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Static limit of quantum chromodynamics

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Several features of the static quark-antiquark potential in quantum chromodynamics are studied. It is shown, first of all, that the potential between fixed color sources exists as the infinite-mass limit of the interaction between a heavy quark and antiquark. The phenomenological consequences of this result are summarized. The potential is then examined as an expansion in the strong-coupling constant α_s using the Wilson loop integral. Infrared properties are elucidated and the power-series expansion in α_s is shown to break down. The quantum color correlations of the quark sources with the gluon field induce singularities which can only be removed by selective resummation. The resulting expression for the potential is nonanalytic in α_s at $\alpha_s = 0$.

I. INTRODUCTION

The static potential between a heavy quark and antiquark is an object of considerable theoretical interest. In quantum chromodynamics (QCD), the absence of quark motion means that attention may be focused primarily on the gauge field sector. While a static potential cannot describe the relativistic binding of the u, d, and s quarks, the experimental discovery of heavy quarks makes it more than a theoretical laboratory. The potential has been measured in the $c\bar{c}$ system where the evidence suggests that it grows approximately linearly with the separation R for $1/M_c \ll R$.

In this paper we shall discuss several features of the static potential in quantum chromodynamics. The work reported here extends and amplifies an earlier letter of ours,[1] and related work by Feinberg,[2] Fischler,[3] and Poggio.[4] The analysis is based in perturbation theory and we will not deal directly with any of the deep questions associated with confinement and the structure of hadrons. The success of asymptotic freedom gives good reason to believe in the usefulness of perturbation theory at short distances ($R < 1$ GeV^{-1}). If confining forces set in at larger distances they will very likely be due to coherent effects associated with vacuum structure[5] and they will presumably not be accessible through perturbation theory.

There are, however, a number of important aspects of the static potential which can be investigated with perturbation theory. The effect of very-short-distance ($<1/M_Q$) quantum fluctuations on the potential and the smoothness of the limit $M_Q \to \infty$ are naturally studied this way. Another interesting question is whether the potential exists as a power-series expansion in the strong-coupling constant $\alpha_s = g^2/4\pi$. It does not. The quantum color correlations between the quark sources and the gluon field induce certain "long-time" singularities which can only be removed by selective resummation. The potential is therefore nonanalytic in α_s at $\alpha_s = 0$. Our analysis of this feature of QCD may also be relevant to statistical-mechanical systems with analogous correlation effects.

The organization of the paper is as follows: In Sec. II, we summarize some features of the Yang-Mills theory in the Coulomb gauge, the gauge which seems most natural for the study of the static limit of the theory.

In Sec. III, we establish a framework for the computation of the static potential. This framework is developed in the infinite-mass limit, a limit which is shown to be nonsingular in Sec. IV. The Wilson loop integral[6] serves as our primary tool for this analysis. The simplified case in which the sources are classical, i.e., uncorrelated with the quantum fields and carrying no quantum labels is considered first. There it is shown that the potential is given by the connected Feynman diagrams. The source-field color correlations in the non-Abelian case prevent the formulation of such a simple final prescription.

In Sec. IV, the limit $M_Q \to \infty$ is analyzed and the features of low-order perturbation theory are reviewed. A Ward identity is established for finite M_Q, which shows that the only ultraviolet-divergent renormalization of the Coulomb-field–quark coupling comes from corrections to the Coulomb propagator. This same Ward identity is used to prove that the limit $M_Q \to \infty$ is nonsingular. This result has an important phenomenological implication: the universality or flavor independence of the static potential.

From this point on, infinitely massive sources are used and the remainder of Sec. IV reviews the one- and two-loop structure of the potential as reported in Refs. 1–4. At one loop, the physical origin of asymptotic freedom is recalled. At two loops and beyond, the potential becomes infrared divergent if the $Q\overline{Q}$ pair is not in a color-singlet state. The two-loop infrared finiteness of the color-singlet potential, first demonstrated by Feinberg,[2] is described and it is conjectured that this feature should be true to all orders.

In Sec. V, we analyze another class of singularities which had been noted in Refs. 1 and 2. These singularities of the color theory are shown to come from large relative times but spatial separations $\lesssim R$ in the coordinate space integrals. They are thus not ordinary infrared divergences and they do not cancel among diagrams. They in fact signal a breakdown of the perturbation expansion. The necessary selective resummation is apparent in the context of the Wilson loop integral, and this is discussed in detail for a simple example. The resummed expansion, necessarily nonanalytic at $\alpha_s = 0$, has a simple physical interpretation. It appears that this reorganization of the computation of the static potential can be carried out to any desired degree of accuracy. However, the formidable combinatoric problems which arise in higher orders make such a result difficult to prove. Section VI contains a summary of our main results and some unresolved questions.

II. THE COULOMB GAUGE

For simplicity, we shall consider only one quark flavor with mass M_Q. The gluon field is $\vec{A}_\mu$ and the Coulomb gauge condition is $\nabla^i\vec{A}^i = 0$. In this gauge, the canonical variables of the gluon field are $\vec{A}^i$ and the transverse electric field $\vec{E}^i_t$. In terms of these and the quark field $\psi(x)$, the Hamiltonian takes the form[7]

$$H = \int d^3x\left[\tfrac{1}{2}(\vec{E}^i_t)^2 + \tfrac{1}{4}(\vec{F}^{ij})^2 + \tfrac{1}{2}(\nabla^i\vec{\phi})^2 + \bar{\psi}(i\nabla\!\!\!/ + M_Q)\psi - g\psi^\dagger\vec{T}\,\alpha^i\psi\cdot\vec{A}^i\right]. \tag{1}$$

The matrices $\vec{T}$ are normalized by $\mathrm{Tr}\,T_aT_b = \frac{1}{2}\delta_{ab}$. For future reference we record the definition of quadratic Casimir operators C_F and C_A of the fundamental and adjoint representations,

$$T_aT_a = C_F\underline{1},\quad f_{acd}f_{bcd} = C_A\delta_{ab}. \tag{2}$$

For a general SU(N), $C_F = (N^2-1)/2N$ and $C_A = N$. The longitudinal electric field $\nabla^i\vec{\phi}$ is a dependent variable determined by the equation

$$(\nabla^2 - g\vec{A}^i\times\nabla^i)\vec{\phi} = \vec{\rho}\,, \tag{3}$$

where $\vec{\rho}$ is the color source

$$\vec{\rho} = g\vec{A}^i\times\vec{E}^i_t - g\psi^\dagger\vec{T}\psi\,. \tag{4}$$

The constraint, Eq. (3), can be inverted in perturbation theory and Feynman rules can be derived by functional or canonical methods.[8] The gluon propagator has an instantaneous part and a transverse part:

$$\begin{aligned} D^{ab}_{\mu\nu} &= i\delta^{ab}\frac{1}{\vec{k}^2},\quad \mu=\nu=0\,,\\ &= i\delta^{ab}\left(\delta_{ij} - \frac{k_ik_j}{\vec{k}^2}\right)\frac{1}{k^2+i\epsilon}\,,\quad \mu=i,\quad \nu=j\,. \end{aligned} \tag{5}$$

A Faddeev-Popov ghost must be included, but it is an instantaneous potential, with propagator $1/\vec{k}^2$, not directly connected with unitarity. The ghost couples only to the transverse component of the gauge field. The usual set of vertices and the quark propagator round out the Feynman rules.

It has been pointed out by Schwinger[9] that in order to ensure operator Lorentz covariance, an additional term must be added to the Hamiltonian. The effect of this term in the canonical derivation of perturbation theory has not been fully explored to our knowledge, but by examination it can clearly be seen to enter only at the two-loop level and beyond. It will have no direct bearing on our work. The ambiguities of the Coulomb gauge for strong-field strength, corresponding to the possibility of solutions to the homogeneous version of Eq. (3), have recently been emphasized by Mandelstam and Gribov.[10] For the part of our work concerned with short distances ($R < \frac{1}{5}$ fermi), asymptotic freedom should make such effects negligible. For larger distances, these problems underscore the limitations of perturbation theory which we have already emphasized.

III. COMPUTING THE STATIC POTENTIAL

In the next section it shall be demonstrated that the limit $M_Q\to\infty$ is free of singularities. Anticipating this result we first address the question of how to compute the potential between two fixed color sources in a singlet state separated by a distance R. The sources can be taken to be spinless, and they couple only to the Coulomb part of the gauge field. The momentum space propagator for the sources is $1/(k_0+i\epsilon)$, which Fourier transforms to $\theta(t)\,\delta^3(x)$. Throughout this section and most of the rest of the paper, we shall find it convenient to work in coordinate space. Since the sources are fixed in coordinate space, this seems simpler and more direct than the equivalent momentum-space analysis.

Let us first consider a simpler problem than QCD in which the sources are classical, that is, uncorrelated with the quantum fields. The simplest such example is electrodynamics where the sources are static pointlike electric charges, but this model is trivial in this context since the only interactions are between the sources and the field. More generally, interactions among $A_\mu(x)$ and other quantum fields can be introduced as a prototype for the Yang-Mills theory.

A gauge-invariant starting point for the computation of the potential is the Wilson loop integral

$$\left\langle P \exp\left(ig \oint dx_\mu A^\mu\right)\right\rangle_0 .$$

The integral is taken about the rectangle of width R and length $T \gg R$, and P is the path ordering symbol. The expectation value is taken in the vacuum, which for a perturbation theory computation is built up perturbatively from the bare vacuum. As pointed out by Fischler,[3] if the ends of the rectangle can be shown not to contribute in the limit $T\to\infty$, the loop integral becomes

$$\left\langle T \exp\left[ig \int_{-T/2}^{+T/2} A_0(0,t)dt - ig\int_{-T/2}^{+T/2} A_0(R,t)dt\right]\right\rangle_0 .$$

This then should give the exponential of the static potential $e^{iV(R)T}$ in the limit $T\to\infty$. Thus

$$V(R) = \lim_{T\to\infty} \frac{1}{iT} \ln \left\langle P \exp\left(ig \oint dx_\mu A^\mu\right)\right\rangle_0 . \tag{6}$$

The vanishing of the contributions from the ends of the rectangle is expected since $F_{\mu\nu}\to 0$ in the limit $T\to\infty$, and therefore the potential A_i becomes gauge equivalent to $A_i = 0$. This fact is easily seen in the perturbation expansion of Eq. (6). Diagrams with lines attaching to either end of the rectangle vanish as $T\to\infty$.

The expansion of Eq. (6) in perturbation theory produces, after the usual cancelation of vacuum disconnected graphs, two classes of diagrams which attach to the sides of the rectangle (the sources). One class is "connected" where connectivity is defined without the inclusion of the source lines, i.e., using only the quantum fields. The other class is "disconnected" in the same sense.

By iterating the connected diagrams in all ways along the sides of the rectangle, all the disconnected graphs will be generated. A given diagram will correspond to a definite time ordering of the various interactions with the source. These are specified by the source propagators which are θ functions in time. The effect of ordering one connected diagram C_1 in all possible ways with respect to another C_2 is to eliminate the θ functions involving the relative time intervals in C_1 with respect to those in C_2. Thus the two sets of time integrations range independently between $-T/2$ and $+T/2$. It is of course crucial here that the sources be uncorrelated. In adding together all time permutations, an overcounting by a factor of $n!$ will occur, where n is the number of identical connected diagrams involved. This must be compensated for by multiplying by $1/n!$.[11]

The upshot of this analysis, which we have only sketched, is that the potential $V(R)$ will be given by the connected diagrams. The disconnected diagrams are iterations which build up the exponential of Eq. (6). We note in passing that the class of connected diagrams includes those which connect only to one side of the rectangle (only one of the sources). These are source self-mass contributions which, as we shall discuss further in the next section, can be regarded as canceled by an appropriate mass counterterm. They do not depend upon quark separation and make no contribution to the potential.

If it can be shown that the large spatial integrations and large relative-time integrations in each connected diagram are convergent (infrared finiteness), then this procedure will give a finite result for the potential. The overall time integration, corresponding to sliding each connected structure as a unit along the rectangle, will give one overall factor of T. It will then follow that the limit $T\to\infty$ in Eq. (6) exists, and the potential will be calculable to any order in the coupling constant. The infrared finiteness for connected diagrams can, in fact, be proved for a wide variety of field theories. In particular, for renormalizable theories even with all the fields massless, a simple power-counting analysis can establish the result to all orders.[12]

It is worth commenting that this type of connectivity result is familiar in many-body theory. It is very similar to the Brueckner-Goldstone linked cluster expansion for the energy of a system of particles at zero temperature. The literature on this problem and the closely related finite-temperature expansion is extensive.[13]

Turning now to the Yang-Mills theory, we again start with the Wilson integral

$$\left\langle \mathrm{Tr} P \exp\left(ig \oint dx_\mu \vec{\mathrm{T}}\cdot\vec{\mathrm{A}}^\mu\right)\right\rangle_0 ,$$

where the contour is the same rectangle of width R and length $T \gg R$. In perturbation theory, it can again be shown that diagrams with gluon lines attaching to the ends of the rectangle vanish in the limit $T\to\infty$.[14] This makes it plausible that in this limit the Wilson integral again takes the form $e^{iV(R)T}$, where $V(R)$ is the static potential energy of a *color-singlet* pair of color sources separated by a distance R. Then, as before,

$$V(R)=\lim_{T\to\infty}\frac{1}{iT}\ln\left\langle \mathrm{Tr}P\exp\left(ig\oint dx_\mu \vec{\mathrm{T}}\cdot\vec{\mathrm{A}}^\mu\right)\right\rangle_0 . \qquad (7)$$

To actually demonstrate this and to give a direct prescription for computing $V(R)$ analogous to the "connectivity" prescription with uncorrelated sources is now a rather more difficult problem. All the complexities stem from the fact that the sources are no longer classical. Color is a quantum label, and the color state of the sources is correlated to the gluonic sector. This leads to a number of interesting consequences. $V(R)$ is no longer given only by diagrams which are connected purely within the quantum field (gluonic) sector and one result of this is that infrared divergences appear which must be shown to cancel. A more important consequence is the appearance of a new class of singularities coming from integrations over large times. They lead to a breakdown of the perturbation expansion and consequently nonanalyticity in α_s $(=g^2/4\pi)$ at $\alpha_s=0$. In the next two sections we analyze these features and outline how the potential could in principle be computed to arbitrary order.

IV. THE INFINITE-MASS LIMIT AND SOME FEATURES OF THE POTENTIAL

We begin with a discussion of the infinite-quark-mass limit and the question of mass dependence in heavy-quark spectroscopy. The Coulomb gauge is particularly convenient for this purpose since the coupling of transverse gluons to the heavy quarks may be neglected when $M_Q\to\infty$. Thus attention may be focused on the fermion–Coulomb-gluon vertex which satisfies a Ward identity similar to QED. This identity can be derived by functional methods[15] and is most easily expressed in momentum space. In a standard notation

$$-q^i\Gamma_i^a(q,p)[1+B(q)]+q^0\Gamma_0^a(q,p)][1+A(q)]=[gT^a-B^a(q,p)]S^{-1}(p)-S^{-1}(p+q)[gT^a-\hat{B}^a(q,p)] . \qquad (8)$$

$S(p)$ is the fermion propagator and the functions $A(q)$, $B(q)$, $B^a(q,p)$, and $\hat{B}^a(q,p)$ are two- and three-point functions involving the ghost as shown in Fig. 1. $B(q)q^{-2}$ is the ghost self-energy while $A(q)$, $B^a(q,p)$, and $\hat{B}^a(q,p)$ are artificial constructs in which an external ghost line terminates.

As the ghost couples only to the transverse gluon, it is possible to factorize the A and B^a amplitudes as follows:

$$B^a(q,p)=q_i B_i^a(q,p) ,$$
$$A(q)=q^{-2}\mathcal{A}(q,q_0) . \qquad (9)$$

It then follows from simple power-counting considerations that B_i^a and $\mathcal{A}$ are primitively convergent. The function $\mathcal{A}$, in fact, vanishes to three loops and we conjecture this to be true to all orders. The primitive convergence of B_i^a and $\mathcal{A}$ can be used along with the Ward identity (8) to show that the only ultraviolet-divergent renormalization of the Coulomb-gluon–quark coupling constant comes from corrections to the Coulomb gluon propagator. This can be shown by differentiation of Eq. (8) with respect to q_0 followed by $q_\mu\to 0$ or by a decomposition of Eq. (8) into O(3) invariants.

We now consider the infinite-mass limit. By mass here, we mean the renormalized mass. All self-energy insertions are to be expanded in powers of $(\not{p}-M_Q)$ with the zeroth-order term absorbed into mass renormalization. Letting the mass tend to infinity, the Ward identity ensures that the combination of fermion vertex Γ_0^a and self-

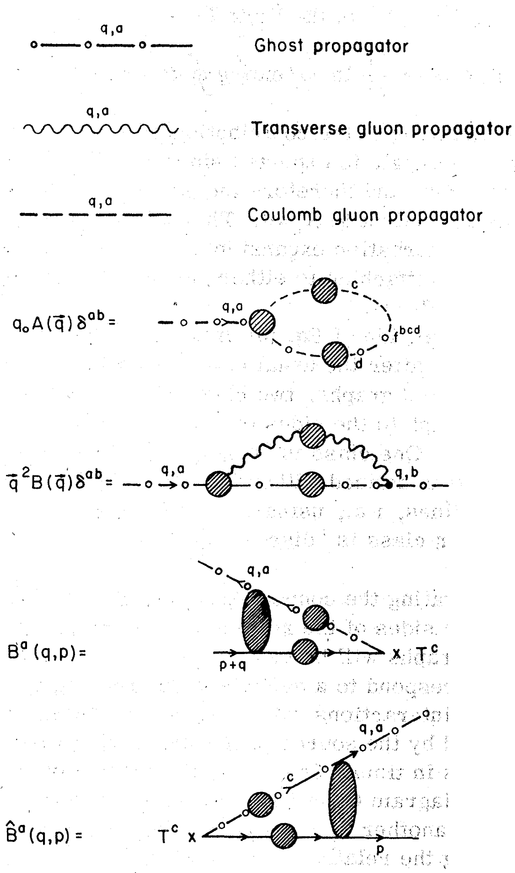

FIG. 1. Definition of amplitudes appearing in the Ward identity [Eq. (8)].

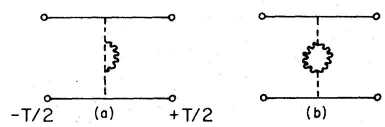

FIG. 2. One-loop corrections to the Coulomb propagator. The solid lines are the fixed sources, separated by a distance $R \ll T$.

energy $S(p)$ parts is finite in this limit. This result follows from the observation that in this limit $\Gamma_i^a(q,p)$ vanishes while $A(q)$ and $B^a(q,p)$ remain primitively convergent. We omit the details of the power-counting analysis required to demonstrate this. It is worth commenting that the finiteness of this limit is not at all apparent in a graph-by-graph analysis of Γ_0^a and $S(p)$. The individual graphs contain $\ln M_Q$ terms which sum to zero in each order as the Ward identity demands.

It is easy to see that the only possible sources of divergence as $M_Q \to \infty$ are the vertex Γ_0^a and the propagator $S(p)$. Apart from these, the computation of the potential involves diagrams which are clearly finite as $M_Q \to \infty$. Diagrams with closed loops of heavy quarks in fact vanish in this limit.[16] Having shown that the combined effect of Γ_0^a and $S(p)$ is finite as $M_Q \to \infty$, it follows that the static potential itself is finite. This result is perhaps expected intuitively, especially in an asymptotically free theory. Our proof used only power counting and did not explicitly involve asymptotic freedom. On the other hand, it is only in an asymptotically free theory that one can be sure that anomalous short-distance behavior does not develop nonperturbatively and destroy the naive power-counting arguments we have employed. A careful treatment of asymptotic freedom and the $M_Q \to \infty$ limit is being developed.[13]

The phenomenological implications of the existence of this limit have been pointed out in Ref. 1. Because the limit is finite, the $Q\overline{Q}$ potential at separation R is insensitive to the quantum fluctuations at distance scales $1/M_Q \ll R$. The same phenomenological potential can be used to describe charmonium and its heavier imitations. We emphasize that this result is quite general. Even though the confining potential describing these systems can almost surely not be derived from perturbation theory, the $M_Q \to \infty$ limit *is* properly studied this way, at least in an asymptotically free theory.

We complete this section by summarizing the structure of the potential through two loops.[1,2,3] The sources are now infinitely massive and the procedure is to compute the Wilson integral and to organize the result into the form $e^{iV(R)T}$. The zero-loop contribution to $V(R)$ is

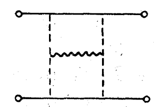

FIG. 3. Two-loop contribution to the static potential involving multiple exchange.

$$V_0(R) = -C_F \alpha_s / R\,, \tag{10}$$

coming from single Coulomb gluon exchange between color sources in the fundamental representation. Its iteration builds up the exponential $e^{iV_0(R)T}$. At one loop, the graphs of Fig. 2 give the entire contribution to the potential, and the dominance of the Coulomb-field self-energy [Fig. 2(a)] over the vacuum polarization [Fig. 2(b)] gives the β function its famous negative sign. The physical mechanism involved here has been described in some detail[1,17] elsewhere. Through one loop

$$V(R) = -C_F \frac{\alpha_s}{R}\left[1 + \frac{11\alpha_s}{24\pi^2} C_A \ln(R\mu) + \cdots\right], \tag{11}$$

where μ is the renormalization scale at which α_s is defined. This structure too can be iterated into an exponential.

At the two-loop level, two new features emerge. First, contributions appear which are more complex than the exchange of a single dressed gluon. To be specific, the contribution of Fig. 3 is proportional to $C_F C_A{}^2 \alpha_s{}^3/R$. Second, infrared divergences appear in the individual graphs shown in Fig. 4. In the case of Fig. 4(a), there is a leading (potential iteration) piece proportional to T^2 and residual pieces proportional to $T \ln T$ as well as T. The other three diagrams all contain $T \ln T$ and T pieces. The $T \ln T$ parts are the infrared divergences, coming from integrations over large times and large spatial distances. In an Abelian model, both the $T \ln T$ and T pieces would completely can-

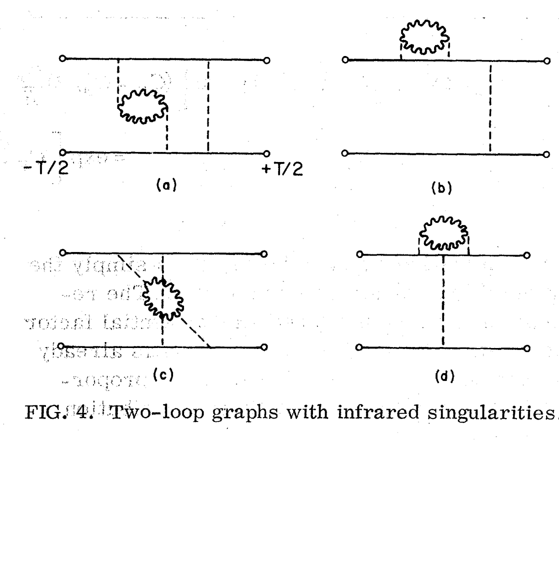

FIG. 4. Two-loop graphs with infrared singularities.

cel pairwise, Fig. 4(a) with (c) and Fig. 4(b) with (d), leaving only the potential iteration part of Fig. 4(a). In QCD, the $T\ln T$ pieces cancel for a *color-singlet* $Q\overline{Q}$ pair. The cancelation is again pairwise but now Fig. 4(a) with (b) and Fig. 4(c) with (d). The remaining piece proportional to T is a contribution to the potential which must be added to other two-loop contributions. From the full two-loop result, one can, for example, extract the two-loop β function. From the foregoing discussion, it is clear that although the potential is well defined through two loops (order α_s^3), it is no longer given only by the connected diagrams.

V. THE BREAKDOWN OF PERTURBATION THEORY

In higher orders, the disconnected diagrams introduce an interesting new feature. Before describing it, we remark that infrared divergences of the type already encountered will continue to exist. We conjecture that they will cancel to all orders for a color-singlet $Q\overline{Q}$ and it is important to prove this. Only then can one be sure, for example, that the short-distance potential ($R \lesssim 1$ GeV^{-1}) is independent of the confinement mechanism and truly described by asymptotic freedom.

The new feature is a divergence coming from large-time ($\sim T$) but finite-distance ($\lesssim R$) integrations. It does not cancel and in fact causes a breakdown of simple perturbation theory. We will illustrate the general problem by examining the simplest (lowest order) situation where it occurs. There the problem can be overcome by a selective resummation. We conjecture that this technique can be systematically applied to any order, but the combinatoric problems become formidable in higher orders.

Consider the disconnected diagram shown in Fig. 5. Without the middle rung, the remaining connected piece is easily evaluated. After performing the x_1 and x_2 spatial integrations, it is proportional to

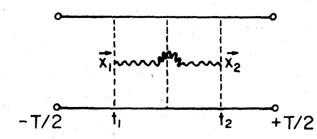

FIG. 5. A contribution to the Wilson loop integral behaving like $T\ln T$.

$$\alpha_s^3 \int_{-T/2}^{+T/2} dt_1 dt_2 \frac{C_F C_A^2}{R^2+(t_2-t_1)^2} = T C_F C_A^2 \alpha_s^3 \int_0^T \frac{dt}{R^2+t^2} \,. \tag{12}$$

The upper limit on the relative-time integration can be taken to infinity and the result is proportional to $T\alpha_s^3/R$. For simplicity, we have made a Wick rotation to Euclidean space-time in writing this expression. The presence of the additional rung leads to an extra numerator factor of $\alpha_s t/R$ in the relative-time integration in Eq. (12). The result is proportional to $T\ln T\alpha_s^4/R$ and it arises from the integration region $t\sim T$ but $x_1, x_2 \lesssim R$.

In an Abelian model, this would of course be canceled by contributions from graphs with the extra rung placed outside the connected piece. In the Yang-Mills theory, however, there is a leftover piece. If the extra rung is outside the connected piece, then the group theory factor resulting from the trace is $C_F \times C_F C_A^2$. If it is inside as in Fig. 5, the result is proportional to $(C_F - C_A/2)\times C_F C_A^2$, and therefore a piece of the form $(-C_A/2)\times C_F C_A^2 \times T\ln T$ remains uncanceled. According to Eq. (7), it would give an infinite contribution to the potential.

This divergence is a signal that the perturbation expansion for the potential has broken down. The breakdown is associated with the presence of infinite-range fields and can be dealt with by selective resummation. Suppose we sum over any number of single Coulomb rungs both inside and outside the connected structure in Fig. 5. The result will be proportional to

$$\int_{-T/2}^{T/2} dt_1 dt_2 \exp\left[C_F \frac{\alpha_s}{R}(T/2-t_2)\right] \exp\left[(C_F - C_A/2)\frac{\alpha_s}{R}(t_2-t_1)\right] \exp\left[C_F\frac{\alpha_s}{R}(t_1+T/2)\right] \frac{C_F C_A^2 \alpha_s^3}{(t_1-t_2)^2+R^2}$$
$$= \exp\left[C_F \frac{\alpha_s}{R} T\right] \times C_F C_A^2 \alpha_s^3 T \int_0^\infty \frac{dt}{t^2+R^2} \exp\left[-\frac{C_A}{2}\alpha_s \frac{t}{R}\right] . \tag{13}$$

The multiplicative exponential factor is simply the iteration of the zeroth-order potential. The resummation has also produced an exponential factor in the relative-time integration which has already been extended to infinity. The result is proportional to T and is therefore a finite contribution to the potential. Note, however, that the exponential $\exp[-(C_A/2)\alpha_s t/R]$ cannot be re-expanded before doing the t integration. That would lead back to the $T\ln T$ divergences of perturbation theory.

A contribution to the potential has been identified which is proportional to

$$C_F C_A{}^2 \frac{\alpha_s{}^3}{R} \int_0^\infty \frac{dx}{1+x^2} \exp[-(C_A/2)\alpha_s x] \, .$$

The exponential integral is nonanalytic in α_s at $\alpha_s = 0$. It contains a logarithmic discontinuity and can be written in the form

$$\int_0^\infty \frac{dx}{1+x^2} e^{-ax} = \ln a \sin a + g(a) \, , \qquad (14)$$

where $g(a)$ is analytic at $a=0$. To leading order in α_s, this contribution to the potential contains the $\alpha_s{}^3$ term mentioned in Sec. IV (Fig. 3). In the next order there is a term proportional to $C_F C_A{}^3(\alpha_s{}^4/R)\ln C_A\alpha_s$ plus a term of order $\alpha_s{}^4$. While there are many other contributions of order $\alpha_s{}^4$, we believe this is the only $\alpha_s{}^4 \ln\alpha_s$ term.

The existence of $\ln\alpha_s$ terms is familiar in quantum electrodynamics where they enter at the level of relativistic corrections in bound-state problems.[18] The physical processes involved are somewhat similar in the two cases. In either case, if the singularities are only logarithmic, it seems likely that perturbation theory can still be used. While there is no experimentally important reason for computing such terms in QCD, it is still interesting to know whether or not the computation could in principle be done to arbitrary accuracy for small α_s.[19] We conjecture that this is the case, that is, that the computation can be organized into a double expansion in $\alpha_s{}^n(\ln\alpha_s)^m$ with $m<n$.

The proof of this conjecture appears to be quite difficult. The $\ln\alpha_s$ terms are associated with disconnected diagrams and it must be shown how to deal with them in general. The selective summation of diagrams must be systematized, perhaps leading to some simple prescription for the computation of $V(R)$. Such a prescription will certainly be more complicated than the connected-diagram prescription for the case of uncorrelated sources.[19]

The physical content of the resummation is clear. In the simple example we considered, it corresponds to reorganizing the expansion about a Coulomb state rather than a state with noninteracting sources. In the ground state, the sources are in a color-singlet state. The intermediate state shown in Fig. 5 consists of one gluon and the sources in an octet state. Since the Coulomb force is attractive in the singlet channel and repulsive in the octet channel, there is an energy gap of order $C_A\alpha_s/R$ between the ground state and the intermediate state. By the uncertainty principle then, the intermediate state can exist for at most a time of order $R/C_A\alpha_s$. That is, the origin of the decreasing exponential on the right-hand side of Eq. (13) and, in fact, the $\alpha_s{}^4\ln\alpha_s$ term comes from $t \approx R/C_A\alpha_s$. It is worth pointing out, that in a bound state this will be a time on the order of the period of the motion, and it is perhaps misleading at this level to say that one is still computing an instantaneous potential.

VI. SUMMARY

Our most important results are the following:

(1) The static potential $V(R)$ between two fixed color sources exists as the infinite-mass limit of the interaction between a heavy quark and antiquark. Once the usual infinities with finite-mass quarks are dealt with, the renormalized quark mass can be taken to infinity. This demonstrates the quark mass independence of the potential in heavy $Q\overline{Q}$ bound states.

(2) The potential between two sources in a color-singlet state is gauge invariant and free of infrared singularities through two loops. The infrared finiteness is conjectured to be true to all orders.

(3) In low-order perturbation theory, the potential is made up of gluonic processes taking place over time intervals of order R. In a bound state (finite mass, moving quarks), this will be a short time interval compared to the period of the motion.

(4) The quantum color correlations of the sources with the gluons gives the potential a much more complicated graphical structure than with uncorrelated sources. In the later case the perturbation theory is given by the connected Feynman graphs (connectivity is defined without the source lines). This is no longer true in the color theory, a consequence of the fact that the sources are not completely classical.

(5) The potential does not exist as a simple power-series expansion in α_s. The expansion must be selectively resummed to avoid long-time singularities in some disconnected graphs. These are not the ordinary infrared singularities already mentioned, which are expected to cancel in each order. The resulting expression contains terms $\alpha_s^n(\ln\alpha_s)^m$, where $m<n$. We conjecture that the selective resummation can be carried out systematically to any order.

In addition to studying the potential in more detail and verifying conjectures, some other extensions of the present work come to mind:

(1) We have only examined the leading (nonvanishing) term in the limit $M_Q \to \infty$. The nonleading terms are important phenomenologically and should be studied. For example, the spin-spin interaction between two heavy quarks in an s state can be written in the form $V_{ss}(R)\vec{\sigma}_1 \cdot \vec{\sigma}_2/M_{Q_1}M_{Q_2}$. Is $V_{ss}(R)$ finite in the infinite-mass limit? If so, it is universal for different flavors, a statement that can be made with confidence even if $V(R)$ is deter-

mined at distance scales R primarily by nonperturbative effects.

(2) The combinatoric problems associated with color correlations have analogs in some statistical-mechanical systems. The spin correlations in the Kondo problem are one example.[20] It is possible that similar techniques with similar consequences could be applied to the computation of the partition function in these systems. Even within QCD, it seems likely that analogous combinatoric problems might arise in different contexts. What effect do they have, for example, in a semiclassical treatment of vacuum structure and the $Q\overline{Q}$ potential?

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[11]An equivalent discussion can also be formulated in momentum space. There the computations bear a strong resemblance to summation techniques used in the eikonal approximation, where dominance of low momenta in internal-loop integrations permits linearization of the fermion propagators. See, for example, M. Levy and J. Sucher, Phys. Rev. 186, 1656 (1969).

[12]Details of this analysis will appear in a future publication by one of us (M.D.).

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[19]The static potential provides a natural framework for the study of QCD perturbation theory in high orders. Clearly the first order of business in such a study is to establish the general form of the expansion, perhaps by verifying our conjecture.

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