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Two-pion contribution to the muon anomaly

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Lower bounds to the pionic contribution to the anomalous magnetic moment of the muon are derived. The bounds incorporate all the existing reliable experimental data on the electromagnetic form factor of the pion or, at will, only a part of them. They also incorporate the phase of the $\pi\pi$ elastic P wave and the theoretical analyticity and normalization properties. The procedure developed in the paper allows a proper error analysis; that is, it allows the assignment of an uncertainty to the bound which depends upon the experimental errors of the data used. A reliable estimate of the bound gives $a_\mu(\pi^+\pi^-) \geq (46.1 \pm 4.0) \times 10^{-9}$. Comments on this result and other possible estimates are made.

I. INTRODUCTION

One of the most important sources of theoretical uncertainty in the computation of the anomalous magnetic moment of the muon is the hadronic contribution to the photon propagator. In spite of its relative smallness compared with the total anomaly, the hadronic contribution is responsible for approximately three quarters of the theoretical error.[16] As we shall see later, the computation involves all the intermediate hadronic states weighted in such a way that the low-energy states are strongly enhanced, so that the two-pion state gives the main part of the hadronic contribution to the anomaly since it dominates over all other intermediate states in that region.

Thus, a more accurate knowledge of the two-pion contribution is necessary if we want to reduce the theoretical uncertainty of the muon anomaly and compare the computed value with experiments. The increasing accuracy of present and future measurements[1] makes the effort of refining the analysis of the two-pion contribution relevant.

At the same time, this analysis leads to the study of mathematically interesting problems related to extrapolation of analytical functions and, in general, to the exploitation of analyticity properties in order to optimize the information that can be extracted from a partial (experimental) knowledge of functions whose analytical structure is known.

To make clear the experimental and theoretical input used in deriving the bound, let us briefly recall the expression of the hadronic contribution to the muon anomaly a_μ(had)

$$a_\mu(\text{had}) = \frac{1}{4\pi^3}\int_{4m_\pi^2}^{\infty} dt\, \sigma_{\text{had}}(t) K(t), \tag{1.1}$$

where $\sigma_{\text{had}}(t)$ is the total cross section for the reaction $e^+e^- \to$ hadrons and the kernel $K(t)$ has the form

$$K(t) = \int_0^1 dx \frac{x^2(1-x)}{x^2 + (1-x)t/m_\mu^2}$$
$$= \tfrac{1}{2} - 4\xi + 4\xi(2\xi - 1)\ln 4\xi + \frac{1 - 8\xi + 8\xi^2}{(1-\xi^{-1})^{1/2}} \ln \frac{1-(1-\xi^{-1})^{1/2}}{1+(1-\xi^{-1})^{1/2}} \tag{1.2}$$

with $\xi = t/4m_\mu^2 > 1$.

But $\sigma_{\text{had}}(t)$ is a sum of terms, each one of them being associated with a possible hadronic intermediate state

$$\sigma_{\text{had}}(t) = \sigma_{\pi^+\pi^-}(t) + \sigma_{4\pi}(t) + \cdots, \tag{1.3}$$

so that, owing to the positivity of the cross sections and the kernel $K(t)$, we can write

$$a_\mu(\text{had}) \geq a_\mu(\pi^+\pi^-) = \frac{1}{4\pi^3}\int_{4m_\pi^2}^{\infty} dt\, \sigma_{\pi^+\pi^-}(t) K(t). \tag{1.4}$$

The cross section $\sigma_{\pi^+\pi^-}(t)$ is related to the electromagnetic form factor of the pion $F(t)$ through the expression[3]

$$\sigma_{\pi^+\pi^-}(t) = \frac{\alpha^2\pi}{3} \frac{(t-4m_\pi^2)^{3/2}}{t^{5/2}} |F(t)|^2. \tag{1.5}$$

In conclusion, the two-pion contribution to the muon anomaly depends on the squared modulus of the electromagnetic form factor of the pion in the following way:

$$a_\mu(\pi^+\pi^-) = \int_{4m_\pi^2}^{\infty} dt\, \rho(t) |F(t)|^2 \tag{1.6}$$

with

$$\rho(t) = \frac{\alpha^2}{12\pi^2} \frac{(t-4m_\pi^2)^{3/2}}{t^{5/2}} K(t), \tag{1.7}$$

the asymptotic behavior of the function $\rho(t)$ being

$$\rho(t) \underset{t\to\infty}{\sim} \frac{m_\mu^2 \alpha^2}{36\pi^2 t^2} = \frac{Z}{t^2}, \quad Z \simeq 1.67\times 10^{-9}\ (\mathrm{GeV})^2. \tag{1.8}$$

The problem we face is to get a lower bound for the quantity $a_\mu(\pi^+\pi^-)$ using the following available information:

(i) The form factor $F(t)$ is a real analytical function in the cut t plane from $4m_\pi^2$ to infinity.

(ii) It is normalized at $t=0$ to the charge of the pion in units of e, i.e., $F(0)=1$.

(iii) The phase of $F(t)$ is known at values of t in which the only intermediate hadronic state is the $I=1$ two-pion state. In that case, the Watson theorem[4] ensures that the phase of $F(t)$ coincides with the $\pi\pi$ P-wave elastic phase shift up to integer multiples of π:

$$\text{phase}\,(F(t)) = \delta_1^1(t) + n\pi, \quad n = 0, \pm 1, \cdots . \tag{1.9}$$

(iv) The modulus of the form factor $|F(t)|$ is quite accurately known on a certain range of the timelike region ($t > 4m_\pi^2$) from measurements of the reaction $e^+e^- \to \pi^+\pi^-$. This range covers essentially the ρ and ω regions and knowing this allows us to compute directly a piece of the integral in (1.4).

(v) The form factor is known on a certain spacelike range from pion electromagnetic experiments $e^-p \to e^- n\pi^+$.

A lot of work has been devoted to the problem of bounding $a_\mu(\pi^+\pi^-)$ (Refs. 5–7), but in each case some of the theoretical properties or experimental information listed above are missing. On the other hand, the important problem of the errors in the experimental input and its repercussion on the bound has been forgotten or incorrectly treated in almost all works dedicated to this subject. However, the error analysis is crucial because the information used as input is, in principle, more than sufficient to determine completely the pionic contribution to the muon anomaly and not only a bound. Therefore, the problem is overdetermined if the experimental input is exact, and only upon taking the errors into account are all the pieces of information compatible. On the other hand, to give a meaningful estimate of the bound requires control of its dependence upon slight modifications in the input compatible with experimental errors, that is, it requires a correct analysis of the errors of the data used.

We think that the present work permits the incorporation of all the available theoretical and experimental information. The bound is given as an explicit functional of the experimental input, which appears in the form of averages of known quantities weighted with smooth functions. This property allows the performance of a proper estimate of the errors affecting the quantities which appear in the bound, respecting the "smoothed" nature of the experimental data and the statistical meaning of the experimental errors.

Finally, it is worthwhile pointing out that the approach presented in this paper is universal, in the sense that it can still be applied when one or several pieces of the experimental input are eliminated or some more information is added. We shall illustrate this point with one particular example in which all information about the modulus of the form factor in the timelike region is ignored.

This paper is organized in the following way: In Sec. II we perform all the mathematical manipulations leading to the expression of the bound. In Sec. III we discuss the experimental input and the choice of a class of functions necessary in the computation of the bound. Finally, in Sec. IV we describe and comment on the results, and we compare them with preceding works.

II. THE BOUND

First of all, we consider the problem of the phase of the form factor. Let us assume that the phase is known from threshold to a certain value of t, $t=t_1$ (see Fig. 1). That is, we assume that the only intermediate hadronic state present at energies lying on the range $(4m_\pi^2, t_1)$ is the $I=1$ two-pion state. In order to incorporate that information, let us build up the associated Omnès-Muskhelishvili function[8]:

$$J(t) = \exp\left[\frac{t}{\pi}\int_{4m_\pi^2}^{\infty} dt' \, \frac{\delta(t')}{t'(t'-t)}\right], \tag{2.1}$$

where

$$\delta(t') = \delta_1^1(t') \text{ for } 4m_\pi^2 \le t' \le t_1, \tag{2.2a}$$

$$\delta(t') \to 0 \text{ when } t' \to \infty. \tag{2.2b}$$

So the phase $\delta(t')$ coincides with the P-wave $\pi\pi$ elastic phase shift and consequently with the phase of the form factor $F(t)$ on the range $(4m_\pi^2, t_1)$ and is arbitrary above t_1 with the only restriction

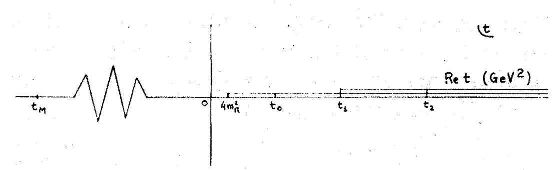

FIG. 1. The analytical structure of the form factors $F(t)$ and $G(t)$. $F(t)$ has a cut from $4m_\pi^2$ to ∞ and $G(t)$ from t_1 to ∞. On the range $(4m_\pi^2, t_1)$ the phase of the form factor is assumed to be known; on (t_0, t_1) modulus and phase are known and from t_1 to t_2 only the modulus is known.

of vanishing at infinity. That implies that $J(t)$ has the phase of $F(t)$ up to $t=t_1$ and then it goes asymptotically to a constant

$$J(t) \underset{t\to\infty}{\sim} \exp\left[-\frac{1}{\pi}\int_{4m_\pi^2}^{\infty} \frac{\delta(t')}{t'}\right]. \tag{2.3}$$

Now the function

$$G(t)=F(t)/J(t), \tag{2.4}$$

which is a kind of "normalized" form factor, is analytical on the whole complex t plane except for a cut going from t_1 up to infinity. Since $J(t)$ can be computed for all values of t, $G(t)$ is known wherever $F(t)$ is known. Moreover, the choice of the subtraction point in the exponent of $J(t)$ (2.1) is such that $J(0)=1$ and the normalization property $F(0)=1$ translates without any change to $G(t)$.

Therefore, the information about the phase of the form factor is completely incorporated when $G(t)$ is assumed to be analytical anywhere except for a branching point at $t=t_1$. We can now reexpress $a_\mu(\pi^+\pi^-)$ in terms of the "normalized" form factor $G(t)$:

$$a_\mu(\pi^+\pi^-)=\int_{4m_\pi^2}^{\infty} dt\, q(t)\,|G(t)|^2, \tag{2.5}$$

where

$$q(t)=\rho(t)\,|J(t)|^2. \tag{2.6}$$

The asymptotic behavior of $J(t)$ (2.3) ensures that the new functions appearing in (2.5), $q(t)$ and $G(t)$, behave asymptotically just as the old ones $\rho(t)$ and $F(t)$, respectively.

We can assume, without loss of generality, that $F(t)$ and consequently $G(t)$ satisfy a once-subtracted dispersion relation. Indeed, if a dispersion relation for $G(t)$ needs more than one subtraction, then the integral (2.5) giving $a_\mu(\pi^+\pi^-)$ diverges. As we look for a lower bound, that asymptotic behavior cannot be assumed and therefore we need to consider only unsubtracted or, at most, once-subtracted dispersion relations. In fact, the function $q(t)$ in (2.5) is strongly peaked at low energies so that the form factor $G(t)$ which will give the lowest value for $a_\mu(\pi^+\pi^-)$ will saturate the maximal asymptotic behavior allowed by the convergence of (2.5), i.e., it will require a subtraction. So we shall start by writing down a once-subtracted dispersion relation for $G(t)$ and, as a matter of fact, we shall see later on that even if we start by assuming that $G(t)$ satisfies an unsubtracted dispersion relation, we are led to the conclusion that a lower bound cannot be reached with such an asymptotic behavior, the saturation of the bound occurring only when the asymptotic behavior of $G(t)$ is such that a subtraction is needed.

The normalization property provides a known subtraction constant at $t=0$, so that we can write

$$\mathrm{Re}\,G(t)=1+\frac{t}{\pi}\int_{t_1}^{\infty} dt'\,\frac{\mathrm{Im}\,G(t')}{t'(t'-t)}, \tag{2.7}$$

where the integral has to be understood in the principal-value sense when $t>t_1$.

Dispersion relation (2.7) contains the analyticity properties of the modified form factor $G(t)$ and therefore is one of the main tools in finding constraints on $a_\mu(\pi^+\pi^-)$. In fact, (2.7) is quite difficult to handle as it stands, owing to the mixed nature of the constraint, local for the real part and integral for the imaginary part, and also owing to the presence of the singular Cauchy kernel. One way of handling analyticity constraints which has been successfully used in another context[9] consists in considering averages of the dispersion relation (2.7) rather than the dispersion relation itself. To do that, let us take a known arbitrary smooth weight function $\omega(t)$ and let us replace (2.7) by

$$\int_{t_M}^{\infty} dt\,\omega(t)\,(\mathrm{Re}\,G(t)-1)=\int_{t_1}^{\infty} dt'\,\overline{\omega}(t')\,\mathrm{Im}\,G(t'), \tag{2.8}$$

where $\omega(t)$ vanishes for values of t between $t=0$ and $t=4m_\pi^2$ and $\overline{\omega}(t')$ is

$$\overline{\omega}(t')=\frac{1}{\pi}\int_{t_M}^{\infty} dt\,\frac{t\,\omega(t)}{t'(t'-t)}. \tag{2.9}$$

Of course, the averaging of the dispersion relation weakens the constraints contained in it, but in exchange we obtain something much easier to handle: one integral sum rule for the form factor with nonsingular kernels. The important point is that, as shown in,[9] it is possible to recover all the constraining power of the dispersion relation (2.7) by varying, at the end, the arbitrary function $\omega(t)$ in a sufficiently large functional space. For the moment $\omega(t)$ will remain a fixed arbitrary function and the only theoretical constraint on $G(t)$ to consider is (2.8).

Let us now assume that we experimentally know $F(t)$, and therefore also $G(t)$ via Eq. (2.4), on a spacelike range running from $t=t_M<0$ up to $t=0$, and the modulus $|F(t)|$ at values of t from $t=t_0$ to $t=t_2$ ($t_0, t_2>4m_\pi^2$, Fig. 1). Since the information on the modulus allows us to compute a piece of the integral (1.6) giving $a_\mu(\pi^+\pi^-)$, let us define clearly the quantity to bound by splitting the integral in (2.5) into two parts:

$$a_\mu(\pi^+\pi^-)=a_\mu^{(0)}+a_\mu^{(1)}, \tag{2.10}$$

where

$$a_\mu^{(0)} = \int_{t_0}^{t_2} dt\, q(t)\, |G(t)|^2 , \qquad (2.11a)$$

$$a_\mu^{(1)} = \int_{4m_\pi^2}^{t_0} dt\, q(t)\, |G(t)|^2 + \int_{t_2}^{\infty} dt\, q(t)\, |G(t)|^2 . \qquad (2.11b)$$

Since $a_\mu^{(0)}$ can be directly computed by injecting the measured values of $|G(t)|$ into (2.11a), the only quantity to be bounded is $a_\mu^{(1)}$ which contains the very low energy and the asymptotic contributions to $a_\mu(\pi^+\pi^-)$.

Before applying the Lagrange-multiplier method to our problem, it is convenient to define

$$\alpha(t) = \mathrm{Re}\, G(t) , \qquad (2.12a)$$

$$\beta(t) = \mathrm{Im}\, G(t), \quad \beta(t) = 0 \text{ for } t < t_1 . \qquad (2.12b)$$

The proper Lagrangian is

$$\mathcal{L} = \int_{4m_\pi^2}^{t_0} dt\, q(t)\, \alpha^2(t) + \int_{t_2}^{\infty} dt\, q(t) [\alpha^2(t) + \beta^2(t)] + \int_{t_0}^{t_2} dt\, \lambda(t) [\alpha^2(t) + \beta^2(t) - M^2(t)] + L \left\{ \int_{t_M}^{\infty} dt\, \omega(t) [\alpha(t) - 1] - \int_{t_1}^{\infty} dt\, \bar{\omega}(t) \beta(t) \right\} , \qquad (2.13)$$

where $M(t)$ is the experimentally known modulus of $G(t)$, L is the Lagrange multiplier associated with the constraint (2.8), and $\lambda(t)$ is the Lagrange multiplier function associated with the constraints coming from the knowledge of $|G(t)|$.

It is straightforward to write down and solve the Lagrange equations

$$\frac{\delta \mathcal{L}}{\delta(\alpha(t))} = \frac{\delta \mathcal{L}}{\delta(\beta(t))} = 0 , \qquad (2.14)$$

which imply that $\lambda(t)$ must be positive in order to have a lower bound and that the functions $\alpha(t)$, $\beta(t)$ saturating the bound are

$$\alpha(t) = -\frac{L\omega(t)}{2q(t)}, \quad 4m_\pi^2 < t < t_0 \qquad (2.15a)$$

$$\alpha(t) = -M(t) \times \mathrm{sign}(L\, \omega(t)), \quad t_0 < t < t_1 \qquad (2.15b)$$

$$\left.\begin{aligned} \alpha(t) &= -\frac{M(t)\omega(t)}{[\omega^2(t) + \bar{\omega}^2(t)]^{1/2}} \times \mathrm{sign}(L) \\ \beta(t) &= \frac{M(t)\bar{\omega}(t)}{[\omega^2(t) + \bar{\omega}^2(t)]^{1/2}} \times \mathrm{sign}(L) \end{aligned}\right\}, \quad t_1 < t < t_2 \qquad (2.15c)$$

$$\alpha(t) = -\frac{L\omega(t)}{2q(t)}, \quad \beta(t) = \frac{L\bar{\omega}(t)}{2q(t)}, \quad t > t_2 . \qquad (2.15d)$$

The value of L can be explicitly worked out by introducing expressions (2.15) into the constraint (2.8). This gives

$$P[\omega, F] = \tfrac{1}{2} L\, Q[\omega] + \mathrm{sign}(L) R[\omega, M] , \qquad (2.16)$$

where the functionals $P[\omega,F]$, $Q[\omega]$, $R[\omega,M]$ have the following form:

$$P[\omega, F] = \int_{t_M}^{0} dt\, \omega(t) G(t) - \int_{t_M}^{\infty} dt\, \omega(t) , \qquad (2.17a)$$

$$Q[\omega] = \int_{4m_\pi^2}^{t_0} dt\, \frac{\omega^2(t)}{q(t)} + \int_{t_2}^{\infty} dt\, \frac{\omega^2(t) + \bar{\omega}^2(t)}{q(t)} , \qquad (2.17b)$$

$$R[\omega, M] = \int_{t_0}^{t_1} dt\, M(t) |\omega(t)| + \int_{t_1}^{t_2} dt\, M(t) [\omega^2(t) + \bar{\omega}^2(t)]^{1/2} . \qquad (2.17c)$$

Since $Q[\omega]$ and $R[\omega,M]$ are positive, Eq. (2.16) has a solution only if

$$|P[\omega, F]| > R[\omega, M] . \qquad (2.18)$$

When condition (2.18) is fulfilled, the solution of (2.16) is

$$\frac{L}{2} = \frac{|P[\omega, F]| - R[\omega, M]}{Q[\omega]}\, \mathrm{sign}(P[\omega, F]) . \qquad (2.19)$$

For functions $\omega(t)$ not satisfying inequality (2.18), the solution is $L = 0$. Then the signs of $\alpha(t)$, $\beta(t)$ on the range (t_0, t_2) are not fixed by Eqs. (2.15b) and (2.15c), and it is possible to saturate the constraint (2.18) by choosing properly the signs on that range and taking out $\alpha(t) = \beta(t) = 0$. Therefore in those cases the analyticity constraint (2.8) is trivially satisfied and does not require the form factor $G(t)$ to be different from zero on the ranges $(4m_\pi^2, t_0)$, (t_2, ∞) and the bound for $a_\mu^{(1)}$ is zero.

In concusion, nontrivial bounds for $a_\mu^{(1)}$ can be obtained only if inequality (2.18) holds. In that case, from (2.11b) and (2.15), we get

$$a_\mu^{(1)} \geq \tfrac{1}{4} L^2 Q[\omega] , \qquad (2.20)$$

and introducing the value of L given in (2.19) into (2.20)

$$a_\mu^{(1)} \geq \frac{(|P[\omega, F]| - R[\omega, M])^2}{Q[\omega]} , \qquad (2.21)$$

which is our final expression for the bound. As remarked in the Introduction, it is given as a simple algebraic function of the quantities $P[\omega,F]$, $R[\omega,M]$ which are smooth averages of the form factor and its modulus on the regions, respectively, where they are experimentally known. Therefore, it is easy to compute the errors on $P[\omega,F]$ and $R[\omega,M]$ coming from the errors of the experimental data and, consequently, to determine the overall uncertainty on $a_\mu^{(1)}$ associated with the experimental errors in the input. By assuming that uncertainty to be small, it is approximately given by the following expression:

$$\Delta a_\mu^{(1)} \simeq 2\left(\frac{a_\mu^{(1)}}{Q[\omega]}\right)^{1/2}(\Delta P[\omega,F]+\Delta R[\omega,M])\,. \tag{2.22}$$

The control of $\Delta a_\mu^{(1)}$ is a crucial point. Indeed, it is perfectly possible to find functions $\omega(t)$ making the bound (2.21) as large as we want, that meaningless result being a consequence of the overdetermined nature of the problem. Of course, such functions $\omega(t)$ also give nonsense results for $\Delta a_\mu^{(1)}$. In order to give a meaning to the bound, it is then necessary to combine the excess in the input with the experimental uncertainty accompanying that input.

Since $\Delta a_\mu^{(1)}$ measures the stability of the result under variations in the input of the order of the experimental errors, a meaningful result is obtained if we require the function $\omega(t)$ to produce small values for the quantity $\Delta a_\mu^{(1)}$. On the other hand, that gives a reasonable estimate of the uncertainty of the result coming from the experimental errors in data.

Therefore, we shall restrict ourselves to functions $\omega(t)$ satisfying inequality (2.18) and making $\Delta a_\mu^{(1)}$ in (2.22) smaller than a fixed value. Then we shall vary $\omega(t)$ within the class of functions satisfying those two conditions in order to optimize the bound.

To conclude this section, let us point out that a similar analysis can be performed if it is assumed that $G(t)$ obeys an unsubtracted dispersion relation. In that case, the normalization condition enters as an independent constraint. What is found is that the solution of the Lagrange equations for the imaginary part $\beta(t)$ grows asymptotically. Therefore, the unsubtracted dispersion relation diverges in contradiction with the starting hypothesis. That means that the function that saturates the lower bound is certainly not to be found within the class of functions satisfying an unsubtracted dispersion relation, according to the conclusion of the qualitative arguments developed in the first section.

III. EXPERIMENTAL INPUT AND WEIGHT FUNCTIONS

We have already explained in the preceding section that the phase of the form factor is automatically incorporated when the P-wave $\pi\pi$ phase shift contained in the function $J(t)$ (2.1) is subtracted. As pointed out in Ref. 10 that phase is experimentally fixed with a high accuracy[11] when the scattering length is fixed. On the other hand, analyticity, unitarity, and crossing, plus the experimental results for other $\pi\pi$ partial waves, permit us to conclude that the P-wave scattering length is $a_1^1 = 0.04\, m_\pi^{-3}$ with an extremely small uncertainty.[10]

In order to fulfill the required threshold behavior and asymptotic vanishing for $\delta(t)$, we parametrize that function in the following way:

$$\frac{2q^3 m_\pi}{\sqrt{t}}\cot\delta(t) = \frac{1}{a_1^1} + q^2 \sum_{j=0}^{N} A_j q^{2j}\,, \tag{3.1}$$

where $q^2 = (t-4m_\pi^2)/4$ and $A_N > 0$. The coefficients A_j are fitted to the experimental P-wave phase shifts for each fixed value of the scattering length. In the following, we shall always assume that a_1^1 has been fixed at $a_1^1 = 0.04 m_\pi^{-3}$ and only at the end of the paper shall we comment on the results obtained when the scattering length is varied.

As pointed out before, the uncertainty on that phase when the scattering length is fixed is extremely small compared to that of the modulus of the form factor, and consequently we shall consider it as exact.

To finish with the phase specifications, we shall assume that the phase of the form factor coincides with the P-wave $\pi\pi$ strong phase shift from threshold to $t_1 = m_\rho^2$. In that range we are sure that the only intermediate hadronic state is the $I=1$ two-pion state so that the modified form factor $G(t)$ is analytic in the complex t-plane except for a cut from t_1 up to infinity.

The modulus of the form factor is quite accurately known over a range of intermediate energies which covers essentially the ρ and ω meson region. For the sake of definiteness, and in order to compare with preceding results,[6] we shall take that range to go from $t_0 = 0.3$ GeV^2 up to $t_2 = 1.0$ GeV^2. There are several parametrizations which well describe the data in that region.[12-14] All of them, in spite of considerable analytical differences, give almost exactly the same values for the modulus of the form factor on the region between t_0 and t_2. So we shall use hereinafter the Benaksas *et al.*[12] parametrization. Introducing it into formula (2.11a), we get for the intermediate energy contribution to the muon anomaly

$$a_\mu^{(0)} = (38.0 \pm 3.0)\times 10^{-9}\,, \tag{3.2}$$

the other parametrizations[13,14] giving almost exactly the same value for $a_\mu^{(0)}$. (For instance, we obtain $a_\mu^{(0)} = 37.9\times 10^{-9}$ when the parametrization due to Barger *et al.*[14] is used.) The estimation of the error appearing in (3.2) has been made according to that performed by Barger *et al.* in Ref. 14.

For the computation of averages on $|F(t)|$ we take the very conservative prescription of a 10% error on the modulus (not on the square of the modulus) at every point t where it is assumed to be known. Therefore, owing to the positivity of the functions appearing in the definition of $R[\omega,M]$, we simply have

$$\Delta R[\omega,M]=0.1\times R[\omega,M], \qquad (3.3)$$

On the spacelike region, we assume that $F(t)$ is known from $t_M=-4.0\ \text{GeV}^2$ to zero according to Bebek *et al.*[15] All data on the spacelike region have been collected in Ref. 15 and a global fit has been performed. Thus, in order to compute averages on $F(t)$, i.e., $P[\omega,F]$, we take the simple pole parametrization given in that paper with a constant error of 5% at every point on the range $(t_M, 0)$. The relative error on the first term in (2.17a), which is the only contribution to $P[\omega,F]$ containing $F(t)$ in the spacelike region, will be in general larger than 5% due to the nonpositivity of the weight function $\omega(t)$:

$$\Delta P[\omega,F]=0.05\times\int_{t_M}^{0} dt|\omega(t)G(t)| . \qquad (3.4)$$

The assumption of a 5% constant uncertainty on data is a crude but realistic estimation. We have performed a more elaborate estimation with errors whose value depends on t and the result is practically unchanged.

Finally, we have to specify the form of the functions $\omega(t)$ that we are going to insert in (2.17) to numerically compute the bound.

If we want to avoid divergences, we must require the functions $\omega(t)$ to vanish asymptotically at least as t^{-2} in order to compensate for the asymptotic behavior of $G(t)$ allowed by the dispersion relation.

Notice that in that case $\bar{\omega}(t)$ has the same asymptotic behavior up to logarithms, and the asymptotic behavior of the form-factor solution of the Lagrange problem (2.15d) agrees with the starting hypothesis of a once-subtracted dispersion relation for $G(t)$. The actual parametrization we have chosen is

$$\omega(t)=\sum_{l=0}^{L} c_l P_l\left(1-\frac{2t}{t_M}\right), \quad t_M\leq t\leq 0 \qquad (3.5a)$$

$$\omega(t)=\frac{1}{(t+t_a)(t+t_b)}\sum_{m=0}^{M} d_m P_m(z), \quad t>4m_\pi^{\,2} \qquad (3.5b)$$

where

$$z=\frac{t-4m_\pi^{\,2}-t_c}{t-4m_\pi^{\,2}+t_c} . \qquad (3.6)$$

The P_l, P_m appearing in (3.5) are Legendre polynomials and t_a, t_b, t_c, c_l, d_m are constants. The function $\bar{\omega}(t)$ can be straightforwardly computed, the result being

$$\bar{\omega}(t)=\sum_{l=0}^{L} c_l\left[\frac{t_M}{\pi t}\delta_{l0}+\frac{2}{\pi}Q_l\left(1-\frac{2t}{t_M}\right)\right] + \frac{2}{\pi t(t_b-t_a)(t+t_a)(t+t_b)}\sum_{m=0}^{M} d_m\Big[t(t_b-t_a)Q_m(z)+t_a(t+t_b)Q_m(z_a)-t_b(t+t_a)Q_m(z_b)\Big] \qquad (3.7)$$

with

$$z_x=\frac{t_x+4m_\pi^{\,2}+t_c}{t_x+4m_\pi^{\,2}-t_c} , \quad x=a,b. \qquad (3.8)$$

All components entering into the computation of the bound have been described. The only thing to do is to insert them into (2.21) and (2.22), to fix a maximal allowed uncertainty $\Delta a_\mu^{(1)}$ and then to vary the function $\omega(t)$ (by varying the constants t_a, t_b, t_c, c_l, d_m) in order to optimize the bound.

IV. RESULTS AND COMMENTS

First, let us forbid the quantity $\Delta a_\mu^{(1)}$ to be greater than 1×10^{-9}. That will produce a total uncertainty on the two-pion contribution to the muon anomaly $a_\mu(\pi^+\pi^-)$ equal to 4×10^{-9}, better than the present phenomenological estimates.[14,16] For a scattering length $a_1^1=0.04 m_\pi^{-3}$, the result is $a_\mu^{(1)}\geq 8.1\times10^{-9}$. That value combined with the value directly computed for $a_\mu^{(0)}$ gives a total result

$$a_\mu(\pi^+\pi^-)\geq(46.1\pm4.0)\times10^{-9}, \qquad (4.1)$$

the function $\omega(t)$ producing the bound (4.1) being determined by means of the parameters listed in column A of Table I. Intermediate results for averages P, R, Q can be found in Table II.

Raszillier *et al.*[6] have recently obtained for the same quanitity a bound of 42×10^{-9} which is to be

TABLE I. The parameters defining the function $\omega(t)$ according to Eqs. (3.5). Columns A and B correspond to bounds (4.1) and (4.2), respectively, where all the information has been incorporated. Columns C and D correspond to the case in which the timelike data on the modulus of the form factor have been ignored.

	A	B	C	D
a_1^1	0.04	0.03	0.04	0.03
t_a	0.2	0.2	0.25	0.25
t_b	0.4	0.4	0.5	0.5
t_c	0.6	0.6	0.6	0.6
c_0	1	1	1	1
c_1	2.829	2.330	3.532	3.583
c_2	−4.854	−2.124	−4.878	−4.806
c_3	0.624	−1.772	−0.279	−0.433
d_0	0.639	0.537	0.647	0.601
d_1	−1.189	−0.921	−0.994	−0.947
d_2	0.619	0.506	−0.213	−0.141
d_3	0.206	0.241	0.903	0.880
d_4	−0.831	−0.893	−0.726	−0.798
d_5	0.600	0.528	0.042	0.037
d_6	−0.458	−0.244	0.055	0.179
d_7	0.371	0.092	−0.043	−0.071
d_8	−0.143	−0.047	0.143	0.052

compared with (4.1). The difference between the two values might be attributed to the fact that the phase of the form factor has not been considered in Ref. 6. On the other hand, they do not treat the error problem in a proper way and they do not consider the very important experimental errors affecting the modulus of the form factor on the timelike region.

It is remarkable that the value in (4.1) practically coincides with the phenomenological estimates. Indeed, assuming for instance that the parametrization of Barger *et al.*[14] or Benaksas *et al.*[12] describes the form factor over the whole timelike region, the results for $a_\mu(\pi^+\pi^-)$ are 48.1 $\times 10^{-9}$ and 47.7×10^{-9}, respectively. Of course, what we get is a bound and not a real estimate. However, the abundance of experimental input

TABLE II. We summarize the final results in rows 6 and 7 and the intermediate quantities P, R, Q defined in (2.17) in the first five rows. The meaning of columns A, B, C, D is as in Table I.

	A	B	C	D
$P[\omega,F]$	−4.28	−3.98	−3.47	−3.48
$\Delta P[\omega,F]$	0.16	0.13	0.18	0.17
$R[\omega,M]$	0.65	0.55	0	0
$\Delta R[\omega,M]$	0.07	0.06	0	0
$10^{-9}Q[\omega]$	1.63	1.25	0.42	0.39
$10^9 a_\mu^{(1)}$	8.1	9.4	28.8	31.1
$10^9\Delta a_\mu^{(1)}$	1.0	1.0	3.0	3.0

entering into the derivation of the bound suggests that it must not be far from the actual value of $a_\mu(\pi^+\pi^-)$. If that conjecture is true, then it becomes important to note that the uncertainty depicted in (4.1) is approximately half of that estimated in phenomenological analysis.

If we now vary a_1^1 it turns out that the bound is a slightly decreasing function of a_1^1, behaving almost linearly for values close to $a_1^1=0.04m_\pi^{-3}$. In order to be more precise, the actual result for a scattering length $a_1^1=0.03m_\pi^{-3}$ is

$$a_\mu(\pi^+\pi^-)\geq(47.4\pm4.0)\times10^{-9}\,, \tag{4.2}$$

the parameters and partial results relevant to the bound (4.2) being listed in column B of Tables I and II.

Since we have an explicit solution (2.15) saturating the bound, we can compute other interesting parameters associated with each of those solutions. In particular, we can compute the "charge radius" of the pion $\langle r_\pi^2\rangle^{1/2}$, which in terms of the modified form factor $G(t)$ has the form

$$\langle r_\pi^2\rangle = 6\left.\frac{dF(t)}{dt}\right|_{t=0} = \frac{6}{\pi}\left[\int_{t_1}^{\infty}dt'\,\frac{\mathrm{Im}G(t')}{t'^2}+\int_{4m_\pi^2}^{\infty}dt'\,\frac{\delta(t')}{t'^2}\right]. \tag{4.3}$$

The actual value of the radius $\langle r_\pi^2\rangle^{1/2}$ when inserting into (4.3) the expression $\mathrm{Im}\,G(t')$ given in (2.15c) and (2.15d) depends strongly on small variations of the function $\omega(t)$. Thus, functions slightly different which give almost exactly the same bound can produce quite different values for the charge radius $\langle r_\pi^2\rangle^{1/2}$. This is easy to understand from the fact that $\langle r_\pi^2\rangle^{1/2}$ is a derivative, that is, a local property which can be modified without introducing any substantial change in the averaged quantities entering into the derivation of the bound. Nevertheless, the values obtained for $\langle r_\pi^2\rangle^{1/2}$ lie between 0.65 and 0.75 fm, that is, in the region of phenomenologically estimated values.[15] This reinforces the idea that in fact the bound can be assumed to be very close to the actual value, and on the other hand, it implies that the bound cannot be substantially improved by imposing the experimental quadratic radius as a new constraint.

In order to illustrate how the method works when a piece of information is ignored, we have computed the bound in a case in which the only experimental input is the phase and the value of the form factor on the spacelike region; i.e., we have eliminated the very important experimental information about the modulus of the form factor on the ρ-meson region. In this case, the quantity $R[\omega,M]$ in (2.21) vanishes, the unequality (2.18) is trivially fulfilled, and $Q[\omega]$ becomes

$$Q[\omega]=\int_{4m_\pi^2}^{t_1} dt\ \frac{\omega^2(t)}{q(t)}+\int_{t_1}^{\infty} dt\ \frac{\omega^2(t)+\overline{\omega}^2(t)}{q(t)}\ . \tag{4.4}$$

The result for a scattering length $a_1^1=0.04m_\pi^{-3}$ and a maximal allowed uncertainty of 3×10^{-9} is

$$a_\mu(\pi^+\pi^-)\geq(28.8\pm3.0)\times10^{-9} \tag{4.5}$$

and for $a_1^1=0.03m_\pi^{-3}$:

$$a_\mu(\pi^+\pi^-)\geq(31.1\pm3.0)\times10^{-9}\,. \tag{4.6}$$

Bounds (4.5) and (4.6) are of the same order of the phenomenological estimates in spite of the fact that all of them are strongly dominated by the experimentally observed ρ-meson enhancement in the modulus of the form factor. They have to be compared with the preceding bounds obtained from a similar input. The best one is that recently derived by Raina and Singh[7] which gives a value of 22.8×10^{-9}. In their paper, Raina and Singh do not incorporate the phase of the form factor as an input, but perhaps the most important difference is that they take as spacelike input the value of the form factor at a few points, that is, local information, and the derivative of the form factor at $t=0$. The value of this derivative is obtained from a fit and therefore it is affected by an artificially small error coming in fact more from the specific form of the chosen parametrization than from the experimental errors.

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