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Application of the Gell-Mann–Okubo formula to the total widths of unstable hadrons

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A calculation of the effects of first-order symmetry breaking on unstable hadrons shows that the total widths should obey the Gell-Mann–Okubo formula. This prediction is accurate to 7 MeV for the vector mesons, to 10 MeV for the tensors, and provides a new constraint on the octet-singlet mixing angle. A new relation is derived for the widths of the $\frac{3}{2}^+$ decimet, which is accurate to 1 MeV.

I. INTRODUCTION

The Gell-Mann–Okubo mass formula[1] in combination with octet-singlet mixing[2] has been remarkably successful in explaining the mass shifts observed within SU(3) multiplets. It is sometimes the principal method for classifying new resonances when spin-parity information is lacking. To obtain an additional tool, it is natural to inquire how hadronic widths are affected by the symmetry breaking. When the total width is thought of as a sum of partial widths each of which has symmetry breaking in the decay amplitudes, the wave functions, and the phase space, this becomes an intractable question. A common approximation is to neglect all symmetry breaking in the amplitude and in the wave functions, but to use the physical masses. Such an approach can fit the partial-width data and does test SU(3) somewhat, but it is not sensitive to SU(3) breaking. The present paper, in contrast, considers only the total hadronic width. The total width is calculated from the location of the pole in the scattering amplitude and is thus put on a more equal footing with the mass. It is then simple to calculate the effect of first-order symmetry breaking. We find that within any multiplet whose members decay via strong interactions the same pattern of symmetry breaking should occur in the widths as in the masses.[3]

This relation is derived in Secs. II and III and then tested in Sec. IV on the seven multiplets that are complete. We find that the error in the width formula (in MeV) is generally the same as the error in the mass formula (in MeV).

One certainly would not expect first-order perturbation theory to work so well. After all, masses are only split by 20%, whereas widths differ by more than a factor of 2 in every case. The actual calculation, however, is of the shift in the complex position. Since the masses are large, this position undergoes a fairly small change (≈20%) so that first-order perturbation theory should be no worse for the widths than for the masses. Of course why it works for the masses is not well understood, but probably due to the fact that the first-order calculation is merely a way of parametrizing the exact calculation in which both octet and 27-plet pieces would occur. The success of the Gell-Mann–Okubo formula amounts to the dominance of the 8 over the 27.

II. PERTURBATION THEORY

A. Perturbation theory with eigenstates of H_0

As usual the strong-interaction Hamiltonian may be split into $H=H_0+V$, where H_0 is SU(3)-symmetric and V transforms like the $I=Y=0$ member of an octet. Each SU(3) multiplet in the spectrum of H_0 is degenerate. The lowest multiplets, viz., the 0^- mesons and the $\frac{1}{2}^+$ baryons, are stable under H_0. The amplitude for scattering these states will have resonance poles on the second sheet at energies

$$E^{(0)}=M^{(0)}-\tfrac{1}{2}i\Gamma^{(0)},$$

corresponding to an SU(3)-degenerate multiplet that is unstable under H_0 with common mass $M^{(0)}$ and common width $\Gamma^{(0)}$.[4] In part B a straightforward calculation of the effect of the perturbation V on the location of these poles is given. Before that calculation, however, we present a less familiar but more direct method by introducing states of complex energy satisfying

$$H_0|E_j^{(0)}\rangle=E^{(0)}|E_j^{(0)}\rangle.$$

These states are not in the usual Hilbert space but in a larger space corresponding to the second sheet of the scattering amplitude. Of course in a Hilbert space a Hermitian operator cannot have complex eigenvalues because

$$\begin{aligned}\langle E^{(0)}|E^{(0)}\rangle E^{(0)}&=\langle E^{(0)}|H_0|E^{(0)}\rangle\\&=E^{(0)*}\langle E^{(0)}|E^{(0)}\rangle\end{aligned}$$

forbids it. But in a larger space, states with complex energies but zero norms are possible. Full details of this description of resonances are contained in Ref. 5. All that is necessary here is that the states come in pairs with conjugate energies and satisfy

$$\langle E_j^{(0)} | E_k^{(0)} \rangle = \langle E_j^{(0)*} | E_k^{(0)*} \rangle = 0 ,$$

$$\langle E_j^{(0)*} | E_k^{(0)} \rangle = \delta_{jk} .$$

In the presence of the symmetry-breaking interaction V, the second-sheet poles occur at nondegenerate energies

$$E_j = M_j - \tfrac{1}{2} i \Gamma_j ,$$

and may be described at states satisfying

$$H | E_j \rangle = E_j | E_j \rangle .$$

To calculate E_j to first order in V then requires diagonalizing the matrix

$$H_{jk} = E^{(0)} \delta_{jk} + \langle E_j^{(0)*} | V | E_k^{(0)} \rangle . \qquad (1)$$

This is the central result. The implications are twofold: First, for a stable multiplet $E^{(0)}$ is real, the term linear in V is real, and consequently there is a first-order shift in the mass, but not in the width. On the other hand, when $E^{(0)}$ is complex, the term linear in V is also complex. Thus if the particles are already unstable under H_0 then V will produce a first-order shift in both the mass and the width.

B. Perturbation theory with eigenstates of K

Using states with a complex energy allows a quick demonstration of the energy shift formula (1), but it is not really necessary to introduce them. It may therefore be worthwhile to derive (1) in a more conventional manner.

The unperturbed Hamiltonian H_0 is SU(3)-symmetric but is extremely complicated because its eigenstates are unstable. Imagine splitting H_0 into two parts by setting $H_0 = K + U$, where K is chosen so that its spectrum consists entirely of stable hadrons. Thus, for example, the pseudoscalar, vector, and tensor mesons are all stable under K and it is U that induces transitions. [In a quark theory K might consist of the kinetic energy of the degenerate quarks plus a phenomenological potential that produces binding. Then U would contain the negative of the phenomenological potential plus the gauge coupling of the quarks to the vector gluons. The details of the split are unimportant because K and U will be summed to all orders in deriving (1).] Let the common (real) mass of a given multiplet be μ and set

$$K | \varphi_j \rangle = \mu | \varphi_j \rangle .$$

Define a projection operator p onto the multiplet and its complement q by

$$p = \sum_j | \varphi_j \rangle \langle \varphi_j | ,$$

$$q = 1 - p .$$

Now the states $|\varphi_j\rangle$ have unphysical masses (μ) and unphysical widths (0). To find the true masses and widths we look for poles of $\langle \varphi_j | (E - H)^{-1} | \varphi_k \rangle$. Let $|\varphi_j'\rangle$ be the linear combination of $|\varphi_j\rangle$ that diagonalizes this resolvent. Then

$$\left\langle \varphi_j' \right| \frac{1}{E - H} \left| \varphi_j' \right\rangle = \frac{1}{E - \langle \varphi_j' | H^{\text{eff}} | \varphi_j' \rangle} ,$$

where H^{eff} is an energy-dependent operator[6]:

$$H^{\text{eff}}(E) = K + p(U+V)p + p(U+V)q \frac{1}{E - K - q(U+V)q} q(U+V)p .$$

The exact masses and widths are solutions of

$$E_j = \langle \varphi_j' | H^{\text{eff}}(E_j) | \varphi_j' \rangle .$$

The first-order symmetry breaking comes from keeping only terms linear in V:

$$H^{\text{eff}}(E) = K + p[U + UqR(E)qU]p + p[UqR(E) + 1]V[1 + R(E)qU]p + O(V^2) ,$$

where

$$R(E) \equiv \frac{1}{E - K - qUq} .$$

Now expand E as a power series in V,

$$E_j = E^{(0)} + E_j^{(1)} + O(V^2) .$$

Then the unperturbed energy is the second-sheet solution of

$$E^{(0)} = \mu + \langle \varphi_j' | U + UqR(E^0)qU | \varphi_j' \rangle .$$

The correction to the energy that is first order in the symmetry breaking is then

$$E_j^{(1)} = \langle \varphi_j' | [UqR(E^0) + 1]V[1 + R(E^0)qU] | \varphi_j' \rangle / N_j , \qquad (2)$$

where the factor N_j comes from the implicit dependence of E on V and is given by

$$N_j = 1 + \langle \varphi_j' | Uq[R(E^0)]^2 qU | \varphi_j' \rangle .$$

Equation (2) is an explicit verification of (1). Note that all orders of U have been summed so that $E^{(0)}$ and $E^{(1)}$ depend only on $H^0 = K + U$ and do not depend on K and U separately.

III. APPLICATION TO SU(3)

The precise pattern of this symmetry breaking can be deduced from the Wigner-Eckart theorem. The matrix element of V in (1) or (2) is just the product of a real Clebsch-Gordan coefficient and a complex, reduced matrix element. For baryon octets there are three complex, reduced matrix elements (including $E^{(0)}$) that determine four complex energies. There is thus one complex constraint whose real and imaginary parts are

$$\tfrac{3}{4} M_\Lambda + \tfrac{1}{4} M_\Sigma = \tfrac{1}{2} M_n + \tfrac{1}{2} M_\Xi \,, \tag{3a}$$

$$\tfrac{3}{4} \Gamma_\Lambda + \tfrac{1}{4} \Gamma_\Sigma = \tfrac{1}{2} \Gamma_n + \tfrac{1}{2} \Gamma_\Xi \,. \tag{3b}$$

The treatment of nonets, like the vector mesons, requires one additional step because the mixing of φ and ω means that H_{jk} is not diagonal. The diagonalization is accomplished by expressing the $I=0$ eigenstates as $\varphi_0 \cos\theta + \omega_0 \sin\theta$ and $-\varphi_0 \sin\theta + \omega_0 \cos\theta$. Because

$$H_{jk} \equiv M_{jk} - \tfrac{1}{2} i \Gamma_{jk}$$

is complex, the mixing angle θ will be complex unless the matrices M and Γ commute. To show this one may use the norm operator of Ref. 5,

$$\Omega \equiv \sum_k |E_k\rangle \langle E_k| + \sum_k |E_k^*\rangle \langle E_k^*| \,,$$

and observe that H and $\Omega H \Omega$ are simultaneously diagonalizable. Consequently $[H, \Omega H \Omega] = 0$, which guarantees that the mixing angle is real. The real and imaginary parts of the constraint equation are then

$$\tfrac{3}{4}(M_\phi \cos^2\theta + M_\omega \sin^2\theta) + \tfrac{1}{4} M_\rho = M_{K^*} \,, \tag{4a}$$

$$\tfrac{3}{4}(\Gamma_\phi \cos^2\theta + \Gamma_\omega \sin^2\theta) + \tfrac{1}{4} \Gamma_\rho = \Gamma_{K^*} \,. \tag{4b}$$

Note that if the symmetry breaking applies to the square of the masses rather than to the masses themselves, then these become equations for M^2 and $M\Gamma$, respectively. Since the agreement of the linear and the quadratic formulas with the data is about the same, only the simpler form (4) will be treated here.

IV. COMPARISON WITH THE DATA

In the following tests of (3) and (4) the review of SU(3) multiplets by Samios *et al.*[7] was used. The masses and widths are from the Particle Data Group[8] and are the pole positions whenever available.

A. Meson nonets

The $J^{PC} = 1^{--}$ nonet is the best known and consists of $\rho(773/152)$, $\phi(1020/4)$, $K^*(892/49.4)$, and $\omega(783/9)$. (Note that 9 MeV is the total hadronic width of the ω; the other 1 MeV is electromagnetic.) It is immediately obvious from these widths that (4b) is reasonably well satisfied no matter what the mixing angle is. The fit which makes the error in (4a) and (4b) the same is $\cos^2\theta = 0.59$. Then (4a) becomes $885 = 892$ and (4b) becomes $42.5 = 49.4$. Thus both relations are satisfied to within 7 MeV.

The 2^{++} nonet consists of $A_2(1310/102)$, $f'(1516/40)$, $K^*(1421/108)$, and $f(1271/180)$. A fit to (4) gives $\cos^2\theta = 0.67$. Then (4a) becomes $1404 = 1421$ and (4b) becomes $91 = 108$. Thus both relations are satisfied to within 17 MeV. It has recently been reported, however,[9] that previous measurements of the f' may have systematically been in error due to neglect of f, f', A_2 interference in the $K\bar{K}$ channel. The new values of mass and width quoted are $f'(1506/66)$. In (4) this gives $\cos^2\theta = 0.74$, which leads to an error of only 10 MeV in both equations.

The 0^{++} nonet has been clarified by Morgan.[10] It is thought to consist of $\delta(970/50)$, $S^*(993/40)$, $\kappa(1250/450)$, and $\epsilon(1200/640)$. The mass formula (4a) cannot be satisfied with such a small ϵ mass. However, both the mass and width of the ϵ are uncertain by as much as 100 MeV. Indeed before Morgan's analysis there appeared to be one ϵ at 900 and another at 1250, in obvious discrepancy with SU(3). He showed that the two states are really one very broad resonance. The resulting large value of Γ_ϵ is in excellent accord with (4b), which demands that $\Gamma_\epsilon > 583$, independently of θ.

The only other meson nonets that are near completion are the 3^{--}, which lacks an isoscalar φ, and the 1^{++}, which is uncertain because of the Q.

B. Baryon octets

There are no really well measured octets on which to test (3), but there are two likely candidates with $J^P = \frac{5}{2}^-$ and $\frac{5}{2}^+$. The $\frac{5}{2}^-$ octet consists of $N(1663/146)$, $\Lambda(1830/95)$, $\Sigma(1773/130)$, and $\Xi(1940/90)$. With these data the Gell-Mann–Okubo mass formula (3a) becomes $1816 = 1802$ and the width formula (3b) becomes $104 = 118$. Thus both relations are satisfied to within 14 MeV.

The $\frac{5}{2}^+$ octet is thought to consist of $N(1688/132)$, $\Lambda(1815/85)$, $\Sigma(1519/100)$, and $\Xi(2038/49)$. (The mass and width of the Ξ are the average of four measurements in the Particle Data Tables with the state of 2129 excluded as the Particle Data Group suggests.) The deviation from the mass formula (3a) is 23 MeV; the deviation from the width formula (3b) is only 2 MeV.

C. Baryon decimets

The only complete decimet is the $\frac{3}{2}^+$ consisting of $\Delta(1211/100)$, $\Sigma(1381/40)$, $\Xi(1533/8)$, and $\Omega(1672/0)$. The approximately equal spacing of these masses was one of the early triumphs of broken SU(3). (Note, however, that these data are the pole positions of Ref. 8. They are not as equally spaced as the earlier Breit-Wigner fits, which were found to be parametrization-dependent.[11]) Obviously the widths are not equally spaced. This result is in accord with (2) for it predicts equally spaced widths only when the unperturbed energy $E^{(0)}$ is complex. Here $E^{(0)}$ is actually real. This is easily seen by using Okubo's general formula[1]

$$M = M^{(0)} + AY + B[I(I+1) - \tfrac{1}{4}Y^2]$$

to calculate the unperturbed mass of the decimet and of its decay products, viz., the $\frac{1}{2}^+$ baryons and the 0^- mesons. One finds $M^{(0)}(\frac{3}{2}^+) = 1375$, whereas $M^{(0)}(\frac{1}{2}^+) = 1155$ and $M^{(0)}(0^-) = 376$. Hence $\Gamma^{(0)}(\frac{3}{2}^+) = 0$ because the threshold is too high. This means, from (2) that $\Gamma^{(1)}(\frac{3}{2}^+) = 0$ and equal spacing is not expected. The widths arise only in second order after the first-order mass corrections have lowered the threshold. Thus the widths should transform like matrix elements of the $I = Y = 0$ part of 8×8, i.e., like a combination of 1, 8, and 27. There are three reduced matrix elements and hence one complex constraint:

$$\tfrac{3}{4}M_\Sigma + \tfrac{1}{4}M_\Omega = \tfrac{3}{4}M_\Xi + \tfrac{1}{4}M_\Delta\,,$$

$$\tfrac{3}{4}\Gamma_\Sigma + \tfrac{1}{4}\Gamma_\Omega = \tfrac{3}{4}\Gamma_\Xi + \tfrac{1}{4}\Gamma_\Delta\,.$$

The mass relation gives $1454 = 1453$, which is a great improvement over the crude equal spacing of the pole positions. The width relations is equally good and gives $30 = 31$.

There is evidence for a $\frac{7}{2}^+$ decimet with only $\Delta(1924/258)$ and $\Sigma(2030/180)$ known. These particles are heavy enough to decay even in the absence of symmetry breaking. Thus both the masses and widths should be equally spaced here. One therefore expects two new states: $\Xi(2136/102)$ and $\Omega(2242/24)$.

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[3]This applies to the electromagnetic splittings among different charge states as well.

[4]Note that when the multiplets are degenerate, all decay channels allowed by SU(3) are open. Thus the equality of the total widths follows from the normalization condition on the Clebsch-Gordan coefficients.

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