Minimal electromagnetic coupling for massive spin-two fields

M. Kobayashi* and A. Shamaly
Theoretical Physics Institute, Department of Physics, University of Alberta, Edmonton, Alberta, Canada T6G 2J1
(Received 17 January 1978)

The minimally coupled spin-2 wave equation is shown to lead to noncausal propagation even if the correct number of constraints are obtained.

It is well known now that fields with spin greater than or equal to unity may show noncausal propagation in the presence of certain types of interactions due to the constraints inherent in the equations of motion.[1-3] In some cases propagation ceases and the equations lose hyperbolicity and are no longer suitable for the description of wave propagation. Such difficulties will lead to inconsistencies when the theory is quantized as had been shown by Johnson and Sudarshan[4] in the case of the minimally coupled spin-$\frac{3}{2}$ field. In Ref. 2 Velo and Zwanziger exhibited that the minimally coupled spin-2 equation loses some constraints, thereby increasing the number of independent field components to six. Later, Tait[5] and Hagen[6] demonstrated that the correct number of constraint equations are obtained by using the correctly symmetrized spin-2 equation of motion. We complete the investigations of Tait and Hagen by deriving the characteristic determinant for their Lagrangian. The purpose of this note is to calculate the characteristic determinant for the minimally coupled spin-2 equation and to show the existence of noncausal modes of propagation.

We start with the equation of motion as in Refs. 5 and 6,

$$L_{\mu\nu} = -(\Pi^2 + m^2)\phi_{\mu\nu} + \tfrac{1}{2}(\Pi_\mu \Pi^\rho \phi_{\rho\nu} + \Pi^\rho \Pi_\mu \phi_{\rho\nu} + \Pi_\nu \Pi^\rho \phi_{\rho\mu} + \Pi^\rho \Pi_\nu \phi_{\rho\mu})$$
$$- \tfrac{1}{2}(\Pi_\mu \Pi_\nu + \Pi_\nu \Pi_\mu) g^{\rho\sigma}\phi_{\rho\sigma} - g_{\mu\nu}\Pi^\rho \Pi^\sigma \phi_{\rho\sigma} + (\Pi^2 + m^2) g_{\mu\nu} g^{\rho\sigma}\phi_{\rho\sigma} = 0\,, \tag{1}$$

where ϕ is the 10-component tensor of rank 2 and m is the mass of the particles with spin 2.[7] This is derived from the Lagrangian proposed by Bhargava and Watanabe[8] by replacing $i\partial_\mu$ with Π_μ:

$$i\partial_\mu \to \Pi_\mu = i\partial_\mu + eA_\mu\,, \tag{2}$$

and by requiring the symmetrization for the Π's. Here the equation of motion is in the quadratic second-order form. Therefore we can specify the values of five components of ϕ along with their time derivatives at all points in space at a given time. The other five components of ϕ and their time derivatives must be derived from these data through Eq. (1). This means that 10 constraint equations should be obtained from Eq. (1), i.e., constraint equations containing higher space derivatives but only first-order time derivatives of the components ϕ.

Of the 10 constraint equations, four are associated with $L_{0\nu}$ and four more are obtained by operating with Π^μ on $L_{\mu\nu}$ of Eq. (1):

$$\begin{aligned} C_\nu &= -\frac{1}{m^2}\Pi^\rho L_{\rho\nu} \\ &= (\Pi^\rho\phi_{\rho\nu} - \Pi_\nu\phi) + \tfrac{3}{2}ie\frac{1}{m^2}(F^{\rho\alpha}\Pi_\alpha\phi_{\rho\nu} \\ &\qquad - F^\rho{}_\nu \Pi^\sigma\phi_{\rho\sigma} - F_{\nu\rho}\Pi^\rho\phi) \\ &\qquad - \tfrac{1}{2}e\frac{1}{m^2}[(\partial_\alpha F^{\rho\alpha})\phi_{\rho\nu} - (\partial^\rho F^\sigma{}_\nu)\phi_{\rho\sigma} - (\partial^\rho F_{\nu\rho})\phi] \\ &= 0\,, \end{aligned} \tag{3}$$

where $\phi = g^{\rho\sigma}\phi_{\rho\sigma}$. The ninth constraint equation is derived from Eq. (3) by operating with Π^ν on it:

$$\begin{aligned} \phi = -\tfrac{2}{3}e\frac{1}{m^4}[&(\partial_\alpha F^{\rho\alpha})\Pi^\sigma\phi_{\rho\sigma} - (\partial^\rho F^{\sigma\alpha})\Pi_\alpha\phi_{\rho\sigma} \\ &+ (\partial^\alpha F_{\alpha\beta})\Pi^\beta\phi] \\ &+ e^2\frac{1}{m^4}(F^{\rho\alpha}F^\sigma{}_\alpha\phi_{\rho\sigma} - \tfrac{1}{2}F_{\alpha\beta}F^{\alpha\beta}\phi)\,. \end{aligned} \tag{4}$$

Here we utilize the relation

$$\Pi^\rho\Pi^\sigma\phi_{\rho\sigma}-\Pi^2\phi=\tfrac{3}{2}m^2\phi\,, \quad (5)$$

which is easily obtained by taking the trace of the equation of motion (1). The tenth constraint equation comes from Eq. (4) by operating with Π_0 on it together with $\Pi_0C_\nu=0$ and $L_{ij}=0$. A tedious but straightforward calculation shows that $\Pi_0\phi$ is expressed as

$$\Pi_0\phi=d_k\Pi_0\phi_{0k}+d_{kl}\Pi_0\phi_{kl}+(\text{no time-derivative terms})\,, \quad (6)$$

in which d_k and d_{kl} are defined later [see Eqs. (11) and (12)]. It is important to note that $\Pi_0\phi$ does not contain a term proportional to ϕ_{00}. By substituting Eqs. (3), (5), and (6) into Eq. (1), we get the true equation of motion:

$$\begin{aligned}L_{\mu\nu}={}&\Pi_0{}^2\phi_{\mu\nu}+\tfrac{1}{2}(\Pi_\mu\Pi_\nu+\Pi_\nu\Pi_\mu)\phi-\tfrac{3}{2}ie\frac{1}{m^2}[\Pi_\mu(F^{\rho\alpha}\Pi_\alpha\phi_{\rho\nu}-F^\rho{}_\nu\Pi^\sigma\phi_{\rho\sigma}-F_{\nu\rho}\Pi^\rho\phi)+(\mu\longleftrightarrow\nu)]\\&+\tfrac{1}{2}e\frac{1}{m^2}\{\Pi_\mu[(\partial_\alpha F^{\rho\alpha})\phi_{\rho\nu}-(\partial^\rho F^\sigma{}_\nu)\phi_{\rho\sigma}-(\partial^\rho F_{\nu\rho})\phi]+(\mu\longleftrightarrow\nu)\}\\&-(\Pi_k{}^2+m^2)\phi_{\mu\nu}-\tfrac{1}{2}m^2g_{\mu\nu}\phi+\tfrac{1}{2}ie(F^\rho{}_\mu\phi_{\rho\nu}+F^\rho{}_\nu\phi_{\rho\mu})\\={}&0\,.\end{aligned} \quad (7)$$

It should be noted that the $\Pi_0\phi$ terms contained in Eq. (7) must always be replaced with Eq. (6).

In order to get the characteristic determinant, we explicitly write the 10 equations of motion by taking the specific frame $\Pi_\mu=(\Pi_0,0,0,0)$:

$$L_{00}=\Pi_0{}^2\phi_{00}+d_k\Pi_0{}^2\phi_{0k}+d_{kl}\Pi_0{}^2\phi_{kl}+\cdots\,, \quad (8)$$

$$L_{0k}=f_{k;0}\Pi_0{}^2\phi_{00}+f_{k;l}\Pi_0{}^2\phi_{0l}+f_{k;ln}\Pi_0{}^2\phi_{ln}+\cdots\,, \quad (9)$$

and

$$L_{kl}=\Pi_0{}^2\phi_{kl}+\cdots\,, \quad (10)$$

where the dots stand for the terms containing no second time derivatives. Here we define d_k, d_{kl}, $f_{k;0}$, $f_{k;l}$, and $f_{k;ln}$ as follows:

$$\begin{aligned}d_k={}&-e\frac{1}{\Delta_1}[i\partial_0(\nabla\times\vec{B})_k+3eF_{jk}E_j]\\&+\tfrac{1}{2}e^2\frac{1}{\Delta_1\Delta_2}\{m^4(\nabla\times\vec{B})_j+\tfrac{3}{2}iem^2[(\nabla\times\vec{B})\times\vec{B}]_j+\tfrac{9}{4}e^2(\nabla\times\vec{B})\cdot\vec{B}B_j\}[3\partial_0F_{jk}+\delta_{jk}\nabla\cdot\vec{E}-(\partial_kE_j)]\,,\end{aligned} \quad (11)$$

$$\begin{aligned}d_{kl}={}&\tfrac{1}{2}e\frac{1}{\Delta_1}[2i(\partial_k\partial_0E_l)+3e(F_{kj}F_{lj}-E_kE_l)]\\&-\tfrac{1}{2}e^2\frac{1}{\Delta_1\Delta_2}\{m^4(\nabla\times\vec{B})_j+\tfrac{3}{2}iem^2[(\nabla\times\vec{B})\times\vec{B}]_j+\tfrac{9}{4}e^2(\nabla\times\vec{B})\cdot\vec{B}B_j\}\{\delta_{jl}[(\nabla\times\vec{B})_k+4\partial_0E_k]+(\partial_kF_{jl})\}\,,\end{aligned} \quad (12)$$

$$f_{k;0}=\tfrac{3}{2}ie(1/m^2)E_k\,, \quad (13)$$

$$f_{k;l}=\delta_{kl}+\tfrac{3}{2}ie(1/m^2)(F_{kl}+E_kd_l)\,, \quad (14)$$

and

$$f_{k;ln}=\tfrac{3}{2}ie(1/m^2)(-E_l\delta_{kn}+E_kd_{ln})\,, \quad (15)$$

with

$$\Delta_1=\tfrac{3}{2}m^4+\tfrac{1}{2}e^2\left\{3|\vec{B}|^2-\frac{1}{\Delta_2}m^4(\nabla\times\vec{B})^2-\tfrac{9}{4}e^2\frac{1}{\Delta_2}[(\nabla\times\vec{B})\cdot\vec{B}]^2\right\}\,, \quad (16)$$

and

$$\Delta_2=m^2(m^4-\tfrac{9}{4}e^2|\vec{B}|^2)\,, \quad (17)$$

in which $\vec{B}$ and $\vec{E}$ are defined by

$$B_i=\tfrac{1}{2}\epsilon_{ijk}F_{jk}\ \text{and}\ E_i=F_{0i}\,. \quad (18)$$

The characteristic determinant is obtained by replacing Π_μ with a Lorentz vector n_μ in the highest derivatives.[9] Moreover, we take the frame $n_\mu=(n_0,0,0,0)$. After lengthy calculations, we obtain the characteristic determinant $D(n_0)$ from Eqs. (8) to (10):

$$D(n_0) = (n_0{}^2)^{10}\left(1 - \tfrac{9}{4}e^2\frac{1}{m^4}\,|\vec{\mathrm{B}}|^2\right). \qquad (19)$$

In covariant form this becomes

$$D(n) = (n^2)^9\left[n^2 + \tfrac{9}{4}e^2\frac{1}{m^4}(n\cdot F^d)^2\right], \qquad (20)$$

where

$$F^d_{\mu\nu} = \epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}. \qquad (21)$$

The characteristic surfaces are normal to n_μ. By solving for n_μ, we find

$$(n^2)^9\left[-n_0{}^2 + |\vec{\mathrm{n}}|^2 + n_0{}^2\left(\tfrac{3}{2}e\frac{1}{m^2}|\vec{\mathrm{B}}|\right)^2\right] = 0. \qquad (22)$$

This has the solutions $n_0 = \pm|\vec{\mathrm{n}}|$, in which the characteristic surfaces are the light cones. The other solutions come from the second factor in the square brackets of Eq. (22),

$$n_0 = \pm|\vec{\mathrm{n}}|/\{1 - [\tfrac{3}{2}e(1/m^2)|\vec{\mathrm{B}}|]^2\}^{1/2}. \qquad (23)$$

The solutions (22) tell us that if $1 - [\frac{3}{2}e(1/m^2)|\vec{\mathrm{B}}|]^2 > 0$, then we have real solutions for n_μ which lie inside the light cones, therefore the characteristic surfaces in this case are spacelike and the propagation is noncausal. If, however, $1 - [\frac{3}{2}e(1/m^2)|\vec{\mathrm{B}}|]^2 < 0$, then n_μ is complex and the equations of motion cease to be hyperbolic.

In conclusion, the minimally coupled spin-2 equations lead to noncausal modes of propagation even though the correct constraint equations are used. To get the correct constraints, we must demand the Lagrange formulation based on the symmetric tensor of rank 2 in the quadratic second-order forms.

We would like to thank Dr. A. Z. Capri for stimulating discussions. One of the authors (M.K.) would like to thank Professor D. D. Betts and Professor Y. Takahashi for the kind hospitality extended to him at the Theoretical Physics Institute of the University of Alberta. This work was supported in part by the National Research Council of Canada.

---

*On leave of absence from the Gifu University, Gifu, Japan 502.

[1]G. Velo and D. Zwanziger, Phys. Rev. 186, 1337 (1969).
[2]G. Velo and D. Zwanziger, Phys. Rev. 188, 2218 (1969).
[3]A. Shamaly and A. Z. Capri, Ann. Phys. (N.Y.) 74, 503 (1972); Lett. Nuovo Cimento 3, 467 (1972); R. A. Krajcik and M. M. Nieto, Phys. Rev. D 13, 924 (1975).
[4]K. Johnson and E. C. G. Sudarshan, Ann. Phys. (N.Y.) 13, 126 (1961).
[5]W. Tait, Phys. Rev. D 5, 3272 (1972).
[6]C. R. Hagen, Phys. Rev. D 6, 984 (1972).
[7]We use natural units and the metric tensor $g_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$. Greek tensor indices run from 0 to 3. Throughout this work we have omitted tensor indices where they are obvious.
[8]S. C. Bhargava and H. Watanabe, Nucl. Phys. 87, 273 (1966); A. J. MacFarlane and W. Tait, Commun. Math. Phys. 24, 211 (1972).
[9]R. Courant and D. Hilbert, *Methods of Mathematical Physics* (Interscience, New York, 1962), Vol. 2, Chap. VI.