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Semiclassical sum rules

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The expression $|\Psi_n(0)|^2 = (2\mu)^{3/2}E_n^{1/2}(dE_n/dn)/4\pi^2$, relating the square of the nth s-wave wave function at the origin to the bound-state reduced mass μ and the excitation energy E_n, is derived semiclassically. The relation is then used to obtain several sum rules for electron-positron annihilation and an expression for the contribution of a given flavor of heavy quark to the photon-nucleon total cross section.

I. INTRODUCTION

Nonrelativistic models have been remarkably successful in describing many properties of mesons composed entirely of heavy quarks. A good deal has been learned by applying simple Schrödinger-equation physics to the charmonium system,[1] and one expects the nonrelativistic approximations used with so much success for that system to be even more reliable for the recently discovered Υ family.[2,3]

It was our interest in families of quark-antiquark states that led us to survey[4] the behavior of simple quantities such as the excitation energy E_n of s-wave states and the squares of their wave functions at the origin $|\Psi_n(0)|^2$ as functions of the bound-state reduced mass μ and the principal quantum number n. These investigations were carried out for potentials of the form $V=ar^\epsilon$ (Ref. 4) and for a potential $V=C\ln(r/r_0)$.[5] (The latter has the interesting property that it gives a level spacing independent of quark mass for which there is some evidence in $Q\overline{Q}$ systems.[3]) In discussing behavior as a function of n, the semiclassical (WKB) approximation[6] was found to be particularly helpful.[4,5,7] A potential-independent relation for the number of narrow quark-antiquark states below flavor threshold also was derived with the aid of the WKB approximation.[8]

In the present article we point out an interesting relation between $|\Psi_n(0)|^2$ and E_n that is *independent* of the potential, as long as that potential is not singular at the origin. This relation follows from an application of the WKB approximation entirely analogous to those of Ref. 4, but the possibility of a more general result was overlooked there. We were led to search for a more general result by an exhortation at the end of an interesting paper by Farrar *et al.*,[9] which discusses the seemingly unrelated subject of sum rules in electron-positron annihilation. The relation between $|\Psi_n(0)|^2$ and E_n obtained here in fact implies the existence of a family of sum rules derived in a somewhat different manner in Ref. 9 and in several other works.[10,11] From these sum rules it has been possible to infer that the mass m_c of the charmed quark is rather low (see also Ref. 11): $m_c = 1.2 \pm 0.1$ GeV/c^2. Moreover, the relation for $|\Psi_n(0)|^2$ permits an immediate (though probably rough) estimate of the contribution of higher $Q\overline{Q}$ vector-meson states in a vector-dominance model[12] for $\sigma_Q(\gamma p)$, the contribution of a given flavor of heavy quark to the photon-nucleon total cross section.

The expression for $|\Psi_n(0)|^2$ is derived in Sec. II. Section III treats the sum rules for electron-positron annihilation, while Sec. IV is devoted to an estimate of $\sigma_Q(\gamma p)$. Section V contains a brief discussion.

II. RELATION FOR $|\Psi_n(0)|^2$

For a two-body nonrelativistic bound state with reduced mass μ, it can be shown[13] that

$$|\Psi_n(0)|^2 = \frac{\mu}{2\pi}\left\langle \frac{dV}{dr}\right\rangle_n . \tag{1}$$

We shall construct a simple semiclassical approximation for $\langle dV/dr\rangle_n$. This may be written

$$\langle dV/dr\rangle \simeq \frac{\int_0^{r_0} dr(dV/dr)[u_{\rm WKB}(r)]^2}{\int_0^{r_0} dr[u_{\rm WKB}(r)]^2}, \tag{2}$$

where r_0 is the classical turning point.

The reduced radial WKB wave function $u_{\rm WKB}(r)$ contains a factor $[E_n - V(r)]^{-1/4}$ times an oscillatory term the square of which approximately averages to $\frac{1}{2}$. Then

$$\left\langle \frac{dV}{dr}\right\rangle_n \simeq \frac{\int_0^{r_0} dr(dV/dr)[E_n - V(r)]^{-1/2}}{\int_0^{r_0} dr[E_n - V(r)]^{-1/2}} . \tag{3}$$

The integral in the numerator of (3) is elementary, as noted in Ref. 4, and yields $2E_n^{1/2}$. We are defining $V(0)\equiv 0$. This derivation fails if $V(0) = -\infty$, but alternative results which apply to certain singular potentials are noted in Ref. 4.

The quantization condition

$$\int_0^{r_0} dr\,\{2\mu[E_n - V(r)]\}^{1/2} = (n-\tfrac{1}{4})\pi \qquad (4)$$

may be differentiated with respect to n:[7]

$$\frac{(2\mu)^{1/2}}{2}\frac{dE_n}{dn}\int_0^{r_0}\frac{dr}{[E_n - V(r)]^{1/2}} = \pi . \qquad (5)$$

But Eq. (5) permits one to evaluate the denominator in Eq. (3). With the help of Eq. (1), we then find that

$$|\Psi_n(0)|^2 = \frac{(2\mu)^{3/2}}{4\pi^2}E_n^{1/2}\frac{dE_n}{dn}. \qquad (6)$$

Equation (6) is our central result.[14] It is a concise summary of expressions obtained previously for power-law potentials.[4] As an illustration, for a linear potential $\langle dV/dr\rangle$ is independent of the energy level and hence so is $|\Psi_n(0)|^2$. Thus $E_n^{1/2}(dE_n/dn) = \text{constant}$, and $E_n \sim n^{2/3}$, which is the correct nonrelativistic result.

III. SUM RULES FOR ELECTRON-POSITRON ANNIHILATION

It is expected that the onset of the production in e^+e^- annihilation of new quark flavors will be signaled by discrete narrow peaks (like ψ, ψ') in the cross section. Then, as the threshold for production of pairs of flavored mesons is passed, the peaks become broader and eventually merge into the multiparticle continuum.

It has been noted by several authors[9-11] that one can write sum rules for leptonic widths of the narrow states below flavor threshold. We shall use Eq. (6) to derive a family of such sum rules.

The leptonic width Γ_n of the nth $^3S_1$ $Q\bar{Q}$ vector meson may be related to the corresponding square of its wave function at the origin[15]: if e_Q denotes the quark charge,

$$\Gamma_n = 16\pi\alpha^2 e_Q^2 |\Psi_n(0)|^2/M_n^2 . \qquad (7)$$

One may then form a weighted sum over the states below flavor threshold:

$$\sum_{\substack{\text{narrow}\\ \text{states}}} \Gamma_n/M_n^p \simeq \int \frac{\Gamma_n dn}{M_n^p} = \frac{4\alpha^2 e_Q^2 m^{3/2}}{\pi}\left[\int_0^{\Delta}\frac{dE\,E^{1/2}}{(2m+E)^{2+p}}\right] \qquad (8)$$

or

$$S_p \equiv \frac{\pi}{\alpha^2 e_Q^2}\sum_{\substack{\text{narrow}\\ \text{states}}}\frac{\Gamma_n}{M_n^p} = (2m)^{1-p}\sqrt{2}\,I_p(\Delta/2m), \qquad (9)$$

where

$$I_p(v) \equiv \int_0^v dy\,\sqrt{y}/(1+y)^{2+p} . \qquad (10)$$

The quark mass m is twice the reduced mass μ. The zero of energy is set at $2m$, so that we have taken $M_n = 2m + E_n$, and flavor threshold ($2M_D$ for the charmonium system) lies at $2m+\Delta$.

The sum rules (9) may be tested for the charmonium system, in which the narrow states consist only of[16]

$$\begin{aligned}&\psi(3095):\ \Gamma_{ee} = 4.8 \pm 0.6\ \text{keV},\\ &\psi'(3684):\ \Gamma_{ee} = 2.1 \pm 0.3\ \text{keV}.\end{aligned} \qquad (11)$$

For each value of $p \geq 0$, we find a range of values of the charmed-quark mass m_c for which Eq. (9) is satisfied. (We take $2M_D = 3730$ MeV.) These ranges are shown in Fig. 1. Notice the very slow increase of the quark mass with increasing p. Very large values of p, which give all weight to the contribution of the ψ, do not make sense in view of the discreteness (and sparse nature) of the spectrum. [Recall that our discussion is a semiclassical one, wherein we approximate the sum in Eq. (8) by an integral. This step is perhaps an unwarranted exercise in boldness for charmonium. We are comforted by the expectation that such a sum will include more states for heavier quarks.[8]] The sum rules for small values of p also appear unreliable, if only for their rapid variation with p. But between $p=3$ and $p=14$, the central value of m_c varies only between 1.1 and 1.3 GeV. We are thus led to the inference that $m_c = 1.2 \pm 0.1$ GeV.

A small value of the charmed-quark mass has been deduced before from related sum rules.[11] We have also encountered the possibility that $m_c \simeq 1.1$ GeV within the context of a potential model

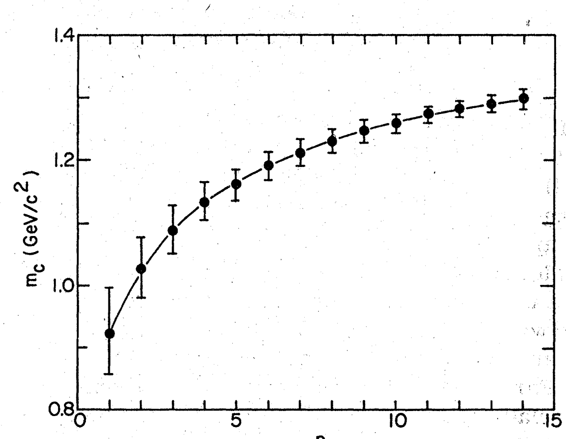

FIG. 1. Ranges of charmed-quark masses m_c implied by the sum rules of Eq. (9).

that reproduced features of both the ψ and Υ families.[5] (This value was favored by the leptonic width of the ψ in that model.)

The sum rule (9) for $p=3$ and $\Delta/2m \ll 1$ is extremely similar to one derived in Ref. 9 in a similar limit, when one neglects effects due to the strong interactions. In Ref. 9, however, the term corresponding to the right-hand side of Eq. (9) is evaluated with the help of a vacuum-polarization Feynman diagram. Evidently some of the information contained therein is of a very general and simple nature since we are able to reproduce it semiclassically.

IV. PHOTOPRODUCTION OF NEW FLAVORS

The suppression of charmed particle production in hadron physics is an obstacle to the study of new-flavor spectroscopy with hadron beams. No such suppression is seen in electron-positron annihilation above charm threshold, and there are suggestions[17,18] that charmed-particle pairs also are photoproduced above threshold, possibly at a rate of several percent of all hadronic interactions.[18]

There have been numerous estimates of the photoproduction of new flavors, both of charm and of the new flavor that is presumably associated with the quarks in the Υ family.[19,20] We would like to focus on just one of these estimates,[20] in which the relation (6) allows the immediate expression of an electromagnetic cross section in terms of a hadronic one. It is not our purpose here to make a critical study of models of photoproduction.

We express $\sigma_Q(\gamma p)$, the contribution of the new flavor to the photon-proton total cross section, as a sum of contributions of vector mesons $\mathcal{V}$. Within a family, each vector meson is taken to have the *same* total cross section $\sigma(\mathcal{V}p)$ for scattering on the proton. Using vector dominance,[12] we then find

$$\sigma_Q(\gamma p) \simeq \alpha\sigma(\mathcal{V}p) \sum_{n=1}^{\infty} \frac{4\pi}{g_n^2} \simeq \alpha\sigma(\mathcal{V}p) \int dn \left(\frac{4\pi}{g_n^2}\right). \tag{12}$$

Here the nth-vector-meson–photon coupling eM_n^2/g_n is related to Γ_n by

$$\Gamma_n = \frac{4\pi}{3}\frac{\alpha^2}{g_n^2} M_n , \tag{13}$$

and to $|\Psi_n(0)|^2$ by

$$\frac{4\pi}{g_n^2} = 48\pi e_Q^2 |\Psi_n(0)|^2/M_n^3 . \tag{14}$$

But now, with the help of Eq. (6), we can evaluate the integral in Eq. (12) in closed form, first transforming it to an integral over E,

$$\sigma_Q(\gamma p) = \alpha\sigma(\mathcal{V}p)\left(\frac{m^{3/2}}{4\pi^2}\right)(48\pi e_Q^2) \int_0^{\infty} \frac{E^{1/2}dE}{(2m+E)^3} . \tag{15}$$

The integral in (15) can be evaluated by elementary means, and leads to the simple result

$$\sigma_Q(\gamma p) = \frac{3\alpha e_Q^2}{4\sqrt{2}} \sigma(\mathcal{V}p) . \tag{16}$$

Equation (16) would underestimate the total photon-nucleon total cross section (> 100 μb) if we were to ascribe it mainly to the coupling of the photon to the ρ and ω families:

$$\sigma_\rho(\gamma p) = \frac{3\alpha}{4\sqrt{2}} \times \tfrac{1}{2} \times \sigma(\rho p) \simeq 50\ \mu\text{b} , \tag{17}$$

$$\sigma_\omega(\gamma p) = \frac{3\alpha}{4\sqrt{2}} \times \tfrac{1}{18} \times \sigma(\omega p) = 5.6\ \mu\text{b} , \tag{18}$$

where we have taken $\sigma(\rho p) = \sigma(\omega p) = 26$ mb.[21] In fact, the coefficient of $\alpha\sigma(\rho p)$ in (17) which is ascribed to the whole ρ family is smaller $[3/(8\sqrt{2}) = 0.27]$ than that expected from the first term alone $(4\pi/g_\rho^2 \simeq 0.4)$ in the sum (12). This certainly indicates the crudeness of our approximations for light quarks. We expect matters to improve somewhat as the quark mass increases, the nonrelativistic approximation gets better, and the semiclassical approximation is more justified.

Let us assume that vector-meson–nucleon total cross sections scale as M_1^{-2}, where M_1 is the mass of the ground state of the $Q\bar{Q}$ system.[22] Equation (16) then predicts the results shown in Table I. It is important to note that the predictions for heavy-quark production apply far above threshold. Considerations such as those in Ref. 23 lead one to expect the charm production cross section to attain half its maximum for photon energies somewhere between 50 and 100 GeV, and the cross section for the production of pairs of quarks in the Υ not to reach half its asymptotic value until at least 200 (and possibly as much as 500) GeV. One can improve these estimates of energy dependence somewhat with the help of the photoproduction sum rules

TABLE I. Cross sections for photoproduction of new flavors (at asymptotic energies).

Quark	Lowest $Q\bar{Q}$ states	$\sigma(\mathcal{V}p)$	e_Q	$\sigma_Q(\gamma p)$
s	ϕ(1020)	14.8 mb	$-\frac{1}{3}$	6.4 μb
c	ψ(3095)	1.6 mb	$\frac{2}{3}$	2.8 μb
b	Υ(9400)	174 μb	$-\frac{1}{3}$	75 nb
t			$\frac{2}{3}$	300 nb

derived in Ref. 24, or with the aid of the more specific models considered in Ref. 19.

V. DISCUSSION

We have derived a semiclassical expression for the square of the s-wave bound-state wave function at the origin in terms of the level density. There is another semiclassical expression which incorporates the level density. It is the relation between the potential and the bound-state energies[25]:

$$r(V)=\frac{2}{(2\mu)^{1/2}}\int_0^V\frac{dE}{(V-E)^{1/2}}\left(\frac{dE_n}{dn}\right)^{-1}. \qquad (19)$$

Substituting Eq. (6) into this relation, one obtains a consistency condition

$$r(V)=\frac{\mu}{\pi^2}\int_0^V\frac{dE}{|\Psi(0)|^2}\left(\frac{E}{V-E}\right)^{1/2}. \qquad (20)$$

The potential thus derived must reproduce the observed energy levels. While we believe the charmonium data are too sparse to permit a test of this relation, bound states of heavier quarks (as in the Υ) may prove rich enough. At present we are exploring alternative means of estimating the quark-antiquark potential (if such a concept makes sense) in a model-independent way with the help of the inverse-scattering formalism.[26]

The relation between sum rules such as those we have derived in Sec. III and duality has been stressed by several authors.[9-11] Duality relates an integral over bound states or resonances to an integral over the continuum. It is amusing that the result (6), based on a simple semiclassical approximation to nonrelativistic quantum mechanics, makes contact with the duality between bound-state and free-quark creation. It would be interesting to know the degree to which such semiclassical arguments are responsible for the success of duality in other contexts.

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