PHYSICAL REVIEW D VOLUME 17, NUMBER 9 1 MAY 1978

Classical space-time concepts in high-energy collisions

F. E. Low
Center for Theoretical Physics, Laboratory for Nuclear Science and Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

K. Gottfried
Laboratory of Nuclear Studies, Cornell University, Ithaca, New York 14853
(Received 19 December 1977)

We show that the observed rapidity dependence of production amplitudes implies a simple classical picture of the longitudinal evolution of the collision.

I. INTRODUCTION

In this paper we discuss the distribution in space and time of the relativistic particles produced in the central region during a high-energy collision. We shall show that the qualitative features of the observed inclusive cross sections imply that the asymptotic final state can be described by a simple semiclassical picture. We first sketch this picture in qualitative terms.

For the sake of definiteness, consider a pp collision in the center-of-mass frame. Choose coordinates so that the colliding particles meet at $t=z=0$, where z is measured along the incident direction. Let ω, m, q_l, and y be the energy, mass, longitudinal momentum, and longitudinal rapidity of the secondary whose trajectory is under discussion.

First note that $\Delta z \Delta q_l \gtrsim 1$, and $\Delta q_l/\omega = \Delta y$ imply that

$$\Delta z \Delta y \gtrsim 1/\omega .$$

This inequality states that one can simultaneously specify the longitudinal coordinate and rapidity of ultrarelativistic objects. The reason is clear from $y \propto \ln q_l$: An uncertainty Δy contains a band of momenta sufficiently broad to permit the construction of a packet that is as narrow as one pleases, provided only that the mean momentum of that band is large enough. Such a packet will then move along a classical trajectory. (The spreading of the packet will be discussed in Sec. III.) A tacit assumption in all this is that production amplitudes only vary appreciably on a logarithmic energy scale—that rapidity is a sufficiently fine-grained variable. On the other hand, the rough energy independence of transverse-momentum distributions means that the transverse localization is of order 10^{-13} cm independent of ω. Hence no classical picture can be associated with the transverse motion.

The argument of the preceding paragraph shows that the longitudinal motion of relativistic secondaries *may* be described by a classical trajectory, but it does not establish in what space-time domain such a description is valid. Our answer to this is best visualized by imagining a set of stationary observers in the c.m. frame. For $t>0$, they see two pulses of hadronic matter receding at the speed of light in opposite directions. The first particle seen to emerge from these pulses is the one of smallest final $|y|$; loosely speaking, it "peels off" the back of one of the receding pulses. Other secondaries follow suit in order of increasing $|y|$. The times t and positions z at which separation occurs are $t\sim(\omega/m)\tau_0$ and $z\sim q_l t/m$, where τ_0 is a characteristic hadronic proper-time scale ($\sim 10^{-23}$ sec). *After* separation, these secondaries move along classical trajectories, thus *appearing as if* they had all been made at the collision point $t=z$

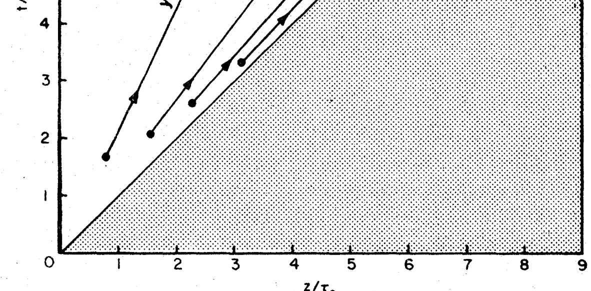

FIG. 1. This figure shows the trajectories of various secondaries in the c.m. system after they are peeled off the back of the pulse receding to $z\rightarrow+\infty$. The dots from which the trajectories stem occur at the times t_d defined in Sec. III.

$=0$ with their finally observed velocities q_i/ω. (This space-time development is illustrated in Fig. 1.)

The picture we have just sketched coincides with the classical model originally proposed by one of us,[1] though we hasten to add that there the picture was also used in the interaction region. The use of classical trajectories in the interaction region is obviously a strong dynamical assumption. The argument to be given here makes the tacit assumption that the dynamics leads to the observed amplitudes, but is otherwise almost totally devoid of dynamical content[2]; it therefore sheds no light on what happens in the interaction region.

II. WAVE-PACKET CONSTRUCTION OF THE ENERGY FLOW

We construct an initial state representing a collision at $z=t=0$ via a superposition of "in" states,

$$\Psi=\int d^3p_1\,d^3p_2\,\phi_a(p_1)\phi_b(p_2)\Psi^{\text{in}}_{p_1p_2}\,. \tag{2.1}$$

One may think of ϕ_a and ϕ_b as Gaussian packets, centered at $\vec{p}_a$ and $\vec{p}_b$, and real so that the collision occurs at the space-time origin:

$$\phi_a(p_1)=\frac{\exp[-(p_{1z}-p_a)^2/2\Delta_l{}^2]\exp(-p_{1\perp}{}^2/2\Delta_\perp{}^2)}{\pi^{3/4}\Delta_l\Delta_\perp{}^2}\,,$$

and similarly for $\phi_b(p_2)$. As explained in Sec. I, the produced particles are to be described by packets having a fixed uncertainty in rapidity. We shall soon show [see Eq. (2.6)] that the uncertainty of the secondary's packet is essentially the same as that of the incident particles. We therefore set

$$\Delta_l=\epsilon_l\omega\,, \tag{2.2}$$

where ω is the energy of the secondary in question, and ϵ_l is the energy-independent rapidity spread of the packet. The transverse-momentum uncertainty is energy independent, and on occasion we parametrize it as

$$\Delta_\perp=\epsilon_\perp m\,. \tag{2.2$'$}$$

As we shall see, both ϵ_l and $\epsilon_\perp$ must be taken as small compared to unity.

We wish to study the emergence of produced particles. A concrete way of doing this is to calculate the expectation value of some local observable after the collision. For charged particles one can use the current density, but this will not do for neutrals. In all cases one can use the stress tensor $T_{\mu\nu}(x)$, and we therefore focus on it.[3]

We wish to calculate the expectation value

$$\langle T_{\mu\nu}(x)\rangle\equiv\langle\Psi|T_{\mu\nu}(x)|\Psi\rangle \tag{2.3}$$

for t sufficiently large so that the collision is over and the secondaries are spatially separated. For such times the appropriate states are the "out" states, Ψ^{out}_n, in terms of which $T_{\mu\nu}$ behaves as if it were constructed from free fields. Thus

$$\begin{aligned}\langle T_{\mu\nu}(x)\rangle&=\sum_{nm}\langle\Psi|\Psi^{\text{out}}_n\rangle\langle\Psi^{\text{out}}_n|T_{\mu\nu}(x)|\Psi^{\text{out}}_m\rangle\langle\Psi^{\text{out}}_m|\Psi\rangle\\&=\sum_{nm}\int\langle n|S|p_1'p_2'\rangle^*\langle n|T_{\mu\nu}(x)|m\rangle\langle m|S|p_1p_2\rangle\phi_a^*(p_1')\phi_a^*(p_2')\phi_b(p_2)\phi_b(p_2)d^3p_1d^3p_2d^3p_1'd^3p_2'\,.\end{aligned} \tag{2.4}$$

For the moment assume there is only one species of secondaries. Then $|m\rangle=|\vec{q}_1\cdots\vec{q}_M\rangle$, $|n\rangle=|\vec{q}_1{}'\cdots\vec{q}_N{}'\rangle$. Since we are concerned with times after the particles have separated, $T_{\mu\nu}$ will only have matrix elements between states of equal multiplicity, and these elements are sums of one-particle matrix elements. Then we can write the spin-averaged tensor as

$$\langle n|T_{\mu\nu}(x)|m\rangle=\delta_{NM}\sum_{i=1}^{N}t_{\mu\nu}(q_i',q_i)e^{i(q_i-q_i')\cdot x}\prod_{j\neq i}(2\pi)^3 2\omega_j\delta^3(\vec{q}_j-\vec{q}_j')\,, \tag{2.5}$$

where $t_{00}(q,q)=2\omega^2$. Thus (2.4) becomes

$$\begin{aligned}\langle T_{\mu\nu}(x)\rangle=\sum_N N\int(dq_1)(dq_1')(dq_2)\cdots(dq_N)\langle\vec{q}_1'\vec{q}_2\cdots\vec{q}_N|\mathfrak{M}|\vec{p}_1'\vec{p}_2'\rangle^*(2\pi)^4\delta^4\Big(P-\sum_k q_k\Big)\langle\vec{q}_1\vec{q}_2\cdots\vec{q}_N|\mathfrak{M}|\vec{p}_1\vec{p}_2\rangle\\\times t_{\mu\nu}(q_1',q_1)e^{i(q_1-q_1')\cdot x}(2\pi)^4\delta^4(P-P'-q_1+q_1')\phi_a^*(p_1')\phi_b^*(p_2')\phi_a(p_1)\phi_b(p_2)d^3p_1d^3p_2d^3p_1'd^3p_2'\,,\end{aligned} \tag{2.6}$$

where we used the identity of the secondaries, defined $P=p_1+p_2$, $P'=p_1'+p_2'$, and $(dq)\equiv d^3q/(2\pi)^3 2\omega$, and introduced the invariant production amplitude $\mathfrak{M}$.

The second δ function in (2.6) guarantees that $\vec{q}_1-\vec{q}_1'$ is constrained by the momentum spread $(\Delta_l,\Delta_\perp)$ of the original packet, and as a consequence ω_1 is close to ω_1'. To take advantage of this, define $\vec{k}=\vec{q}_1-\vec{q}_1'$, $\vec{q}=\frac{1}{2}(\vec{q}_1+\vec{q}_1')$. Since the uncertainties are assumed to satisfy $(\Delta_l^2+\Delta^2)\ll\omega^2(q)$, we have

$$\omega_1 \equiv \omega(q+\tfrac{1}{2}k) \simeq \omega(q) + \tfrac{1}{2}\vec{k}\cdot\vec{v}_q\,,$$
$$\omega_1' \equiv \omega(q-\tfrac{1}{2}k) \simeq \omega(q) - \tfrac{1}{2}\vec{k}\cdot\vec{v}_q\,, \quad (2.7)$$

where $\vec{v}_q = \vec{q}/\omega(q)$ is the velocity of a particle of mass m and momentum $\vec{q}$.

In Sec. IV we shall show that the production amplitudes vary negligibly over the momentum spread of our packets.[4] Accepting this, and recalling the definition of the cross section σ_{ab}^s for $a+b\to s+$ anything, we can then write (2.6) as

$$\langle T_{\mu\nu}(x)\rangle = \sum_s \frac{d\sigma_{ab}^s}{d^3q}\,\frac{t_{\mu\nu}^s(q)}{2\omega_s(q)}\,\rho_q^s(x) F_{ab}\, d^3q\,, \quad (2.8)$$

where F_{ab} is the flux $4E_aE_bv_{ab}$, and the density $\rho_q^s(x)$ is given by

$$\rho_q^s(x) = \int \frac{d^3k}{(2\pi)^3}\, e^{i(q_1-q_1')\cdot x}\phi_a^*(p_1')\phi_b^*(p_2')\phi_a(p_1)\phi_b(p_2) \times (2\pi)^4\delta^4(P-P'-q_1+q_1')\,d^3p_1\cdots d^3p_2'\,. \quad (2.9)$$

In (2.8) we have removed the restriction that there is only one kind of secondary by the indicated sum on s.

The normalization of (2.8) and (2.9) can be clarified by calculating the total number of events in the collision, and the energy of the final state. To do this, we note from (2.1) that the amplitude for producing a final state Ψ_n^{out} is $(\Psi_n^{\text{out}},\Psi)$, and that the probability of any event is therefore

$$\mathcal{P} = \sum_n |(\Psi_n^{\text{out}},\Psi)|^2\,. \quad (2.10)$$

Now

$$(\Psi_n^{\text{out}},\Psi_{p_1p_2}^{\text{in}}) = -i(2\pi)^4\delta^4(p_n-p_1-p_2)\langle n|\mathfrak{M}|\vec{p}_1\vec{p}_2\rangle, \quad (2.11)$$

and the total cross section is

$$\sigma_T = \frac{(2\pi)^4}{F_{ab}}\sum_n \delta^4(p_1+p_2-p_n)\,|\langle n|\mathfrak{M}|\vec{p}_1\vec{p}_2\rangle|^2\,.$$

Assuming again that $\mathfrak{M}$ varies but slightly over the packet, we have

$$\mathcal{P} = (2\pi)^4\sigma_T F_{ab}\int d^3p_1 d^3p_2 d^3p_1' d^3p_2'\delta^4(P-P') \times \phi_a^*(p_1)\phi_b^*(p_2)\phi_a(p_1)\phi_b(p_2)\,. \quad (2.12)$$

Following the collision the energy density per event is thus

$$\langle T_{00}(x)\rangle_{\text{per event}} = \frac{1}{\sigma_T}\sum_s \int \frac{d\sigma_{ab}^s}{d^3q}\,\omega_s(q)\,\hat{\rho}_q^s(x)\,d^3q\,, \quad (2.13)$$

where

$$\hat{\rho}_q^s(x) = \frac{\sigma_T F_{ab}}{\mathcal{P}}\,\rho_q^s(x)\,, \quad (2.14)$$

and where we have used $t_{00}^s(q) = 2[\omega_s(q)]^2$. By virtue of (2.12) and (2.9),

$$\int d^3x\,\hat{\rho}_q^s(x) = 1\,. \quad (2.15)$$

That (2.13) is correctly normalized is now verified by evaluating the total energy following the collision:

$$\int d^3x\,\langle T_{00}(x)\rangle_{\text{per event}} = \int d^3q \sum_s \frac{1}{\sigma_T}\,\frac{d\sigma_{ab}^s}{d^3q}\,\omega_s(q)\,,$$

and this, by the energy-conservation sum rule, is the incident energy. Q.E.D.

III. THE SPACE-TIME DISTRIBUTION $\rho_q^s(x)$

In the preceding section we showed [recall Eq. (2.8)] that the expectation value of the energy-momentum tensor $T_{\mu\nu}(x)$ following the collision can be expressed as an integral over several factors: the one-body inclusive cross section, the momentum-space density $t_{\mu\nu}^s(q)$, and the function $\rho_q^s(x)$. The latter has the unambiguous physical interpretation of giving the space-time distribution of secondaries of type s and momentum $\vec{q}$. If one were to study the final state by means of some other local observable, say a scalar or vector density, the final result would again have the form of (2.8), but with $t_{\mu\nu}^s(q)$ replaced by the appropriate momentum-space density. In all cases $\rho_q^s(x)$ would enter and have the interpretation stated above.

The study of $\rho_q^s(x)$ is facilitated by introducing

$$G(\vec{k},\vec{q}) = \int \phi_a^*(p_1')\phi_b^*(p_2')\phi_a(p_1)\phi_b(p_1) \times (2\pi)^4\delta^4(P-P'-q_1+q_1')\,d^3p_1\cdots d^3p_2'\,, \quad (3.1)$$

in terms of which

$$\rho_q^s(x) = \int \frac{d^3k}{(2\pi)^3}\, e^{i[\vec{k}\cdot\vec{x}-(\omega_1-\omega_1')t]}G(\vec{k},\vec{q})\,. \quad (3.2)$$

For sufficiently short times it is legitimate to replace ω_1 and ω_1' by (2.7), and then (3.2) becomes

$$\rho_q^s(\vec{x},t) = \int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot(\vec{x}-\vec{v}_q t)}\,G(\vec{k},\vec{q})\,. \quad (3.3)$$

This confirms the statement made in Sec. I: The produced particle appears just as if it had left the origin at time $t=0$ with its final velocity.

Although (3.3) makes it appear as if the particle travels along the classical trajectory $\vec{x}=\vec{v}_q t$ from $t=0$, it must be remembered that our basic result (2.8) is only valid once the particle in question is sufficiently separated so that its interactions may be ignored. We therefore must calculate the detachment time t_d. For this purpose we go to the particle's rest frame, where the detachment time is the time required for it to be separated by a distance of order one Fermi from its nearest neighbor. Assuming a logarithmic multiplicity, nearest neighbors are separated by a finite rapidity gap δ, and so the rest frame detachment time τ_0 will be of order 10^{-23} sec. In a fixed frame this corresponds to a time $t_d \sim [(\omega(q)/m]\tau_0$, and therefore to a position $(q_l/m)\tau_0$, as stated in Sec. I and depicted in Fig. 1.

We must still ascertain whether the step from (3.1) to (3.2) is valid for the time span of interest to us; i.e., whether the spreading of the packet is ignorable. The time interval T over which (3.2) holds is found by requiring the corrections to the exponent to be small, i.e.,

$$|\tfrac{1}{24}T(\vec{k}\cdot\vec{\nabla}_q)^3\omega(q)| \sim 1\,,$$

or (3.4)

$$T \lesssim \frac{8\omega^5}{\vec{k}\cdot\vec{q}[\omega^2k^2-(\vec{k}\cdot\vec{q})^2]}\,.$$

In estimating this, remember that we are only concerned with energetic secondaries, and for these $q_l^2 \gg q_\perp^2$, $\omega \simeq q_l$. Setting $k_l \sim \Delta_l \equiv \epsilon_l\omega$, $k_\perp \sim \Delta_\perp \equiv \epsilon_\perp m$ as in (2.3), and setting[5] $q_\perp = 0$, one finds

$$T \lesssim \frac{8\omega}{m^2\epsilon_l(\epsilon_l^2+\epsilon_\perp^2)}\,,$$

or (3.5)

$$T \lesssim \frac{8}{m\tau_0\epsilon_l(\epsilon_l^2+\epsilon_\perp^2)}t_d\,.$$

This result shows that by appropriate choice of ϵ_l and $\epsilon_\perp$ one can construct final-state packets that maintain their shape and move along the classical trajectory for a time far longer than that required for the detachment of secondaries.[6]

We must point out that when detachment as defined above occurs, the wave packets of neighboring particles still overlap. At detachment the distance between neighbors is $(v_1-v_2)t_d \simeq \tau_0\delta/\cosh y$, whereas our wave packets have a size $\Delta z \simeq 1/\Delta_l \simeq (m\Delta y \cosh y)^{-1}$ where $\Delta y = \epsilon_l$. Thus separation of the packets (as compared to cessation of interaction) only occurs after a time or order $t_d/\epsilon_l m\tau_0\delta$. As one sees from (3.5), even over this longer time interval spreading of the packets may be neglected.

From the foregoing one might suppose that one could localize the packets so well that "visible" separation and detachment of secondaries occur at the same time. To achieve this one would set $\Delta y = \epsilon_l \simeq 1$, and argue that this will not destroy the validity of Eq. (2.8) because in the central region one-body inclusive cross sections do not depend on y. In the following section we show that this is a false argument. The detachment region is necessarily blurred.

IV. THE OFF-FORWARD MUELLER AMPLITUDE

We now turn to the variation of the production amplitudes. What appears in (2.6) is a nonforward Mueller-type absorptive part,

$$A = \sum_N \int (dq_2)\cdots(dq_N)(2\pi)^4\delta^4(P-\Sigma q_k) \times\langle q_1'q_2\cdots q_N|\mathfrak{M}|p_1'p_2'\rangle^* \langle q_1q_2\cdots q_N|\mathfrak{M}|p_1p_2\rangle\,. \tag{4.2}$$

Let us take a collision axis defined by the centers of the wave packets ϕ_a and ϕ_b. We may then characterize all momenta by their rapidity and transverse components. We assume the nonforward Mueller amplitude A resembles the forward amplitude in that it is a smooth function of $p_\perp$'s, varying over a scale comparable with hadron masses, and hence large compared with our $\Delta p_\perp$'s. Similarly, we assume that A is a smooth function of rapidities, varying over a scale of order unity except in the triple-Regge region. Even there our analysis is valid provided $\epsilon_l \ll |y_{\max}-y|$. Of course, our analysis fails in the resonance region, $|y_{\max}-y| = O(s^{-1})$.

We thus conclude that our kinematic description is valid for almost all final particles, provided $\epsilon_l \ll 1$. This restriction on ϵ_l is necessary even if $d\sigma/dq$ is essentially y-independent, because the Mueller discontinuity will, in general, depend also on $y_{q_1} - y_{q_1'}$. Thus wave-packet separation at the particle-separation point cannot be achieved by the suggestion at the end of Sec. III, for that requires $\epsilon_l \sim 1$.

ACKNOWLEDGMENTS

One of us (F.E.L.) is grateful to C. E. De Tar for helpful conversations concerning three-body amplitudes. The other (K.G.) acknowledges the hospitality of the Aspen Center for Physics where part of this work was carried out. This work was supported in part by the U. S. Department of Energy under Contract No. EY-76-C-02-3069 and in part by the National Science Foundation.

[1]K. Gottfried, Phys. Rev. Lett. 32, 957 (1974).

[2]For an analysis of the space-time structure of the multi-peripheral-parton model, see J. Kogut and L. Susskind, Phys. Rep. 8C, 75 (1973); J. Koplik and A. H. Mueller, *ibid.*, Phys. Rev. D 12, 3638 (1975); L. Bertocchi, in *High Energy Physics and Nuclear Structure—1975*, proceedings of the Sixth International Conference, Sante Fe and Los Alamos, edited by D. E. Nagle *et al.* (AIP, New York, 1975); L. Caneschi, I. G. Halliday, and A. Schwimmer, CERN Report No. CERN TH-2389, 1977 (unpublished). Kinematical aspects of space-time structure in collisions are also discussed in B. Durand and L. O'Raifeartaigh, Phys. Rev. D 13, 99 (1976).

[3]There is in general no local particle-density operator, but one could carry our procedure through with an effective density operator constructed from Newton-Wigner states.

[4]One might worry that form factors in $t_{\mu\nu}(q,q')$ depart significantly from unity, but this is not so. According to (2.7), $t\equiv(q-q')^2\simeq k_\perp^2+(1-v_q^2)k_l^2$, and since $k_\perp^2\sim\Delta_\perp^2\ll m^2$, and $(1-v_q^2)k_l^2\ll m^2$, t remains very small throughout. Thus we may safely substitute $t_{\mu\nu}(q)\equiv t_{\mu\nu}(q,q)$ for $t_{\mu\nu}(q,q')$ in (2.8).

[5]As long as $q_\perp^2\ll\omega^2$, $q_\perp\neq 0$ does not alter the conclusion that $T>t_d$.

[6]The definition of spreading that we have used is specific to inelastic processes. That is, we have not used the $\vec{q}$ distribution, as given by $d\sigma/dq$ in (2.8), to compute a total spreading. Rather, we have attached a wave packet density, in a natural way, to each momentum $\vec{q}$. This has the effect of slightly retarding the spread, compared to what we would have obtained had we first projected states of given $\vec{x}$ and $\vec{q}$ with approximate Newton-Wigner functions. However, the difference would not be significant.