|^2 - 2|\mathrm{Re} z_1| + 1)^{1/2}}. \qquad (3.19)$$

A detailed inspection of the right-hand side of (3.14) shows that for $|\mathrm{Re} z_1| > 1$, the B_k will approach the right-hand side of (3.19) arbitrarily closely. This means that

$$|\gamma_l| \leq \frac{|1 - z_1|}{(|z_1|^2 - 2|\mathrm{Re} z_1| + 1)^{1/2}}, \quad l \geq 1. \qquad (3.20)$$

For $\mathrm{Re} z_1 \geq 1$ the right-hand side is unity, which implies that all ϵ_l $(l \geq 1)$ are positive. If we compare (3.20) with (3.7) we find

$$\frac{|1 - z_1|}{(|z_1|^2 - 2|\mathrm{Re} z_1| + 1)^{1/2}} \geq \frac{3}{2}\left|\frac{1 - z_1}{z_1}\right|(|z_1| - \tfrac{1}{3}), \qquad (3.21)$$

which leads to the bound

$$|\mathrm{Re} z_1| \leq 1 + (\tfrac{2}{3})^{1/2}. \qquad (3.22)$$

The bound (3.20) is in fact valid for all l and for an arbitrary class if $|\mathrm{Re} z_1| > 1$. In order to demonstrate this, let us consider the general situation we described in Sec. I, i.e., $\gamma_{n_1} = \gamma_{n_2} = \cdots \gamma_{n_\mu} = 0$. If $|\mathrm{Re} z_1| > 1$ we already know that (3.20) holds for $l \geq n_\mu + 1$. We also know that (3.8) holds for all l. We write once again $\gamma_l = e^{i\Phi_l}\mu_l$, with μ_l real. The phase, Φ_l, is a constant for $n_i < l < n_{i+1}$, but can differ from one block to the next. Within each block, say for $n_i < l < n_{i+1}$, we follow the same procedure of successive bounds as in (3.16)–(3.20), but now for μ_l. Since the phases Φ_l disappear from the unitarity relation, we obtain

$$|\gamma_l| = |\mu_l| \leq \frac{|1 - z_1|}{(|z_1|^2 - 2|\mathrm{Re} z_1| + 1)^{1/2}}, \quad \text{all } l. \qquad (3.23)$$

We then find, from the unitarity relation for $l = n_\mu$,

$$|1 - z_1| = \left|\frac{(n_\mu + 1)\gamma_{n_\mu+1} + n_\mu \gamma_{n_\mu - 1}}{2n_\mu + 1}\right|$$
$$\leq \frac{|1 - z_1|}{(|z_1|^2 - 2|\mathrm{Re} z_1| + 1)^{1/2}} \qquad (3.24)$$

or

$$[|z_1|^2 - 2|\mathrm{Re} z_1| + 1]^{1/2} \leq 1,$$

which is valid for $|\mathrm{Re} z_1| > 1$. So for arbitrary class n we have the bounds (3.12) and (3.24), and for class 2 the tighter bounds (3.13) and (3.21). These bounds are shown in Fig. 1.

In the rest of this section, we shall be concerned with the analytic properties of class n amplitudes. Let us first of all assume that the corrections to the asymptotic form (3.3) are negligible. Then we

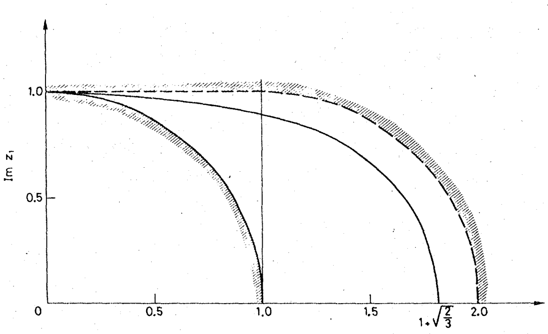

FIG. 1. Allowed regions for the zero, z_1. For class n amplitudes this region is bounded by the unit circle and the interrupted curve. For class 2 we must stay inside the solid curve. The regions in the other quadrants can be obtained by reflection with respect to the $\mathrm{Re} z_1$ and $\mathrm{Im} z_1$ axes.

find from (1.3)

$$F(z)=-\frac{1}{2i}\frac{z-z_1}{1-z_1}\sum_{l=0}^{\infty}(2l+1)\epsilon_l P_l(z)$$
$$\approx-\frac{1}{2i}\frac{z-z_1}{1-z_1}\left[C\sum_{l=0}^{\infty}(2l+1)Q_l(u)P_l(z)+R_{n-1}(z)\right], \quad (3.25)$$

where $R_{n-1}(z)$ is a polynomial of degree $n-1$. The remaining sum in (3.25) can be evaluated and we obtain

$$F(z)\approx-\frac{1}{2i}\frac{z-z_1}{1-z_1}\left[\frac{C}{u-t}+R_{n-1}(z)\right], \quad (3.26)$$

so that in this approximation $F(z)$ has a pole at $z=u$. We shall now show that, even if the nonlinear term in (3.1) is retained, the nearest singularity of F must be a pole. The argument appears in the paper of Itzykson and Martin.[5] Let F and F' be two elastically unitary amplitudes with the same modulus, and let $F+F'^*$ be a polynomial. We write

$$D(z)=\sum_{l=0}^{\infty}(2l+1)\,\mathrm{Re}F_l P_l(z), \quad (3.27a)$$

$$A(z)=\sum_{l=0}^{\infty}(2l+1)\,\mathrm{Im}F_l P_l(z). \quad (3.27b)$$

If z_0 is the nearest singularity of $D(z)$, then D is analytic inside a unifocal ellipse, $\epsilon(z_0)$, with semimajor axis z_0. Because of elastic unitarity, $A(z)$ is analytic in a larger ellipse, $\epsilon(z_1)$, with semimajor axis $2z_0^2-1=z_1$. We now use the fact that F and F' have the same modulus so that

$$[D(z)-D'(z)][D(z)+D'(z)]=-[A(z)-A'(z)][A(z)+A'(z)]. \quad (3.28)$$

Now, if $\Delta D=D-D'$ and $\Delta A=A-A'$, we have

$$2D\Delta D=-(2A-\Delta A)\Delta A+(\Delta D)^2. \quad (3.29)$$

Because ΔD and ΔA are polynomials, the right-hand side of (3.29) is analytic in $\epsilon(z_1)$ and therefore, the left-hand side must also be analytic in $\epsilon(z_1)$. This is possible only if the singularities of D inside $\epsilon(z_1)$ are poles which are canceled by zeros of ΔD. In the case of class 2 amplitudes it can be easily verified that ΔD has only one zero at $z=u$ and therefore, the only singularity allowed is a pole in D at $z=u$. In the examples we show in Sec. II, the partial waves clearly exhibit this dominant pole behavior.

If $\mathrm{Im}\,z_1$ vanishes, the expression (3.26) is exact since the nonlinear term in (3.1) vanishes. In this case we obtain an exact solution of the iteration (3.1), namely

$$\epsilon_l=Q_l(u)/Q_1(u), \quad (3.30)$$

where we have taken $\tau_l=+1$ for all $l\geq 2$. It is a simple exercise to show that in this case all partial waves with $l>1$ vanish, so that we are left with a single amplitude with only S and P waves. For $|\mathrm{Im}z_1|\ll 1$, the two amplitudes have almost identical S and P waves and differ only in the fact that the small asymptotic tails are related to one another by complex conjugation. It is interesting to note that, for $\mathrm{Im}z_1=0$, z_1 is no longer a zero of $F(z)$. This is because the position of the pole at $z=u$ coincides with $z_1=\mathrm{Re}z_1=u$. We find in fact that

$$A(\mathrm{Re}z_1)=\frac{1}{2}\frac{1}{(\mathrm{Re}z_1-1)Q_1(\mathrm{Re}z_1)}. \quad (3.31)$$

IV. NUMERICAL RESULTS

In this section we shall present some results obtained with the method discussed in the previous sections. We have limited our numerical work to the case of class 2 amplitudes, for a few special choices of the signs, τ_l, which occur in (3.1). However, the main features of our construction are quite clear from the few examples we shall show. As expected, we obtain regions in the complex z_1 plane in which all necessary conditions for a Crichton-type ambiguity are satisfied.

Let us first of all choose $\tau_l=+1$ for $l\geq 2$. It is clear that in this case there will be no solutions with $C=0$ since the latter would imply that the ϵ_l vanish for $l\geq 1$ so that (3.6) could not be satisfied. However, we have obtained solutions with $C\neq 0$, and the corresponding regions in the z_1 plane (regions I) are indicated in Fig. 2. There is a large region for $\mathrm{Re}z_1>1$. This region is bounded by the line $\mathrm{Re}z_1=1$ (where the amplitudes are entire functions), by curves on which $\mathrm{Im}\gamma_0=0$ (where the two amplitudes are related to one another by the trivial ambiguity), and by the line $\mathrm{Im}z_1=0$ (where the two amplitudes become polynomials of degree 1, being then equal to one another). At the point where the curves $\mathrm{Im}\gamma_0=0$ and the line $\mathrm{Im}z_1=0$ intersect, the real part of the amplitude vanishes

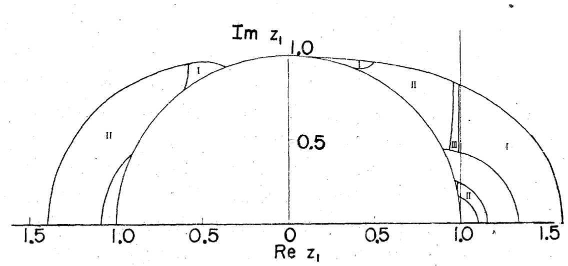

FIG. 2. Some of the regions in the complex z_1 plane where, for each z_1, a phase-shift ambiguity can be obtained. The regions I, II, and III correspond to different choices of the signs τ_l in Eq. (3.1).

completely [since $F'(z)=F(z)=-F^*(z^*)$], and we find amplitudes $\delta_0=0$, $\delta_1=\pi/2$ ($\mathrm{Re}z_1=1.6052$), and $\delta_0=\delta_1=\pi/2$ ($\mathrm{Re}z_1=1.3417$).

With the same choice of the τ_l we also obtain two small regions for $|\mathrm{Re}z_1|<1$. Let us consider the region in the first quadrant. This is bounded below by the unit circle, above by a line on which $\mathrm{Im}\gamma_0=0$, and on the right by a line on which we have $\epsilon_1=1$ but also

$$1+\frac{\mathrm{Im}^2 z_1}{(\mathrm{Re}z_1-1)^2}\epsilon_2(2-\epsilon_2)=0\,. \qquad (4.1)$$

If, with the same choice of τ_l, we try to find a solution for a z_1 further to the right, condition (3.4) will be violated for $l=2$.

It is worth noting that in region I, $0<\mathrm{Re}z_1<1$, the imaginary part in the forward direction is rather small for some z_1 values; it is even smaller than the Itzykson-Martin bound. In Ref. 5, it was shown that if the amplitude is an entire function of order 0 (including also polynomials) there is no ambiguity if $\mathrm{Im}F(1)<\frac{111}{80}=1.32$. We have in fact obtained ambiguities with $\mathrm{Im}F(1)$ as small as 1.28, showing that the conjecture that the Itzykson-Martin bound holds also for functions with singularities in the complex z plane is incorrect.

In region II of Fig. 2, we choose $\tau_2=-1$, $\tau_l=+1$ for $l\geq 3$. Here we obtain as a special case the *SPD* polynomials (for $\mathrm{Re}z_1=\frac{4}{5}$). This region is bounded on the right by a curve on which the left-hand side of (3.4) vanishes for $l=3$, and this time we can move further by choosing also $\tau_3=-1$. This process can be continued all the way to the line $\mathrm{Re}z_1=1$. In region III ($\tau_2=\tau_3=-1$) we have as a special case polynomials of degree 3; in region IV ($\tau_2=\tau_3=\tau_4=-1$), which is in fact too narrow to be shown in Fig. 2, we have polynomials of degree 4, etc. In the limit $\mathrm{Re}z_1\to 1$ we approach the entire function.

We have also generated regions in the z_1 plane in which, on the $\mathrm{Im}z_1=0$ line, the amplitudes reduce to *SPD* polynomials. There are two such regions, one for $\mathrm{Re}z_1<0$, one for $\mathrm{Re}z_1>1$, which are also called II in Fig. 2. In these regions the τ_l are chosen such that $\tau_2=-1$, $\tau_l=+1$ for $l\geq 3$. The region we obtain for $\mathrm{Re}z_1>1$ can again be continued to the unit circle by appropriate changes of the τ_l: Near the unit circle we must have $\tau_2=+1$, $\tau_3=\tau_4=-1$, $\tau_l=+1$ for $l\geq 5$, and again the number of negative τ_l increases as we approach the line $\mathrm{Re}z_1=1$ where again in the limit we obtain entire functions. Once again we find polynomials of increasing degree in this region, this time starting with degree 4.

Of course, it is possible, by other choices of the signs τ_l, to obtain many more regions in the z_1 plane. We have also found that these regions are similarly shaped in the case of class 3 amplitudes.

We shall now discuss the properties of the partial waves which we obtain. Let us start with Table I where we show δ_l and δ_l' for amplitudes in region I, $\mathrm{Re}z_1>1$, for various choices of the position of the zero z_1. We present the amplitudes $F(z)$ and $-F'^*(z^*)$ so that the δ_l and δ_l' are equal for $l\geq 2$ (see the discussion in Sec. II). In this region we have $u>1$, and so the amplitudes have a pole for $z>1$. It is therefore not surprising that all δ_l for $l\geq 2$ have the same sign. As $\mathrm{Re}z_1\to 1$, u increases and moves further away from the physical region, which of course has a clear effect on the decrease with l of the phase shifts. For $\mathrm{Im}z_1=0$ we find, as explained previously, a single amplitude with only S and P waves.

In Table II we give δ_l and δ_l' for representative values of z_1 in the region $0<\mathrm{Re}z_1<1$. Here we have $u<-1$ and $Q_l(u)$ now has the sign $(-1)^{l+1}$. This oscillating behavior is reflected in the sign of δ_l for sufficiently large l. In region I, z_1 is very close to the unit circle, and the modulus of the phase shifts decreases very slowly with l. In region II we pass the line $\mathrm{Re}z_1=\frac{4}{5}$, where C changes sign, and as a consequence the large l phase shifts change sign, too. The sign of C can be determined in this region from (2.16) since, close to the line $\mathrm{Re}z_1=\frac{4}{5}$, ϵ_2 will be well approximated by $CQ_2(u)$. In Table II we also give some solutions from region III, where we find an *SPDF* polynomial ambiguity. This $L=3$ solution is, of course, among the ambiguities obtained by Berends and Ruysenaars.[3] They have obtained several sets of class 2 ambiguities, but only for one of these is $|z_1|$ is greater than 1. This set, corresponding to Eq. (30) of their paper with parameter values $-1.097\leq x\leq -0.5465$, is in fact the set of $L=3$ ambiguities we obtain in region III. We have thus constructed a continuous connection between the *SPD* and *SPDF* ambiguities which can be continued to polynomials of arbitrary degree by moving closer and closer to the line $\mathrm{Re}z_1=1$. A continuous path between the *SPD* and *SPDF* case was found previously by an entirely different method.[8]

For some of the amplitudes in Tables I and II we have plotted the differential cross sections in Fig. 3. We expect the differential cross section to be dominated by the contribution from the pole at $z=u$. Indeed, the differential cross sections have either backward or forward peaks and, because we consider class 2 only, very little structure elsewhere. For solution A in Fig. 3, u is very close to -1 and the cross section almost vanishes, except for the large backward peak and a small contribution in the forward direction. For this

TABLE I. Phase shifts of amplitudes F and F' that correspond to the same differential cross section. For each pair of amplitudes we give the values of z_1, u, C, Im$F(1)$, and sinμ. Re$z_1 > 1$ only. The labels C, D, and E correspond to those in Fig. 3.

	C	Region I		
Rez_1	$1+10^{-5}$	1.1	1.2	1.3
Imz_1	0.7	0.65	0.6	0.5
u	4.9×10^5	5.325 0	3.000 0	2.133 33
c	1.1×10^{19}	191.133 73	36.524 23	14.018 05
Im$F(1)$	2.1367	2.3123	2.473 7	2.730 7
sinμ	3.19	3.43	3.86	4.10
δ_0, δ_0'	0.099 15 −1.059 21	0.140 76 −1.025 20	0.156 67 −0.942 07	0.260 49 −0.923 70
δ_1, δ_1'	0.911 84 0.658 97	0.982 80 0.740 64	1.057 84 0.834 71	1.167 36 0.943 85
δ_2, δ_2'	−0.223 58	−0.204 51	−0.180 36	−0.152 33
δ_3, δ_3'	−0.028 92	−0.036 41	−0.038 81	−0.039 18
δ_4, δ_4'	−0.000 53	−0.003 77	−0.006 56	−0.009 17
δ_5, δ_5'	-0.188×10^{-6}	−0.000 33	−0.001 04	−0.002 10
δ_6, δ_6'	-0.179×10^{-11}	-0.292×10^{-4}	−0.000 17	−0.000 49
δ_7, δ_7'	-0.171×10^{-16}	-0.258×10^{-5}	-0.266×10^{-4}	−0.000 11
δ_8, δ_8'	-0.164×10^{-21}	-0.230×10^{-6}	-0.428×10^{-5}	-0.264×10^{-4}
δ_9, δ_9'	-0.159×10^{-26}	-0.206×10^{-7}	-0.697×10^{-6}	-0.624×10^{-5}
$\delta_{10}, \delta_{10}'$	-0.154×10^{-31}	-0.186×10^{-8}	-0.114×10^{-6}	-0.148×10^{-5}

	D	Region I			E
Rez_1	1.4	1.45	1.5	1.5	1.5
Imz_1	0.4	0.3	0.2	0.1	0
u	1.80	1.650	1.580	1.52	1.500
c	8.464 12	6.511 52	5.663 47	5.032 96	4.829 10
Im$F(1)$	2.917 1	3.083 2	3.162 5	3.297 3	3.821 9
sinμ	4.51	4.60	4.82	4.67	4.64
δ_0, δ_0'	0.298 54 −0.805 64	0.391 59 −0.775 57	0.423 78 −0.671 74	0.571 10 −0.701 93	0.666 47 −0.666 47
δ_1, δ_1'	1.281 05 1.075 14	1.381 06 1.172 53	1.479 11 1.281 98	−1.562 70 1.365 30	−1.461 96 +1.461 96
δ_2, δ_2'	−0.116 03	−0.087 99	−0.056 69	−0.030 07	0
δ_3, δ_3'	−0.032 92	−0.026 63	−0.017 66	−0.009 74	0
δ_4, δ_4'	−0.009 13	−0.008 11	−0.005 64	−0.003 26	0
δ_5, δ_5'	−0.002 54	−0.002 50	−0.001 83	−0.001 11	0
δ_6, δ_6'	−0.000 71	−0.000 78	−0.000 60	−0.000 39	0
δ_7, δ_7'	−0.000 20	−0.000 25	−0.000 20	−0.000 14	0
δ_8, δ_8'	-0.577×10^{-4}	-0.783×10^{-4}	-0.677×10^{-4}	-0.478×10^{-4}	0
δ_9, δ_9'	-0.166×10^{-4}	-0.251×10^{-4}	-0.229×10^{-4}	-0.170×10^{-4}	0
$\delta_{10}, \delta_{10}'$	-0.479×10^{-5}	-0.806×10^{-5}	-0.778×10^{-5}	-0.608×10^{-5}	0

TABLE II. The same as Table I, but now, for z_1 such that $\mathrm{Re}z_1 < 1$. The labels A and B correspond to those in Fig. 3.

	A		Region II		B	
$\mathrm{Re}z_1$	0.4	0.5	0.6	0.7	0.8	0.9
$\mathrm{Im}z_1$	0.95	0.9	0.9	0.85	0.8	0.75
u	−1.104 17	−1.12	−1.425	−1.708 3	−2.4	−4.725
c	0.654 07	0.575 89	1.286 80	1.342 84	0	−32.372 42
$\mathrm{Im}F(1)$	1.402 4	1.535 4	1.505 5	1.649 7	1.801 4	1.964 0
$\sin\mu$	2.28	2.15	2.60	2.64	2.78	2.97
δ_0, δ_0'	−0.244 34	−0.126 56	−0.173 39	−0.087 58	−0.017 95	0.044 23
	−0.717 50	−0.887 64	−0.832 84	−0.940 63	−1.012 43	−1.052 84
δ_1, δ_1'	0.577 69	0.644 45	0.671 26	0.730 78	0.787 72	0.847 11
	0.429 79	0.419 25	0.481 31	0.500 72	0.538 10	0.591 13
δ_2, δ_2'	−0.262 32	−0.270 45	−0.248 35	−0.248 59	−0.244 98	−0.236 94
δ_3, δ_3'	0.096 48	0.087 29	0.039 05	0.019 33	0	−0.016 52
δ_4, δ_4'	−0.056 80	−0.047 36	−0.015 04	−0.005 80	0	0.001 38
δ_5, δ_5'	0.031 80	0.025 65	0.005 48	0.001 68	0	−0.000 14
δ_6, δ_6'	−0.018 99	−0.014 83	−0.002 09	−0.000 50	0	0.134×10^{-4}
δ_7, δ_7'	0.011 19	0.008 46	0.000 80	0.000 15	0	-0.134×10^{-5}
δ_8, δ_8'	−0.006 74	−0.004 93	−0.000 31	−0.000 05	0	0.135×10^{-6}
δ_9, δ_9'	0.004 06	0.002 87	0.000 12	0.142×10^{-4}	0	-0.137×10^{-7}
$\delta_{10}, \delta_{10}'$	−0.002 46	−0.001 69	−0.000 05	-0.437×10^{-5}	0	0.139×10^{-8}

		Region III	
$\mathrm{Re}z_1$	0.97	0.980 59	0.99
$\mathrm{Im}z_1$	0.7	0.7	0.7
u	−15.363 3	−2.426 3	−48.01
c	−836.797 80	0	7.277×10^{4}
$\mathrm{Im}F(1)$	2.112 07	2.120 98	2.128 71
$\sin\mu$	3.11	3.14	3.16
δ_0, δ_0'	0.115 96	0.110 40	0.105 13
	−1.104 48	−1.088 96	−1.074 75
δ_1, δ_1'	0.896 39	0.901 91	0.906 76
	0.631 57	0.641 16	0.649 76
δ_2, δ_2'	−0.231 56	−0.228 74	−0.226 23
δ_3, δ_3'	−0.026 97	−0.027 72	−0.628 33
δ_4, δ_4'	0.000 29	0	−0.000 26
δ_5, δ_5'	-0.863×10^{-5}	0	0.240×10^{-5}
δ_6, δ_6'	0.259×10^{-6}	0	-0.230×10^{-7}
δ_7, δ_7'	-0.789×10^{-8}	0	0.224×10^{-9}
δ_8, δ_8'	0.242×10^{-9}	0	-0.220×10^{-11}
δ_9, δ_9'	-0.747×10^{-11}	0	0.217×10^{-13}
$\delta_{10}, \delta_{10}'$	0.232×10^{-12}	0	-0.215×10^{-15}

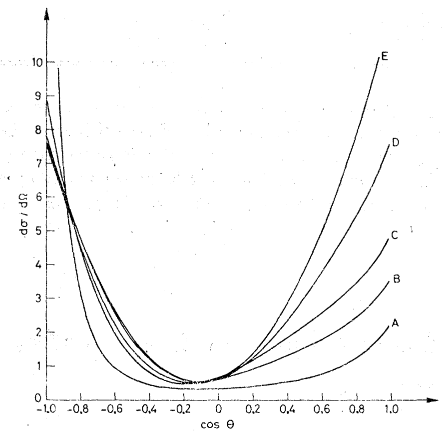

FIG. 3. The differential cross sections corresponding to some of the amplitudes in Tables I and II. The labels A–E correspond to those in the Tables.

amplitude the total cross section is very small. As we go from solution A to solutions B and C, the pole moves further away from the physical region and the backward peak becomes less pronounced. For solutions D and E, the pole lies to right of the physical region and a forward peak appears in the differential cross section.

For completeness, we have calculated the $\sin\mu$ parameter of Martin.[9] Martin showed that if the functional of the differential scattering cross section

$$\sin\mu = \sup_{1,2} \int \frac{d\Omega_3}{4\pi} \frac{|F(1,3)|\,|F(2,3)|}{|F(1,2)|}$$

is smaller than 0.79, then no nontrivial ambiguity exists.

We have obtained examples with $\sin\mu$ values that are reasonably small, but they never approach the theoretical bound closely.

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