Periodic Euclidean solutions and the finite-temperature Yang-Mills gas

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We present explicit periodic solutions for SU(2) Euclidean gauge theory and briefly consider the contribution of the corresponding finite-temperature configurations to the partition function of the Yang-Mills gas.

I. INTRODUCTION

In an earlier work[1] we showed that finite-action classical solutions of non-Abelian Euclidean field theory[2] are relevant in an apparently different physical context, that of an equilibrium ensemble of Yang-Mills fields at a finite temperature. In both cases the continuation $it \to \tau$ is made.[3] This substitution converts the field-theoretic vacuum generating functional into the functional integral for the partition function, where τ is confined to the range $0 \leqslant \tau \leqslant \beta$ $(\beta = 1/kT)$,

$$Z = \int [d\phi] \exp\left[\int_0^\beta d\tau \int d^3x\, \mathcal{L}_E(\phi, \partial_\mu \phi)\right]. \quad (1.1)$$

It is reasonable to assume that the dominant contributions to Z come from fields that are near classical fields $\phi(\vec{x}, \tau)$ which minimize the (Euclidean) action and which obey an additional restriction characteristic of finite-temperature field theory,

$$\phi(\vec{x}, \tau + \beta) = \phi(\vec{x}, \tau). \quad (1.2)$$

We proved in Ref. 1 that for the pure SU(2) Yang-Mills theory the space of periodic Euclidean solutions $A_\mu(\vec{x}, \tau)$ can be divided into an infinite set of homotopy classes, corresponding to the classes of mappings $S^2 \times S^1 \to S^3$. Thus at finite temperatures, topologically distinct field configurations exist just as in the zero-temperature case.[4]

In this paper we present explicit periodic solutions for the SU(2) case and begin our study of the contribution such finite-temperature configurations (which will be referred to as "calorons") make to the thermodynamics of the Yang-Mills gas.

Periodic Euclidean solutions also arise in other interesting physical situations, e.g., the path-integral treatment of thermal (Hawking) radiation in spacetimes with event horizons,[5] and the study of the large-order perturbation theory behavior of the ground-state energy.[6]

II. PERIODIC EUCLIDEAN SOLUTIONS

We consider pure SU(2) Yang-Mills theory in Euclidean space:

$$\begin{aligned} &\mathcal{L}_E = (1/2g^2)\,\mathrm{Tr}(F_{\mu\nu})^2\,, \\ &F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]\,, \\ &A_\mu = g(\sigma^a/2i) A_\mu^a\,. \end{aligned} \quad (2.1)$$

The equations of motion are

$$D_\mu F_{\mu\nu} = \partial_\mu F_{\mu\nu} + [A_\mu, F_{\mu\nu}] = 0\,. \quad (2.2)$$

Using the by-now familiar (Lorentz gauge) ansatz[7,8]

$$A_\mu = i\bar{\sigma}_{\mu\nu}\partial_\nu \ln\phi\,, \quad (2.3)$$

one finds that the equations of motion, Eq. (2.2), are satisfied provided that

$$\Box\phi = 0 \quad (2.4)$$

or

$$\Box\phi = c\,\phi^3\,. \quad (2.5)$$

We look only for solutions which obey the self-dual (or anti-self-dual) condition

$$^*F_{\mu\nu} \equiv \tfrac{1}{2}\epsilon_{\mu\nu\alpha\beta}F_{\alpha\beta} = \pm F_{\mu\nu}\,. \quad (2.6)$$

The (Euclidean) action and Pontryagin index[4] are defined by

$$S = -(1/2g^2)\int d^4x\,\mathrm{Tr}(F_{\mu\nu})^2\,, \quad (2.7a)$$

$$q(A) = (1/16\pi^2)\int d^4x\,\mathrm{Tr}(^*F_{\mu\nu}F_{\mu\nu})\,. \quad (2.7b)$$

When Eq. (2.6) is satisfied, $q(A) = \mp(g^2/8\pi^2)S$. In the finite-temperature case $\int d^4x$ is always to be understood to mean $\int_0^\beta d\tau \int d^3x$. With A_μ written as in Eq. (2.3), S takes the very convenient form[8,9]:

$$S = -(1/2g^2)\int d^4x\,\Box\Box\ln\phi\,. \quad (2.8)$$

The partition function is simply

$$Z = \int [dA_\mu]\exp(-S)\,, \quad (2.9)$$

where S is given by Eq. (2.7a) or (2.8).

For the finite-temperature situation[1,10] we seek solutions to Eqs. (2.4) or (2.5) which are periodic in τ, i.e., obey Eq. (1.2). In addition, there are the conditions that ϕ be real, finite, and strictly positive in order that A'_μ, given by (2.3), be properly behaved.

There are several ways to arrive at the solutions which we shall present. One method is to start from the zero-temperature multipseudoparticle solutions of 't Hooft and Jackiw, Nohl, and Rebbi.[8] 't Hooft takes, for the solution of (2.4),

$$\phi = 1 + \sum_{i=1}^{n} \lambda_i{}^2/(x - y_i)^2 , \tag{2.10}$$

which Jackiw *et al.* have generalized to the conformally covariant form

$$\phi = \sum_{i=1}^{n+1} \lambda_i{}^2/(x - y_i)^2 . \tag{2.11}$$

Both solutions give $q = n$ and thus each may be interpreted as a configuration of n pseudoparticles.

To form an expression periodic in τ, let

$$\phi = \sum_{k=-\infty}^{\infty} \lambda^2[(\vec{x} - \vec{x}_0)^2 + (\tau - \tau_k)^2]^{-2} . \tag{2.12}$$

Comparing with (2.11), we see that the pseudoparticle "sizes" λ_i and spatial "positions" $\vec{y}_i$ have all been set equal, and the sum is now over an infinite number of poles.

If the poles are at

$$\tau_k = \tau_0 + k\beta , \tag{2.13}$$

ϕ is clearly periodic. The sum can be performed, and we find

$$\begin{aligned}\phi = (\pi\lambda^2/\beta|\vec{x} - \vec{x}_0|) \sinh(2\pi\beta^{-1}|\vec{x} - \vec{x}_0|) \\ \times[\cosh(2\pi\beta^{-1}|\vec{x} - \vec{x}_0|) \\ -\cos(2\pi\beta^{-1}(\tau - \tau_0))]^{-1} .\end{aligned} \tag{2.14}$$

Note that ϕ satisfies Eq. (2.9), is real, positive, and $\sim|\vec{x}|^{-1}$ as $|\vec{x}| \to \infty$.

However, in the zero-temperature limit ($\beta \to \infty$),

$$\phi \to \lambda^2[|\vec{x} - \vec{x}_0|^2 + (\tau - \tau_0)^2]^{-1} ,$$

which corresponds to a field which is gauge equivalent to $A_\mu = 0$ and hence carries zero topological charge.[11] The simplest finite-temperature configuration with a nontrivial zero-temperature limit comes from considering

$$\begin{aligned}\phi = 1 + (\pi\lambda^2/\beta|\vec{x} - \vec{x}_0|) \sinh(2\pi\beta^{-1}|\vec{x} - \vec{x}_0|) \\ \times [\cosh(2\pi\beta^{-1}|\vec{x} - \vec{x}_0|) \\ -\cos(2\pi\beta^{-1}(\tau - \tau_0))]^{-1} .\end{aligned} \tag{2.15}$$

More generally, if we put N poles in the "physical region" $0 \le \tau - \tau_0 \le \beta$,

$$\tau_k = \tau_0 + k\beta/N ,$$

we simply replace β in Eqs. (2.14) and (2.15) by β/N. We can also change the sign in the denominator by the shift $\tau_0 \to \tau_0 - \beta/2$.

Defining

$$\begin{aligned}\bar{r} = 2\pi N\beta^{-1}|\vec{x} - \vec{x}_0| , \\ \bar{\tau} = 2\pi N\beta^{-1}(\tau - \tau_0) ,\end{aligned} \tag{2.16}$$

and

$$\bar{\lambda} = 2\pi N\beta^{-1}\lambda ,$$

we thus consider the following periodic solutions to $\Box\phi = 0$:

$$\phi = 1 + (\bar{\lambda}^2/2\bar{r}) \sinh\bar{r}(\cosh\bar{r} - \cos\bar{\tau})^{-1} . \tag{2.17}$$

Since we have reduced the problem of solving the original nonlinear equation for A_μ, Eq. (2.2), to that of solving the *linear* homogeneous equation $\Box\phi = 0$, we can generate more general solutions by superposing solutions of the form (2.17) with arbitrary scale size, pole location, and pole structure

$$\phi = 1 + \sum_i (\bar{\lambda}_i{}^2/2\bar{r}_i) \sinh\bar{r}_i(\cosh\bar{r}_i - \cos\bar{\tau}_i)^{-1} . \tag{2.18}$$

The solution in Eq. (2.17) can also be written

$$\begin{aligned}\phi = 1 + (\bar{\lambda}^2/4\bar{r})[\coth(\bar{r}/2 + i\bar{\tau}/2) \\ + \coth(\bar{r}/2 - i\bar{\tau}/2)] .\end{aligned} \tag{2.19}$$

An approach which would have led us directly to this form of the periodic solutions comes from seeking solutions to $\Box\phi = 0$ which are spherically symmetric (around any chosen point):

$$\phi(\vec{x}, \tau) = \phi(r = |\vec{x}|, \tau) . \tag{2.20}$$

Then,

$$\Box\phi = \ddot{\phi} + r^{-2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial\phi}{\partial r}\right) \tag{2.21}$$

where the dot signifies differentiation with respect to τ. Setting

$$\phi = r^{-1}f(r, \tau) , \tag{2.22}$$

we obtain

$$\Box\phi = r^{-1}(\ddot{f} + f'') , \tag{2.23}$$

where the prime signifies differentiation with respect to r. Hence, a general solution to $\Box\phi = 0$ is

$$\phi = r^{-1}[f_1(r + i\tau) + f_2(r - i\tau)] , \tag{2.24}$$

where f_1, f_2 are arbitrary functions. Requiring ϕ to be real implies f_1 and f_2 real or $f_1^*(z) = f_2(z^*)$. Combining this with the requirements that ϕ be dimensionless, periodic, finite, and positive, we are led to the coth term given in Eq. (2.19). Of course any overall multiplicative factor is unspecified by this approach.

In what follows we propose to use the name "caloron" to refer to the finite-temperature configurations corresponding to the periodic solutions (2.17) or (2.18).

III. PROPERTIES OF THE CALORON SOLUTIONS

Given the solutions to Eq. (2.4) presented in the last section, it is straightforward to calculate A_μ from Eq. (2.3), $F_{\mu\nu}$ from (2.1), and then the action from (2.7a). Of primary interest to us is the contribution of any solution to the partition function, Eq. (2.9), since all thermodynamic quantities follow from this.

It is simplest to calculate the contribution of our classical periodic Euclidean fields to S by using Eq. (2.8) which may in fact be further simplified for our case.[12] For the basic solution, Eq. (2.17), it is possible to exactly integrate[9] S over a finite spatial volume, $V=\frac{4}{3}\pi R^3$. The resulting general expression is very involved and will not be given. For thermodynamic applications we are interested in the large-volume limit for which the exact expression for the action becomes[13]

$$S \underset{T\neq 0,\, R\to\infty}{\sim} \frac{8\pi^2 N}{g^2}(1-10\pi^2N^2k^2T^2\lambda^4/R^2)\,, \tag{3.1}$$

where the validity of the expansion depends on the assumptions that $R \gg (1/kT)$ and the second term in the parentheses is small compared to 1.[14]

From Eqs. (3.1) and (2.7) and following, we see that when $V\to\infty$ $(R\to\infty)$, $q(A)=\pm N$, i.e., the caloron (anticaloron) carries an integer winding number.

IV. AN INTRODUCTION TO THE THERMODYNAMICS OF THE YANG-MILLS GAS

In this section we do not attempt a complete treatment of the finite-temperature Yang-Mills gas. Such a treatment requires an investigation of other modes, especially those consisting of both calorons and anticalorons. In addition we have not calculated the quantum corrections to the caloron contribution. Neglecting such corrections may grossly underestimate the true caloron contributions. Polyakov[15] has shown in some simple pseudoparticle examples that the exponentially small effects characteristic of the semiclassical approximation are greatly enhanced when quantum corrections are included.

What we shall do in this section is to use the basic caloron contribution to construct, at least formally, the partition function in the "dilute-gas" approximation.[16] The only quantum corrections which will be (approximately) included are those arising from zero modes. Without further investigation of other modes contributing to the partition function and a study of thermal fluctuations for arbitrary temperature, it seems premature to calculate thermodynamic quantities or correlation functions.

The state of the finite-temperature SU(2) Yang-Mills gas will be specified by giving the occupation numbers $n_+(q_i)$, $n_-(q_i)$ of the caloron, anticaloron modes carrying winding number $+q_i$, $-q_i$, respectively.[17] In the partition function we must sum (integrate) over all physically distinct configurations. Then, assuming the system is in contact with a constant temperature reservoir with which it can exchange (quasi-) particles, one would use the grand potential to calculate thermodynamic functions.[18]

What we would like to calculate is the contribution to the partition function from a distinguishable physical configuration consisting of $n_+(q_i)$ calorons with winding number q_i and $n_-(q_i)$ anticalorons with winding number $-q_i$, where all "positions" and scale sizes are arbitrary. Since we have not found periodic solutions corresponding to multi-caloron-anticaloron states, this cannot be done exactly. What is typically done at zero temperature is to make a "dilute-gas" approximation[16]—assuming all calorons (i.e., instantons at zero temperature) and anticalorons are far apart and hence approximately noninteracting. Then one simply superposes assumed nonoverlapping calorons and anticalorons. In addition we only include configurations carrying topological charge ± 1.[19,15] The validity of the dilute-gas approximation is a matter of conjecture until a more exact treatment is possible. Nevertheless, if we apply this same set of approximations in our case, it leads to an expression of the form

$$\begin{aligned} Z_{\substack{\text{dilute}\\ \text{gas}}} &= \sum_{n_+,\,n_-=0}^{\infty} [V\beta\, D\exp(-S_{\text{cl}})]^{n_++n_-} \times \frac{\exp[i(n_+-n_-)\theta]}{n_+!\,n_-!} \\ &= \exp[2\cos\theta\; V\beta\, D\exp(-S_{\text{cl}})]\,. \end{aligned} \tag{4.1}$$

The combinatorial factors $n_+!$ and $n_-!$ arise because we only wish to count each topologically distinguishable configuration once, and these are labeled by the net topological charge n_+-n_- (recall that we are only including calorons and anticalorons carrying ± 1 unit of charge). The angle θ parametrizes the mixing of states due to thermal and quantum fluctuations (the latter alone produces this phase freedom at zero temperature).

S_{cl} is the classical part of the one-caloron contribution, i.e., Eq. (3.1) with $N=1$. The full one-caloron contribution including quantum corrections is represented by the factor $V\beta D\exp(-S_{cl})$. Including only the zero-mode part of the quantum corrections and assuming the usual renormalization-group transformation to the running coupling constant $\bar{g}^2(1/\lambda\mu)$, we have

$$V\beta D\exp(-S_{cl})\approx\int d^3x\int_0^\beta d\tau\int\frac{d\lambda}{\lambda^5}(8\pi^2/\bar{g}^2)^4\exp\left[-\frac{8\pi^2}{\bar{g}^2}(1-10\pi^2k^2T^2\lambda^4/R^2)\right]. \tag{4.2}$$

At finite temperatures the dominant scale size is determined by the temperature $\lambda\lesssim\beta$.

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[9]As discussed in Ref. 8, Eq. (2.8) holds only away from the singularities of ϕ. One must exclude from the integration infinitesimal neighborhoods around each singularity. See also J. Giambiagi and K. Rothe, Nucl. Phys. B129, 111 (1977).

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[11]In the *nonzero*-temperature (infinite-volume) limit, the configuration described by Eq. (2.14) does carry an integer topological charge $q(A)$. The order in which the limits $V\to\infty$, $T\to 0$ are taken is critical.

[12]Some especially useful relations are

$$\int d^4x\,\Box\,\Box\ln\phi=\int d^3x\,\nabla^2\nabla^2\int_0^\beta d\tau\ln\phi$$
$$=4\pi N\left(\bar{r}^2\frac{\partial}{\partial\bar{r}}\nabla^2\int_0^{2\pi}d\bar{\tau}\ln\phi\right)\Bigg|_{\bar{r}=0}^{\bar{r}=\bar{R}}$$
$$=4\pi N\int_0^{2\pi}d\bar{\tau}\left(\bar{r}^2\frac{\partial}{\partial\bar{r}}\nabla^2\ln\phi\right)\Bigg|_0^{\bar{R}}.$$

[13]It is perhaps worth emphasizing that we are integrating over a finite range in τ, $0\leqslant\tau-\tau_0\leqslant\beta$. The action would be infinite if the integration range were unrestricted.

[14]Expanding S for small temperatures, such that $kT\ll 1/R$ as $R\to\infty$, one finds

$$S\to(8\pi^2N/g^2)(1+10\pi^2N^2k^2T^2\lambda^2+10\pi^2N^2k^2T^2\lambda^4/R^2),$$

where $R\gg\lambda$ is also assumed.

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[17]Since we have shown in Sec. III that $q(A)$ is an integer in the infinite-volume limit, we may use the topological charge to label contributions to the functional integral.

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