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Stopping-power formula for magnetic monopoles

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A stopping-power formula is presented for magnetic monopoles with charge $|g| = 137e/2$ and $|g| = 137e$. Close- and distant-collision effects are considered and the estimated accuracy in an absolute sense is $\pm 3\%$ for $\beta \gtrsim 0.2$ and $\gamma \lesssim 100$.

In an earlier paper[1] the total stopping power and the restricted energy loss of magnetic monopoles with charge $|g| = 137e$ was estimated by using the classical impact-parameter approach of Fermi[2] as modified by Tompkins[3] to be appropriate to magnetic charges. Close-collision effects were taken into account by suitably choosing a quantum-mechanical minimum impact parameter. While this approach works reasonably well for small electric charges,[4] it fails miserably for extremely large charges where significant deviations of the Rutherford from the Mott cross sections are encountered.[5] (For $Z_e = 100$ and $\beta \sim 1$ the correct stopping-power formula is ~20% larger than that given by the Bethe-Bloch formula; refer to Ref. 5 for a discussion of this point.) One might expect higher-order quantum electrodynamics to similarly affect magnetic monopoles. The absence of a good theory for electron-monopole interactions in the relativistic regime has thus far prevented the analysis of this problem. Recently, Kazama, Yang, and Goldhaber[6] (KYG hereafter) have used the Dirac equation for an electron moving in the magnetic field of a fixed monopole to obtain the differential scattering cross section. If the magnetic-monopole charge-to-mass ratio is comparable to or smaller than the corresponding nuclear value, as one would expect for a 't Hooft-type monopole,[7] then the use of the KYG cross section should be as reliable as the Mott cross section for calculating the close-collision contribution to the stopping power. Both of these cross sections should become inadequate to this task for $\gamma > 100$, beyond which spin effects, internal-structure contributions, primary-particle bremsstrahlung, radiative corrections, and kinematical complications become important.

By using the tabulated values for the KYG cross section[6] one finds, after an appropriate angular integration, that the close-collision energy loss is

$$\left.\frac{dE}{dx}\right|_c = \frac{2\pi N g^2 e^2}{mc^2}\left[\ln\frac{w_m}{w_0} + K(|g|)\right], \qquad (1)$$

where

$$K(|g|) = \begin{cases} 0.406 & \text{for } |g| = 137e/2\,, \\ 0.346 & \text{for } |g| = 137e\,. \end{cases} \qquad (1a)$$

N is the electron density, mc^2 is the electron rest energy, $-e$ is the charge of the electron, and g is the charge of the monopole. The monopole velocity is $\beta c, \gamma = 1/(1-\beta^2)^{1/2}$ and $w_m = 2mc^2\beta^2\gamma^2$ provided $m\gamma \ll M$ where M is the monopole mass. The energy transfer below which it is no longer valid to use the free electron approximation is denoted by w_0. It is interesting to note that the QED correction, $K(|g|)$, is independent of the monopole velocity and is only mildy dependent on charge. This follows from the velocity independence of the KYG to the classical cross-section ratio.[6]

In order to obtain the total energy loss, the distant-collision energy loss must be added to the close-collision loss as given by Eq. (1). The analogous problem for incident electric charges is solved by treating the interaction between the particle and the atom in the first Born approximation.[8] All collisions are characterized by momentum transfer, and Bethe's generalized sum rule allows an unambiguous joining between the distant collisions (which are treated in the dipole approximation) and the close collisions (which are treated in the free electron approximation). The situation is somewhat more complicated for incident magnetic charges due to the complexity of the quantum-mechanical formalism for these particles. I therefore adopt the semiclassical approach of Landau[9] in which the distant collisions are considered from the point of view of classical macroscopic electrodynamics but are characterized by momentum transfer rather than impact parameter. This characterization is made possible by interpretation of the vector $\vec{k}$ which appears in the Fourier transforms of the classical fields as the wave vector of an exchanged photon.

The magnetic-monopole analog of Eq. (85.15) of Ref. 9 is easily shown to be

$$\left.\frac{dE}{dx}\right|_d = \frac{4\pi N g^2 e^2}{mc^2}\left[\ln\frac{\gamma\beta c q_0}{\Omega_m} - \tfrac{1}{2} + \xi^2/(2\beta^2\gamma^2\omega_p^2)\right], \qquad (2)$$

where $\omega_p = (4\pi N e^2/m)^{1/2}$ is the plasma frequency of the medium, $\hbar q_0$ is the maximum momentum transfer for which the above treatment is valid, and $\xi = \omega(0)/i$, where $\omega(q)$ is defined by

$$\omega^2(q)[\epsilon(\omega(q)) - 1/\beta^2] = c^2 q^2 , \tag{3}$$

where $\epsilon(\omega)$ is the complex dielectric constant of the medium. The quantity Ω_m plays the role of the mean excitation frequency and is defined by

$$\ln\Omega_m = \frac{2}{\pi\omega_p^2}\int_0^\infty \omega \,\mathrm{Im}[\epsilon(\omega)] \ln(\omega^2 + \xi^2)^{1/2} d\omega . \tag{4}$$

It has been assumed that the medium is nonpermeable, i.e., $\mu = 1$.

In those cases for which there are two roots to Eq. (3) with $q = 0$, that value of $\omega(0)$ with the largest imaginary part is to be used in the definition $\xi = \omega(0)/i$. Hence, if $\beta^2 < 1/\epsilon_0$ [where $\epsilon_0 = \epsilon(0)$], $\xi = 0$, and if $\beta^2 > 1/\epsilon_0$, ξ is defined by $\beta^2\epsilon(i\xi) = 1$. For conductors, $\epsilon_0 = \infty$, so that the latter value of ξ should always be used.

If it assumed that there exists a value of q_0 for which both the close- and distant-collision approximations are valid (with $w_0 = \hbar^2 q_0^2/2m$), then the stopping-power formula is obtained by adding Eq. (1) to Eq. (2):

$$\left.\frac{dE}{dx}\right|_m = \frac{4\pi N g^2 e^2}{mc^2}\left[\ln\frac{2mc^2\beta^2\gamma^2}{I_m} + K(|g|)/2 - \tfrac{1}{2} - \delta_m/2\right], \tag{5}$$

where the magnetic mean ionization potential and density-effect correction are given by

$$\ln I_m = \frac{2}{\pi\omega_p^2}\int_0^\infty \omega\,\mathrm{Im}[\epsilon(\omega)] \ln\hbar\omega \, d\omega \tag{6}$$

and

$$\delta_m = \frac{2}{\pi\omega_p^2}\left\{\int_0^\infty \omega\,\mathrm{Im}[\epsilon(\omega)]\ln\left(1+\frac{\xi^2}{\omega^2}\right)d\omega - \frac{\pi}{2}\xi^2\frac{(1-\beta^2)}{\beta^2}\right\}. \tag{7}$$

The corresponding electric stopping power is given by (to the first Born approximation)

$$\left.\frac{dE}{dx}\right|_e = \frac{4\pi N Z_1^2 e^4}{mv^2}\left(\ln\frac{2mc^2\beta^2\gamma^2}{I_e} - \beta^2 - \delta_e/2\right) , \tag{5$'$}$$

with the electric mean ionization potential and density-effect correction

$$\ln I_e = \frac{2}{\pi\omega_p^2}\int_0^\infty \omega\,\mathrm{Im}\left[-\frac{1}{\epsilon(\omega)}\right]\ln\hbar\omega\, d\omega \tag{6$'$}$$

and

$$\delta_e = \frac{2}{\pi\omega_p^2}\left\{\int_0^\infty \omega\,\mathrm{Im}\left[-\frac{1}{\epsilon(\omega)}\right]\ln\left(1+\frac{\xi^2}{\omega^2}\right)d\omega - \frac{\pi}{2}\xi^2(1-\beta^2)\right\} . \tag{7$'$}$$

The dielectric constant is usually expressed as a sum over the oscillators of a given *atom*,[8]

$$\epsilon(\omega) = 1 + \omega_p^2 \sum_n \frac{f_n}{\omega_n^2 - \omega^2 - i\Gamma\omega} , \tag{8}$$

where f_n is the oscillator strength of the nth transition, $\hbar\omega_n$ is the transition energy, and Γ is the damping constant. In the limit $\omega_p \to 0$ it is clear that $\mathrm{Im}\epsilon \to \mathrm{Im}(-1/\epsilon)$, and therefore $I_m \to I_e$ and $\delta_m \to \delta_e$ (for $\beta \sim 1$) in this limit. Thus, the magnetic ionization potential is the same as the electric ionization potential in gases for all β and the density effect corrections are also the same for $\beta \sim 1$.

To investigate the relationships of these quantities in condensed media, let us restrict our attention to nonconductors for simplicity. (Sternheimer and Peierls[10] show that the distinction between conductors and nonconductors is of no practical concern for the case of incident electric charges; this is probably not true for magnetic charges because of the absence of longitudinal screening.) In this case ϵ_0 is finite and there is a sharp dividing velocity below which there is no density effect correction, namely $\beta_0 = 1/\sqrt{\epsilon_0}$. I_e is then just the experimentally determined ionization potential. [If one demands equality between Eq. (5′), with $\delta_e = 0$ for $\beta < 1/\sqrt{\epsilon_0}$ and experimental results, shell corrections are included in I_e; however, for large enough velocities these shell corrections are small enough to be neglected for all but the heaviest absorbers.[8]] Sternheimer[11] has expressed I_e in terms of f_n and ω_n:

$$\ln I_e = \sum_n f_n \ln[\hbar\,\omega_n(1 + \omega_p^2 f_n/\omega_n^2)^{1/2}] . \tag{9}$$

By making use of the small damping expression for $\mathrm{Im}[\epsilon(\omega)]$,[8]

$$\lim_{\Gamma\to 0} \mathrm{Im}[\epsilon(\omega)] = \pi\omega_p^2 \sum_n f_n \delta(\omega^2 - \omega_n^2) , \tag{10}$$

the analogous expression for I_m is obtained:

$$\ln I_m = \sum_n f_n \ln\hbar\omega_n . \tag{11}$$

Since the f_n, ω_n refer to isolated atoms it is clear that the low-velocity energy loss for monopoles is the same per g/cm² for the same type of material in either gaseous or condensed form while this is not true for electric charges. The difference in behavior is due to the absence of a longitudinal interaction between the monopole and atomic electrons. As a consequence there is no dielectric screening such as that experienced in the interaction between electric charges in a condensed medium.

The relationship between I_m and I_e can be expressed as

$$I_m = I_e \exp(-D/2) \ , \tag{12}$$

where

$$D = \sum_n f_n \ln(1 + \omega_p^2 f_n / \omega_n^2) \ . \tag{13}$$

Sternheimer[11] has estimated the magnitude of D for several solids. He finds that $D(\mathrm{Li}) = 0.34$, $D(\mathrm{C}) = 0.22$, $D(\mathrm{Al}) = 0.056$, $D(\mathrm{Fe}) = 0.14$, $D(\mathrm{Cu}) = 0.13$, $D(\mathrm{Ag}) = 0.09$, $D(\mathrm{Sn}) = 0.05$, and $D(\mathrm{W}) = 0.07$. Sternheimer emphasizes that these values are extremely uncertain because of the sensitivity of D to the distribution of energies of the outer shells. However, it seems that the amount of error incurred in the dE/dx formula is of the order of 1% or less if one uses I_e in place of I_m for absorbing media heavier than carbon.

In a similar way it can be shown that the error is quite small if one uses δ_e in place of δ_m. Hence, in the regime for which it is legitimate to separate the problem into the distant and close collisions as has been done above, Eq. (5) gives the correct value for dE/dx to within 1% with[10,12]

$$I_m \approx I_e = \begin{cases} (12Z_2 + 7)\ \mathrm{eV}, & Z_2 \le 13 \quad (14a) \\ (9.76 Z_2 + 58.8 Z_2^{-0.19})\ \mathrm{eV}, & Z_2 \ge 13 \quad (14b) \end{cases}$$

(Z_2 is the atomic number of the absorber), and[10]

$$\delta_m \approx \delta_e = \begin{cases} \ln\beta^2\gamma^2 - 2\ln\left(\dfrac{I_e}{\hbar\omega_p}\right) + a(X_1 - X)^m - 1 \quad (X_0 < X < X_1)\ , & (15a) \\ \ln\beta^2\gamma^2 - 2\ln\left(\dfrac{I_e}{\hbar\omega_p}\right) - 1 \quad (X > X_1)\ , & (15b) \end{cases}$$

and $\delta_m = 0$ if $X < X_0$, where $X = \log_{10}\beta\gamma$ and a general expression for X_1, X_0, a, and m is given in Ref. 10 (the accuracy in dE/dx for electric charges is claimed to be ~1% with this general expression).

The validity of the various assumptions and approximations, both implicit and explicit, which led to the derivation of Eq. (5) will now be considered.

(a) The existence of an intermediate momentum transfer q_0 which satisfied both the close-collision and distant-collision approximations relies on the monopole velocity being much larger than the atomic electron velocities. The dipole approximation is implicit in any application of macroscopic electrodynamics so it must be true that $q_0 a_0$ is small (a_0 is the atomic size). Fano[8] shows that for incident electric particles the applicability of Bethe's generalized sum rule[13] is consistent with such a q_0 only if the above velocity constraint holds. For small velocities, shell corrections become important. Shell corrections of the 1% level accrue at[8] $\beta = 0.075$ for $Z_2 = 6$, at $\beta = 0.2$ for $Z_2 = 26$, and at $\beta = 0.475$ for $Z_2 = 82$. Although a generalized sum rule has not been derived for a magnetic-particle interaction, it is not totally absurd to assume the existence of one, just as Bohr did in 1913[14] for electric particles. (In physical terms, the sum rule implies that on the average, a momentum transfer $\hbar q$ which is greater than $\hbar q_0$ results in an energy transfer of $\hbar^2 q^2/2m$; Bohr reasoned that this must be so on the basis of the ratio of the collision time to the orbit time, which is quite small for these intermediate-momentum-transfer collisions.)

(b) Another implicit assumption in the use of classical electrodynamics is the linear relationship between $\vec{D}$ and $\vec{E}$. This is basically the result of a first-order Born approximation, and an idea of the size of the errors which arise from this assumption can be gleaned from the distant-collision Z_1^3 corrections calculated by Ashley, Ritchie, and Brandt,[15] Hill and Merzbacher,[16] and Jackson and McCarthy.[17] These corrections are due to the polarization of the atoms and amount to a fractional correction of the order $(\alpha Z_1 \hbar\omega_0)/(mc^2\beta^3)$ where α is the fine-structure constant and $\hbar\omega_0$ is the atomic energy. If one naively replaces $Z_1\alpha/\beta$ by $g\alpha/e$, which is the usual prescription for comparison of magnetic to electric particle properties, this correction is of the order of 2% for $\hbar\omega_0 = 100$ eV and $\beta = 0.1$ if $g = 137e$. However, it is apparent that this is an enormous overestimate of the distant-collision monopole correction when one considers the directions of the forces involved. For electric particles the interaction is longitudinal and hence either drags the electron closer to the incident particle or pushes it away. The electric field due to the monopole is transverse and only shifts the electron sideways which, to a first approximation, cannot change the separation of the monopole and electron and, therefore, should not affect the energy transfer. In fact, the symmetry of the problem actually forbides a g^3 correction (or any odd-power correction) to the distant-collision energy loss. For these reasons we can feel quite safe in using the macroscopic approach. [This is particularly well suited to the energy loss problem in which $\epsilon(\omega)$ already contains the quantum-mechanical sums which ensure the correspondence between the classical and quantum-mechanical treatments.]

(c) Lindhard[18] has pointed out that close-collision polarization effects lead to corrections of the same order as the distant-collision corrections for electric particles. For the same reasons given above, this correction should not be relevant for monopoles.

(d) By setting $\mu(\omega)=1$ we have demanded that the electron spin have negligible interaction with the magnetic monopole for distant collisions. Landau and Lifshitz[9] emphasize that it is justifiable to set $\mu=1$ for incident electric charges since matter does not exhibit magnetic properties at frequencies important with regard to ionization and excitation losses. This statement also applies for the magnetic monopoles. Of course, at low velocities the spin interaction dominates, since the electric field at the electron due to the monopole is proportional to β. However, ionization ceases before this happens. To be more quantitative, the energy loss rate to electric dipole transitions induced by the spin–magnetic-field interaction is $\sim\gamma^2\bar{\lambda}^2/(2\beta^2 b_1^2 \ln b_{\max}/b_1)$ (where $\bar{\lambda}=\hbar/mc$) times smaller than that induced by the electric-field interaction, where b_1 is the impact parameter which separates the close from the distant collisions. (We do not need to consider those transitions for which only spin flip occurs because of the extremely small amount of energy absorbed in these transitions.) Since b_1 must be larger than $\approx 3a_0 = 3\bar{\lambda}/(\alpha Z_2^{1/3})$ in order for the distant-collision dipole approximation to be valid, the above fraction is smaller than $(\gamma Z_2^{1/3}/1000\beta)^2$. For $Z_2=82$ this amounts to 1% or more for $\beta<0.04$ or $\gamma>25$. Therefore, for the largest part of the range of velocities considered in this paper, it is of no consequence to completely neglect the electron spin for distant collisions.

(e) Whenever one uses a scattering cross section to predict event rates it is assumed that the incident beam has a cross-sectional area which completely covers the scattering center. In quantum-mechanical jargon one requires an approximate plane wave with lateral extent greatly exceeding the scattering center size. In order to apply the Mott cross sections and KYG cross sections to the energy loss problems, the lateral extent of the electron waves, which are of the order a_0, must be much larger than the effective size of the scatterer. Bloch[19] has shown that the relevant parameter for electric charges is $Z_1\alpha/\beta$, and that for $Z_1\alpha/\beta \ll 1$ Bethe's formula[13] obtains and for $Z_1\alpha/\beta \gg 1$ Bohr's formula[14] obtains. Bloch's correction, which bridges the gap between the two results, is given by $\psi(1)-\mathrm{Re}\,\psi(1+iZ_1\alpha/\beta)$ [$\psi(Z)$ is the logarithmic derivative of the gamma function[20]] and this expression is added to the contents of the parentheses in Eq. (5′). Since the strength of the monopole-electron force is roughly $g\beta/Z_1 e$ times as strong as the analogous electric particle force, it seems reasonable that the Bloch parameter should become $g\alpha/e$ for monopoles. In any case, it pushes dE/dx in the correct direction for large values of $|g|$ since the Bloch correction is negative (if we multiply a cross section by a flux which does not cover the whole scattering center, we will be overestimating the amount of scattering). For $|g|=137e/2$, $\psi(1)-\mathrm{Re}\,\psi(1+ig\alpha/e)=-0.248$, and for $|g|=137e$, $\psi(1)-\mathrm{Re}\,\psi(1+ig\alpha/e)=-0.672$.

Including the Bloch correction $B(|g|)$, the monopole stopping-power formula is

$$\frac{dE}{dx}=\frac{4\pi N g^2 e^2}{mc^2}\left[\ln\frac{2mc^2\beta^2\gamma^2}{I}+K(|g|)/2-\tfrac{1}{2}-\delta/2-B(|g|)\right], \tag{16}$$

where

$$B(|g|)=\begin{cases}0.248, & |g|=137e/2 \quad (17a)\\ 0.672, & |g|=137e \quad (17b)\end{cases}$$

and K, I, and δ are given by Eqs. (1a), (14), and (15), respectively. In view of the relatively large number of ~1% errors mentioned above and the uncertainty of the Bloch correction, Eq. (16) is probably accurate to within ~3% for $\gamma<100$ and

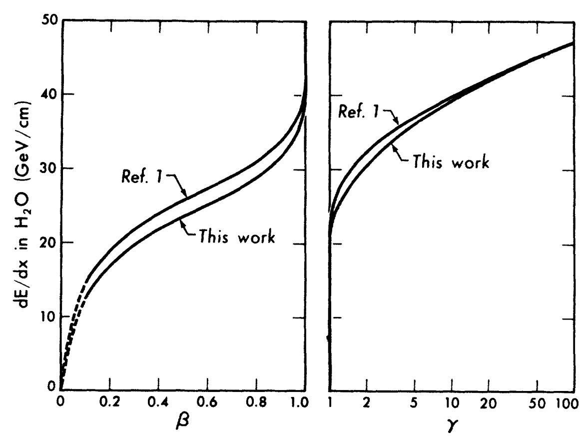

FIG. 1. Theoretical stopping power for magnetic monopole with $g=137e$ in water. Bottom curve is taken from Eq. (16) using Sternheimer's density-effect corrections. Upper curve is calculated using the technique of Ref. 1. The curves correspond to average values for the stopping power.

β large enough so that shell corrections can be safely ignored. This places a lower limit on β from ~0.1 to 0.5 depending on the atomic number of the absorber.

In Fig. 1, dE/dx is plotted as a function of β and γ for a monopole with $g=137e$ in water. Shell corrections will probably become important for $\beta < 0.1$, but interpolation between $\beta=0$ and $\beta=0.1$ should give reliable results since the monopole ionization rate is a monotonically increasing function of velocity. The parameters to be used for water in Eq. (16) were taken from detailed calculations by Sternheimer[21]: $I=74$ eV, $C=-3.47$, $a=0.519$, $m=2.69$, $X_1=2$, and $X_0=0.23$. For comparison, the technique from Ref. 1 has also been used to compute dE/dx. The separation of the two curves at low velocities is due primarily to the Bloch correction. The two curves join at large γ due to the different manner in which the density effect correction was calculated.

The curves of Fig. 1 correspond to the average energy loss. This should be the same as the most probable energy loss for thick detectors as determined by the requirement

$$G=0.15(Z_2/A_2)\,\frac{\text{cm}^2}{\text{g}}\,\rho t\left(\frac{g}{e\beta\gamma}\right)^2 \gg 1\ , \tag{18}$$

where A_2/Z_2 is the absorber nucleon-to-electron ratio, and ρt is the absorber thickness in g/cm^2. If $G \ll 1$, one would expect to be in the Landau regime and the most probable energy loss would cease to increase with γ at some point. Refer to Fano's review article[8] for a summary of energy-loss fluctuations for electric charges. Condition (18) was obtained from the analogous electric-particle criterion by replacing $Z_1 e$ by $g\beta$ which, to a first approximation, describes the behavior of monopoles relative to that of electric charges.

To further improve the calculation of dE/dx for monopoles it will be necessary to rigorously derive the magnetic analog to Bethe's generalized sum rule and to Bloch's corrections. Owing to the complexity of the electron-monopole interaction in a quantum-mechanical treatment, these seem to be nontrivial exercises.

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