PHYSICAL REVIEW D VOLUME 17, NUMBER 8 15 APRIL 1978

String representation for a field theory with internal symmetry

Roscoe Giles, Larry D. McLerran, and Charles B. Thorn
Center for Theoretical Physics, Laboratory for Nuclear Science and Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139
(Received 5 December 1977)

It is possible to represent certain quantum field theories as theories of interacting strings when both are defined on a suitable null-plane lattice. This representation is discussed for scalar field theories with internal degrees of freedom and quartic self-couplings. The internal-symmetry structure of the lattice string is that of an appropriate two-dimensional statistical-mechanical vertex model. The internal degrees of freedom are frozen out in the continuum limit (confinement) unless the vertex model is critical. The simplest internal symmetry, U(1), corresponds to Pauling's model of two-dimensional ice, which is critical. We compute the excitation energies of the internal degrees of freedom of the F model, a generalization of the ice model. The F model is characterized by a parameter $\Delta < 1$, and ice corresponds to $\Delta = 1/2$. It is critical for $-1 < \Delta < 1$. For $\Delta < -1$ charged states have infinite energy in the continuum limit. For $-1 < \Delta < 1$ the spectrum of long-wavelength excitations is that of a free one-dimensional semiperiodic boson field defined on a finite space. When the space is a ring there is a conserved topological charge as well as ordinary charge. The effect of these excitations on the resulting dual model is to contribute one degree of freedom reducing the critical dimension of space-time by 1.

I. INTRODUCTION

Recently,[1] a method was developed for implementing a string interpretation of relativistic quantum field theory. This interpretation is made by reorganizing the standard Feynman graph expansion for the Green's functions of the field theory in a combined topological–strong-coupling expansion on a suitable lattice. The result of this reorganization is that the sum of all Feynmann graphs can be rewritten as a dual string expansion, the leading term of which is the dual resonance model (DRM).[2]

The string representation for $\lambda\phi^4$ theory with ϕ a real scalar field has been described in detail by Thorn.[3] In this article we consider scalar field theories with internal symmetry. We concentrate on the simplest such theory, namely $\lambda(\phi^\dagger\phi)^2$, i.e., ϕ is taken complex. The only difference between the graphical expansion of this theory and of the neutral scalar field theory is that each line carries an orientation specifying the direction of charge flow, with the vertices constrained to force charge conservation. We therefore have a counting problem similar to that faced by workers in statistical mechanics.

The topological expansion starts with planar graphs and for these graphs the counting problem is to find the total number of ways to place arrows on the bonds of a two-dimensional grid subject to charge conservation at each vertex. The answer to this counting problem is precisely the partition function for Pauling's two-dimensional ice model,[4] which has been obtained, at least in the thermodynamic limit, by Lieb.[5]

For our purposes, we need to know all the low-lying excited states of the ice model, which has not been completely worked out in the solid-state literature. We accordingly calculate these excited states in this article. In fact, we consider the more general two-dimensional F model for ferroelectrics.[6] There is a close connection between these models and one-dimensional Heisenberg spin systems,[7] so our results also apply to these.

If the dimensions of the two-dimensional lattice are $M\times N$, we take a continuum limit M, $N\to\infty$ with N/M fixed, and we must keep all dependence on the shape ratios N/M which survive in this limit. We are therefore interested in finite-size effects which are usually neglected in solid-state applications.

The F model has a symmetry under the interchange of roles of M and N (duality) which puts constraints on the allowed shape dependence. We include a discussion of the implications of this symmetry.

We find that the long-wavelength structure of the F model can be summarized by a free boson quantum field theory on a *finite* two-dimensional space. This boson equivalence is *exact*, *even for the shape dependence of the continuum F model*. The boson field is an *angle*, i.e., the wave functional is periodic under $\phi\to\phi+P$, where P depends on the parameters of the F model.

Implicit in our results is the statement that the massless Thirring model defined on a finite space-time is completely equivalent to a free-boson field theory defined on the same finite space-time. When the space is closed, i.e., space-time is a

 © 1978 The American Physical Society

cylinder, both the charge and axial charge of the Thirring model are conserved. In the boson version, the charge is just the integral of $\dot{\phi}$ over the space. The axial charge is just the topological quantum number which is the number of times ϕ winds around its space as the cylinder is encircled.

This article is organized as follows. In Sec. II we discuss in a general way the relationship between the internal symmetry of the field theory and the structure of the corresponding statistical-mechanical model. In Sec. III we work out the low-energy spectrum of the F model in detail. In Sec. IV we discuss the implications of duality and apply it to determine the parameters of the ground-state energy. Finally in Sec. V we discuss the Bose-Fermi equivalence in this model and describe the properties of the dual model derived from $\lambda(\phi^\dagger\phi)^2$ theory.

II. GENERAL CONSIDERATIONS

The equivalence between neutral $\lambda\phi^4$ theory, with $\lambda<0$, and the standard dual model has been discussed in some detail in Ref. 3. In this section we briefly review these results and consider in a general way how the internal-symmetry structure of a scalar field theory is mapped onto that of its corresponding dual model.

The mapping between field theory and dual models is most conveniently expressed as an identification of Feynman graphs with terms of the dual loop expansion. We shall discuss field theories of quartically self-coupled real or complex scalar fields $\{\phi_a,\ a=1,\ldots,n\}$ in $d+2$ space-time dimensions. We define the field theory in terms of Green's functions computed in a mixed coordinate-momentum representation on the null plane[1]:

$$P^+=\frac{p^0+p^{d+1}}{\sqrt{2}},\qquad -i\tau=\frac{x^0+x^{d+1}}{\sqrt{2}},\qquad \vec{x}_\perp=(x^1,x^2,\ldots,x^d)\,. \tag{2.1}$$

We assume that the bare mass matrix is diagonal with respect to internal degrees of freedom, so the scalar propagator is

$$D_{ab}(P^+,\tau-\tau',\vec{x}_\perp-\vec{x}_\perp')=\delta_{ab}\theta(P^+(\tau-\tau'))\,\frac{1}{4\pi|P^+|}\left(\frac{P^+}{2\pi(\tau-\tau')}\right)^{d/2}\exp\left[-\frac{P^+}{2(\tau-\tau')}(\vec{x}_\perp-\vec{x}_\perp')^2-\frac{(\tau-\tau')}{2P^+}m^2\right]. \tag{2.2}$$

This propagator carries positive P^+ forward in τ and negative P^+ backward. We build Green's functions out of time-ordered Feynman graphs where each line, by convention, carries positive P^+. Note that the propagator factorizes into a space-time part identical to that of neutral scalar field theory and the δ_{ab} in internal indices. In the case of complex fields, each propagator also has an orientation depending on the direction of charge [U(1)] flow.

We assume that $\mathcal{L}_I$ has the structure

$$-\frac{1}{4!}\lambda V_{abcd}\phi_a\phi_b\phi_c\phi_d \tag{2.3a}$$

for real fields, or

$$-\tfrac{1}{4}\lambda V_{abcd}\phi_a^*\phi_b\phi_c^*\phi_d \tag{2.3b}$$

for complex fields. After Wick rotation, each vertex picks up a factor -1 from $-i\int dx^+\to-\int d\tau$.

Finally, we regulate infrared and ultraviolet divergences by replacing integrals over P^+ and τ in each graph by sums over discrete variables:

$$P^+=ka,\quad k=1,2,\ldots\qquad \tau=lb,\quad l=1,2,\ldots\,. \tag{2.4}$$

The exclusion of $P^+=0$ and $\tau=0$ regulates infrared and ultraviolet divergences, respectively. The ratio of lattice spacings $a/b\equiv T_0$ is a fundamental dimensional parameter of the regulated theory and will be identified as the string tension $1/2\pi\alpha'$ of the corresponding dual model.

In the lattice theory, each propagator gives

$$\int dP^+D_{ab}(P^+,\tau,\vec{x}_\perp)\to\delta_{ab}\sum_{k>0}\frac{1}{4\pi k}\left(\frac{T_0}{2\pi}\frac{k}{l}\right)^{d/2}\times\exp\left(-\frac{T_0}{2}\frac{k}{l}\vec{x}_\perp^{\,2}-\frac{m^2}{2T_0}\frac{l}{k}\right) \tag{2.5}$$

and each vertex gives

$$-\lambda\int d\tau\, d\vec{x}_\perp 2\pi\delta\left(\sum P^+\right)V\to\sum_l\left(\frac{-2\pi\lambda}{T_0}\,\delta_{\Sigma k=0}\right)\times V_{abcd}\int d\vec{x}_\perp\,. \tag{2.6}$$

Note that a and b appear only in the combination T_0. As discussed in Ref. 1, continuum field theory divergences arise, in amplitudes involving finite P^+ and τ, from large integer factors of order P^+/a and τ/b. The parameter T_0 plays a role

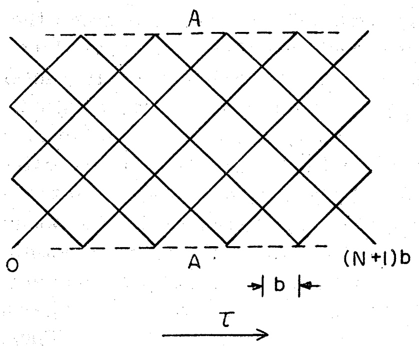

FIG. 1. The fishnet graph for the bare closed-string propagator. The vertices on the dashed lines A are identified.

analogous to that of the renormalization point μ in other regularization schemes.

It was pointed out in Ref. 1 that, for any Green's function carrying finite P^+ and spanning finite τ, there is a maximum order in perturbation theory (in λ) which can occur. The maximum corresponds roughly to graphs in which every line carries minimum $P^+=a$ and propagates a minimum $\tau=b$. These graphs are roughly of order $P^+\tau/ab$. The sum over all graphs can be organized about this limit and the terms put in one-to-one correspondence with the dual expansion on the same lattice.[8] The building blocks of this lattice dual expansion are closed- and open-string propagators to which we now turn.

Closed- and open-string graphs are fishnets of minimal propagators as shown in Figs. 1 and 2, respectively. Closed strings have the topology of a cylinder while open strings are planar. Both graphs can be represented on a τ-P^+ plane, as shown in Figs. 1 and 2. The closed string has periodic boundary conditions in P^+, the open string has free ends in P^+. We take $P^+=Ma$ and $\tau=Nb$ so there are M lines propagating (horizontally) through the closed-string diagram. These graphs correspond to bare propagators D of their respective strings. The Feynman graph *is* the corresponding latticized free-string functional integral. We note that the amplitude for each graph factors into a factor arising from transverse spatial degrees of freedom identical to that of neutral ϕ^4 and a factor from internal degrees of freedom.

Our primary interest here is the spectrum of low-lying free-string states in the continuum limit. The spectrum is most easily extracted from the trace (sum over identical initial and final states) of the propagator. We shall therefore concentrate on configurations obeying periodic boundary conditions in τ. The trace Z may be identified as the partition function of a two-dimensional classical system at fixed temperature or of a one-dimensional quantum system at temperature $1/\tau$. This one-dimensional quantum system is the closed or open lattice string[8] of $P^+=Ma$. We may write Z as

$$Z(M,N)=\sum_r e^{-NE_r(M)}$$
$$=\sum_r e^{-\tau P_r^-(M)}\,, \qquad (2.7)$$

where $\{r\}$ is a set of quantum numbers labeling states of the one-dimensional system. E_r is the dimensionless energy of the lattice string state r. We identify P^- as the coefficient $(1/b)E(M)$ of τ in the exponentially decreasing term associated with state r. In the continuum limit, $M=P^+/a\to\infty$ and we expect

$$E_r(M)\underset{M\to\infty}{\sim}\alpha M+\beta_r+\frac{\gamma_r}{M}+\cdots\,, \qquad (2.8)$$

giving

$$P_r^-(P^+)\sim\alpha\frac{P^+}{ab}+\beta_r\frac{1}{b}+\frac{T_0\gamma_r}{P^+}\,. \qquad (2.9)$$

The terms proportional to α and β are divergent and noncovariant in the continuum limit.

As discussed by Giles and Thorn,[8] the bulk coefficient α (which is the free energy per site of the two-dimensional statistical-mechanics analog) is the same for all closed or open low-lying excitations and corresponds to an unobservable infinite phase in each P^+ channel of the continuum Minkowski-space dual theory. β_r is a finite lattice energy which must be zero for states which survive the continuum limit if the resulting continuum theory is to be covariant and dual. In the case of no internal degrees of freedom, β is a surface energy which depends only on the number of boundaries of a free string state and vanishes for the closed string. In a lattice string theory, β may be arranged to vanish by adding a

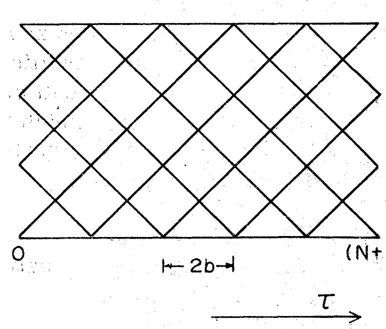

FIG. 2. The fishnet graph for the bare open-string propagator.

local surface counterterm to the action which defines the theory.[8] For a dual model derived from neutral $\lambda\phi^4$, β of the open string depends on the coupling λ and vanishes only for a suitable choice of bare coupling (see Sec. IV):

$$\lambda_{\text{critical}} = -32\pi^2\left(\frac{T_0}{4\pi}\right)^{1-d/2} . \tag{2.10}$$

The term γ_r is finite and covariant and gives a mass squared:

$$m^2 = 2T_0\gamma_r . \tag{2.11}$$

For the neutral case, the string ground states have

$$\gamma_{\text{closed}} = -\frac{\pi d}{6} , \qquad \gamma_{\text{open}} = -\frac{\pi d}{24} , \tag{2.12}$$

giving the usual tachyons of the dual resonance model.

Because the partition function Z of string states factorizes, the contribution of internal degrees of freedom to the spectrum is additive and the problem of internal symmetry can be considered in isolation. The internal partition function $Z_I(N,M)$ may be regarded as the partition function of a two-dimensional assembly of links (propagators) each with n possible states, corresponding to $a=1,\ldots,n$, and, in the case of complex fields, with orientation corresponding to the direction of charge flow. Each configuration of links is weighted by the appropriate product of V_{abcd}'s over its vertices. This sort of problem is that of a two-dimensional statistical-mechanical vertex model, and it is well known that such models can have relatively complex phase structure depending on the vertex couplings V_{abcd}.[9,10]

As we have characterized them so far, the two-dimensional vertex models arising from scalar field theories may be quite complicated. However, they share one important simplifying feature. This is self-duality in the statistical-mechanical sense. The partition function on an $N\times M$ lattice is identical, up to boundary conditions, to that of the $M\times N$ lattice which is a result of a rotation by 90°. For the closed string, which has periodic boundary conditions in both directions, the partition function is exactly self-dual:

$$Z_{\text{closed}}(M,N) = Z_{\text{closed}}(N,M) . \tag{2.13}$$

For the open string, a duality transformation maps the open-string partition function into the closed-string propagator summed over independent initial and final states. The implications of self-duality are discussed in more detail in Sec. IV.

We may translate our questions about the string spectrum to the statistical-mechanics language of vertex models. We are particularly interested in the β and γ terms in the energy. Unfortunately, the most commonly addressed problem in the statistical-mechanics literature is the calculation of the bulk free energy per site, α. β and γ are finite-size effects associated with the boundaries, and are of no interest in the thermodynamic limit. For the ground state, β is simply a surface energy. Nonzero γ reflects the existence of long-range correlations between the boundaries. Thus, the ground state can have nonzero γ only at a critical point.

Excited states may or may not have the same β as the ground state. If the lowest excited states have a different β, there is a finite gap in the lattice energy and thus no long-range correlatations. In the continuum limit, such a finite gap on the lattice corresponds to an infinite gap in P^-. Thus, if the vertex model is not critical, all excitations of internal symmetry in the continuum dual theory are frozen out—this may be aptly called confinement.

In this paper, we shall consider only the simplest of vertex models arising from field theory. Consider the case of a simple complex scalar field ϕ. The corresponding vertex model has only one type of link whose only degree of freedom is its orientation. We regard each link as an arrow. Vertices are restricted by the requirement that two arrows must come in and two must go out. This model is the two-dimensional ice model.[4,5] The description as ice is apparent if we identify vertices as oxygen atoms and arrows as polar hydrogen bonds. An arrow pointing toward a given vertex indicates that the corresponding hydrogen atom is associated with the oxygen atom at that vertex—hence the "ice condition" that two arrows enter and two leave each vertex. In our interpretation, the "ice condition" is simply charge conservation. The six possible vertex configurations are indicated in Fig. 3.

A straightforward generalization of the ice model is the F model[6] where vertices shown in Fig. 3(a) are given some weight, v, and those of Fig. 3(b) retain weight 1. The vertices in Fig. 3(a) have net polarization in some channel while those in Fig. 3(b) do not. This is a model of a two-dimensional ferroelectric.[9] In the next section we shall consider this generalization of the ice model in some detail. It is evident that, though the F model is self-dual, there is no field theory whose internal-symmetry structure leads to it in the dual limit. This is obvious because the distinction between the vertices of Figs. 3(a) and 3(b) is meaningful only relative to the *plane* in which they are

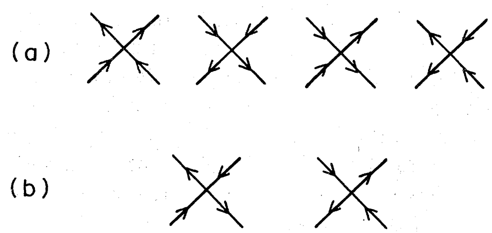

FIG. 3. The six vertices for the F model. Those in (a) have charge flow in some planar channel, those in (b) do not.

embedded—no such distinction arises in the original field theory.

However, the F model may represent a simple approximation to the more complicated structure of a U(N) theory for large N. In the U(N) case, each link has N degrees of freedom while vertices, in addition to conserving charge, contract particle-antiparticle indices in all possible ways. Thus, we can regard each term of the internal-symmetry partition function as a collection of oriented nonoverlapping loops. Each such term contributes to the partition function a factor N^L where L = the number of loops on the graph. For large N, the leading internal-symmetry diagram is *planar* in the sense that all loops are of minimal area. Diagrams which deviate from this planarity might be approximated by allowing different vertex weights as in the F model.

We close with the observation that though every self-dual internal-symmetry structure on the lattice need not correspond to a field theory, they certainly lead to dual models in the continuum limit and are therefore interesting in their own right.

III. LOW-LYING EXCITATIONS OF THE F MODEL

In this section, we consider the F model in some detail and relate its phase structure and low-lying excitation spectrum to the corresponding U(1)-symmetric string model. The version of the F model derived in Sec. II is almost identical to that considered by many solid-state physicists as a two-dimensional ferroelectric model.[9] The two versions have the same vertex weights, but the conventional solid-state F model is defined on a rectangular- rather than diamond-shaped lattice. The thermodynamic properties of the F model with periodic boundary conditions in space and time on the rectangular lattice can be computed exactly. The model was first solved by Lieb[5] in the ice case, $v=1$ and by Lieb and by McCoy and Wu[11] in the general case. Correlation functions have also been computed.[12] Our discussion will be primarily a review and a translation of these results to a language more suitable to the problem at hand.

First, we argue that the quantities of greatest interest to us are insensitive to the differences between the rectangular or diamond lattice. The structure of the low-lying spectrum in a critical phase depends on excitations of wavelengths very large compared to the lattice spacing. The symmetry of the local vertices under 90° rotations leads to isotropy of low-lying excitations in the continuum limit. Therefore, we expect that terms of order $1/M$ in the ground-state energy and in the excitation spectrum will be correctly given by the rectangular lattice.

In contrast, it is to be expected that the surface terms, β, will be sensitive to the local structure of the lattice near boundaries and will not agree numerically for the diamond and rectangular lattices. Note, however, that for the ground state at least β must be renormalized (by an appropriate choice of λ_0) to zero in order to obtain a dual model. Therefore the numerical value of β in the ground state does not affect the structure of the resultant continuum model. In a phase where there is a finite gap $\Delta\beta$ between the ground state and first excited state, the value of $\Delta\beta$ computed on the rectangular lattice is expected to differ from that of the diamond lattice. However, the existence of such a gap reflects the phase structure of the system and is expected to be independent of the lattice orientation. Any finite gap has the same consequences in the continuum limit—the freezing out of the internal degrees of freedom of the string. The F model can be solved exactly, in the special case $v=1/\sqrt{2}$, on both the diamond and rectangular lattices. On both lattices, β vanishes for the low-lying excited states and the spectra of excitations whose energies are of order $1/M$ are identical.

In the remainder of this section, we consider the F model defined on an $N\times M$ rectangular lattice with periodic boundary conditions in space and time (Fig. 4) with six vertices as shown in Figs. 5(a) and 5(b).

Analysis of the F model reveals that there are two very different cases, depending on the value of v. For $v<\frac{1}{2}$, the excitation spectrum of the one-dimensional string theory (viz., the rows) has a finite gap and gives confinement of charge in the continuum limit. For *all* $v>\frac{1}{2}$, the system is critical; there are long-range correlations and a nontrivial spectrum of states with excitation energies of order $1/M$.

These results can be anticipated qualitatively before plunging into the actual calculation. In the limiting case $v=0$, there are only two possible graphs for the partition function. These correspond to the two ways of arranging the vertices

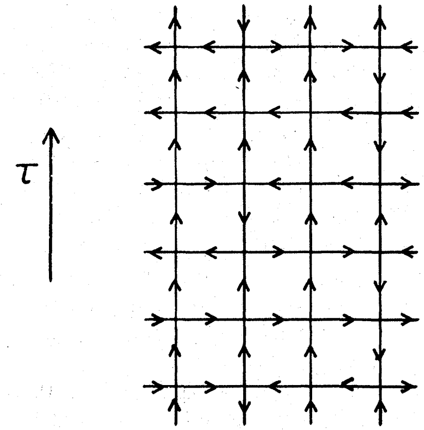

FIG. 4. Rectangular lattice for the F model.

of Fig. 5(b) in such a way that they alternate along every row and column. The corresponding partition function is $Z = 2$ and the energy is

$$E = \lim_{N\to\infty} \frac{1}{N}(-\ln Z(N,M)) = 0\,. \tag{3.1}$$

The directions of arrows on the graph are the directions of local current flow. Thus the vertical arrows between rows of vertices correspond to the charge density. We see that along each horizontal row, these arrows alternate direction, so that in the continuum limit the charge density of this state is zero.

If v is small but nonzero, we can imagine computing the partition function perturbatively. On a given row we can introduce two vertices from the set of Fig. 5(a) in such a way that the net charge of the row is 2. It is clear that at least two such vertices must be located somewhere on each of the adjacent rows though not necessarily in the same column. Therefore, every row has two such vertices and the energy is greater than that of the ground state by a term of order $2\ln(1/v)$. This is the gap to which we have referred. Because there are $M(M-1)$ such states on each row which all have energies of order $2\ln(1/v)$ and which mix

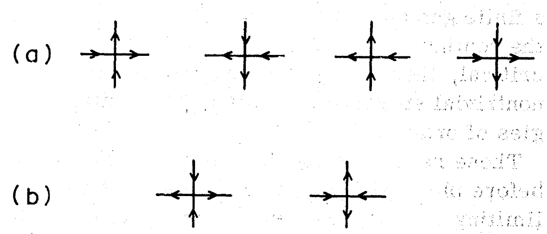

FIG. 5. The six vertices for the rectangular version of the F model.

with each other in the partition function, we expect that the actual spectrum will be a band of finite width about $2\ln(1/v)$. Qualitatively, as v increases the center of this band is lowered until, at some point (namely $v=\frac{1}{2}$) there is a level crossing and therefore critical behavior. To obtain more quantitative results, we proceed to describe the exact solution of the model.

The solution of the F model is most naturally described in terms of the one-dimensional lattice quantum theory associated with the two-dimensional vertex model as described in Sec. II. We focus first on a given row of vertical arrows. Such a row can be in 2^M possible "states" since each arrow can be up or down. Each arrow can be represented conveniently as an elementary spin-$\frac{1}{2}$ object. We may define the transfer operator T as that $(2^M\times 2^M)$-dimensional matrix whose matrix elements between row states is the sum over all ways of going from the initial row states to the final one through one row of vertices, weighted as in the partition function. In the spin basis, T can be represented in terms of the set of Pauli spin operators $\{\vec{\sigma}_j,\ j=1,\dots,M\}$. The partition function is

$$Z(M,N) = \mathrm{Tr}[T(M)^N]\,. \tag{3.2}$$

An eigenvalue t of the transfer operator corresponds to energy

$$E = -\ln t\,. \tag{3.3}$$

The transfer operator is the generator of discrete time translations on the lattice, so the negative of its logarithm is the analog of the Hamiltonian.

The exact solution of the F model follows from an analysis of this transfer matrix. The transfer matrix can be constructed for the general ($v\neq 0$) F model by use of techniques discussed in Refs. 5 and 9. The total z component of the spin S_z is conserved, that is, the transfer matrix element vanishes between states of different spin. This is simply charge conservation.

The expression for T in terms of σ matrices is complicated and is not directly useful to us here. The interested reader is referred to Ref. 9, where the T matrix is constructed and studied in great detail. In this reference, the observation is made that the transfer matrix, expressed in terms of σ matrices, commutes with the Hamiltonian of an anisotropic Heisenberg spin model[11]:

$$H_{\mathrm{Heis}} = -\tfrac{1}{2}\sum_i (\sigma_i^x\sigma_{i+1}^x + \sigma_i^y\sigma_{i+1}^y + \Delta\sigma_i^z\sigma_{i+1}^z)\,, \tag{3.4}$$

where

$$\Delta \equiv 1 - \frac{1}{2v^2}\,. \tag{3.5}$$

Since H and T commute, they can be simultaneously diagonalized.

The eigenstates of T are given by the Bethe ansatz.[13] Any state with m arrows down and $M-m$ arrows up can be labeled by the positions of its m down arrows: $i_1 < i_2 < \cdots < i_m$. The Bethe ansatz is that the wave function takes the form

$$\psi^{(m)}(i_1, i_2, \ldots, i_m) = \sum_P A_P \exp\left(i \sum_l k_{P_l} i_l\right), \tag{3.6}$$

where the sum, $\sum_P$, is over all permutations of the distinct momenta $k_1, \ldots, k_m$. It can be shown that ψ vanishes if any two momenta are identical. Moreover, it can be shown that ψ diagonalizes H when the momenta k_i satisfy the nonlinear matrix equation[13]

$$k_i = \frac{2\pi}{M} I_i(m, \{\lambda\}) - \frac{1}{M} \sum_{j=1}^{m} \theta(k_i, k_j). \tag{3.7}$$

The set of numbers $\{I_i\}$ are integers for odd m and half-odd integers for even m. The set of numbers $\{I_i\}$ completely specifiy the momenta k_i and, hence, the state $\psi(m, \{I\})$. The matrix θ is

$$\theta(k_i, k_i) = 2 \tan^{-1} \frac{\Delta \sin\frac{1}{2}(k_i - k_j)}{\cos\frac{1}{2}(k_i + k_j) - \Delta \cos\frac{1}{2}(k_i - k_j)}. \tag{3.8}$$

The eigenvalue of the transfer operator corresponding to $\{k_i\}$ is[9]

$$t\{k_i\} = \exp\left[-\frac{M}{2} \ln 2(1-\Delta)\right] \times \left[\prod_j \Lambda(k_j) + \prod_j \Lambda(-k_j)\right], \tag{3.9}$$

where we define

$$\Lambda(k) \equiv \frac{2\Delta - 1 - e^{ik}}{1 - e^{ik}}.$$

The special case $\Delta = 0$ is trivial to solve, yet gives us some insight into the structure of the nontrivial critical cases $0 < |\Delta| < 1$. For convenience, we will take both m and $M/2$ to be even integers.

For $\Delta = 0$, the functions $\theta(k_i, k_j)$ all vanish, and we have

$$k_j = \frac{2\pi}{M} I_j,$$

where each I_j is a half-odd integer.

Two Bethe states defined by k_j's which differ by a multiple of 2π give the same state (3.6) so we may take $-\pi < k_j < \pi$, whence $\{I_j\}$ must be some subset of

$$\left\{-\left(\frac{M-1}{2}\right), -\left(\frac{M-1}{2}\right)+1, \ldots, \left(\frac{M-1}{2}\right)-1, \frac{M-1}{2}\right\}.$$

The eigenvalue of the transfer operator corresponding to a given set of k_j's is

$$t = 2(-1)^{m/2} \prod_{i=1}^{m} \cot\left(\frac{k_i}{2}\right). \tag{3.10}$$

The low-lying states correspond to the maximal values of t. Note that t is negative unless there are an even number of negative k_j's.

To understand the significance of negative t, we return to the diamond lattice, which is the appropriate one for our interpretation. For that lattice, one must include two steps in time to obtain the full dynamics of the theory. For the case $\Delta = 0$, this two-step transfer matrix has positive eigenvalues, and gives the same low-lying excitation spectrum as T^2 does in the rectangular lattice. We conclude that the fundamental quantity is T^2, not T, so the negative sign is inconsequential. We accordingly take the absolute value of (3.10) and write

$$E(\{k\}) = -\ln 2 - \sum_j \ln\left|\cot\tfrac{1}{2} k_j\right|. \tag{3.11}$$

As long as $|k_j| < \pi/2$, $-\ln|\cos\frac{1}{2} k_j|$ is negative. Therefore the ground state has all possible modes with $|k_j| < \pi/2$ occupied. This corresponds to $m = \frac{1}{2} M$ with the set of half-odd integers chosen to be

$$I_j^{(0)} = j - \tfrac{1}{2} - \tfrac{1}{2} m, \quad j = 1, \ldots, m \tag{3.12}$$

as illustrated in Fig. 6. Note that this state is neutral: $Q = M - 2m = 0$. The asymptotic form of the ground-state energy for large M may be easily computed using the Euler-Maclaurin formula

$$E_0(M) \underset{M\to\infty}{\sim} -M \ln 2 - \frac{\pi}{6M} + O\left(\frac{1}{M^2}\right). \tag{3.13}$$

Note that there is no gap term and that the term of order $1/M$ has the same coefficient γ as we have found for each transverse dimension in the neutral

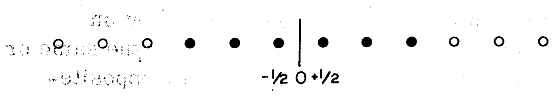

FIG. 6. The I_j's for the ground state. All available values are indicated by closed and open circles, the closed circles being the ones actually taken.

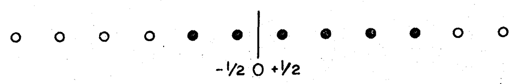

FIG. 7. The I_j's for the lowest state in a nonzero chirality sector.

string model. This will be seen to be true for all Δ and signals the fact that the internal degrees of freedom of a critical phase are equivalent to transverse dimensions in their contribution to the ground-state mass.

We are interested in excited states whose energies are of order $1/M$ relative to the ground-state energy as $M\to\infty$. The set of mode numbers $\{I_j\}$ which characterize such states necessarily differs from the ground-state set $\{I_j^{(0)}\}$ only for a finite (as $M\to\infty$) number of I_j's near the boundaries, $|k_j|\sim\pi/2$. The energy associated with such a mode

$$I=\pm\left(\frac{M/2-1}{2}+n\right)$$

is, to order $1/M$,

$$\epsilon_n=\frac{2\pi}{M}\left(n-\tfrac{1}{2}\right). \tag{3.14}$$

It is clear that we can express the low-lying excitation energies of the $\Delta=0$ problem as sums over such ϵ_n's.

First consider the ground state in a given (m even) charge sector $Q=-4k$. The set of I_j's corresponding to this state is similar to $I_j^{(0)}$ but with k modes removed from each boundary. The corresponding excitation energy is

$$E_0(Q)=-2\sum_{n=-k+1}^{0}\epsilon_n=\frac{2\pi k^2}{M}=\frac{\pi}{2}\,\frac{Q^2}{4M}. \tag{3.15}$$

For $Q=+4k$ we obtain the same result. We denote the set of modes for the ground state of the charge Q sector by $I^{(Q)}$.

We may construct any set of mode numbers corresponding to a state in the charge Q sector by adding "particle-hole" pairs to the set $I^{(Q)}$. Here a "particle" refers to a mode number I not in the set $I^{(Q)}$, while "hole" refers to the deletion of some I_j contained in $I^{(Q)}$. Any particle-hole pair may be characterized by its momentum:

$$P\equiv k_{\text{particle}}-k_{\text{hole}}\ . \tag{3.16}$$

Particle-hole pairs having energies of order $1/M$ fall naturally into two classes depending on whether the particle and hole are near the same or opposite boundaries of $I^{(Q)}$. The only opposite-side particle-hole excitations we need to consider are those which correspond to moving a block of l particles from one side to the other (Fig. 7). These have energy $(2\pi/M)l^2$. The momentum of such a state is

$$P=l\pi\left(1-\frac{Q}{M}\right). \tag{3.17}$$

In the large-M limit all low-lying states will have total momentum of the form $P=l\pi+O(1/M)$ so that the number l gives a good quantum number in the limit. We define this quantum number by $Q_5=2l$ and call it the "axial charge" or "chirality" of the state. The energy associated with a sector of chirality Q_5 is

$$E(Q_5)=2\pi\,\frac{Q_5^{\,2}}{4M}\ . \tag{3.18}$$

Equation (3.14) gives $E=|P|$ for each same-side particle-hole pair. For any *set* of particles and holes on the same side, we have

$$E=|P|\ , \tag{3.19}$$

where P is the total momentum carried by particle-hole pairs on that side.

Thus we have the following description of low-lying states. States fall into sectors defined by the two quantum numbers Q and Q_5. The ground-state energy in the (Q,Q_5) sector is

$$E_0(Q,Q_5)=-\frac{\pi}{6M}+\frac{\pi}{2}\,\frac{Q^2}{4M}+2\pi\,\frac{Q_5^{\,2}}{4M}\ . \tag{3.20}$$

The ground state of the (Q,Q_5) sector corresponds to a set of I_j's which form an unbroken sequence of length $(M-Q)/2$ whose center is offset from zero by $Q_5/2$. Excitations within the $(Q-Q_5)$ sector correspond to the introduction of particles and holes in this chain and have excitation energy:

$$\Delta E_{\text{particle hole}}=|P_L|+|P_R|\ , \tag{3.21}$$

where P_L, P_R are the total momenta of particle-hole pairs on the left and right side, respectively.

The analysis of the general case $|\Delta|<1$ is immensely more complicated than that of the case $\Delta=0$. For m, $M\to\infty$, the nonlinear matrix equation (3.7) can be approximated by an integral equation to leading order in $1/M$. Yang and Yang[14] have analyzed the structure of this equation, and proved that it gives the true ground-state energy density in each charge sector, solving it exactly for $Q=0$.

The low-lying spectrum may be computed by retaining the $1/M$ corrections to the replacement of the sum by an integral and allowing finite Q and Q_5 and "particle-hole" pairs near the ends of the set I. The perturbation series in $1/M$ so developed is an asymptotic one for the energy. The procedure is outlined in more detail in the Appendix.

A remarkable property of the general case is that, after lengthy calculations, it is found that the low-lying spectrum is very similar to that of the

$\Delta=0$ case. Again, sectors are characterized by quantum numbers Q and Q_5. The ground state in the (Q, Q_5) sector has energy

$$E_0(Q,Q_5)=-\frac{\pi}{6M}+(\pi-\mu)\frac{Q^2}{4M}+\frac{\pi^2}{\pi-\mu}\frac{Q_5{}^2}{4M}, \tag{3.22}$$

where we have defined $\Delta=-\cos\mu$, $0<\mu<\pi$.

Again we have further excitations defined as sets of same-side particle-hole pairs whose energies are identical to those in the $\Delta=0$ case[15]:

$$\Delta E_{\text{particle hole}}=|P_R|+|P_L|, \tag{3.23}$$

where P_R and P_L are the total momenta on the left and right sides. Notice that, since $\theta(k_i,k_j)$ is antisymmetric, the total momentum can be written in terms of I_j's.

We see that the only Δ dependence of the low-lying spectrum comes in the reciprocal variation in the slopes of the Q^2 and $Q_5{}^2$ terms in the energy. As we discuss in detail in the next section, this reciprocity is a reflection of duality. Q and Q_5 are interchanged under the duality transformation.

This result is quite reasonable. If we perform a Jordan-Wigner transformation on the Heisenberg Hamiltonian (3.4), we arrive at an equivalent fermion theory. In the $\Delta=0$ case, this theory is a free, massless lattice field theory which is chirally symmetric for all M. In the $\Delta=0$ case, Q is the vector charge while Q_5 is the axial charge. (Note that $Q_5=N_L-N_R$.) For $\Delta\neq0$, the theory is not exactly chirally symmetric. For large M, the matrix elements of the chirality-breaking term between low-lying states are small and the Hamiltonian goes over to that of the massless Thirring model.[16] Again Q and Q_5 are the fermion charge and axial charge. In two dimensions, the vector and axial-vector currents are dual to each other (in the sense of differential geometry):

$$J_5^\mu=\epsilon^{\mu\nu}J_\nu .$$

For $\Delta<-1$, as we have mentioned, there is a gap. Yang and Yang compute this gap explicitly between the ground states of the various charge sectors:

$$\Delta\beta(Q)\propto|Q| .$$

Johnson, Krinsky, and McCoy[12] have computed correlation functions of the F model (as a limit from an eight-vertex model) and find no long-range correlations for $\Delta<-1$.

As we have indicated in our general discussion in Sec. II, the significance of this gap is that in the continuum limit all internal-symmetry excitations have infinite energy. The only allowed charge state is the (neutral) ground state. This is our paradigm of confinement: Internal-symmetry excitations have finite energy only if the corresponding two-dimensional statistical-mechanical system is critical. For the F model we have two phases:

$v<\frac{1}{2}$, $\Delta<-1$ confinement $(Q=0)$,

$v>\frac{1}{2}$, $-1<\Delta<1$ $Q\neq0$ states have finite energy.

In a field theory with U(N) symmetry we might speculate that $1/N$ plays a role similar to v; certainly the limit $N\to\infty$ freezes the system into a locally singlet state. If this is the case we could hazard the conjecture that there is a critical N_c such that the theory confines U(N) degrees of freedom for $N>N_c$, but not for $N<N_c$.

IV. DUALITY: CONSTRAINTS ON LOW-ENERGY SPECTRUM

The F model for an $M\times N$ two-dimensional lattice has a symmetry under the interchange of M and N.[17] This symmetry is simplest when the lattice has periodic boundary conditions in both dimensions and is valid for either the diamond or rectangular array. Thus, in terms of the transfer matrix, we have the relation

$$Z(M,N)\equiv \mathrm{Tr}[T(M)^N]=\mathrm{Tr}[T(N)^M] . \tag{4.1}$$

In our interpretation, the eigenvalues of $T(M)$ are identified with $e^{-E_r(M)}$ where $E_r(M)$ are the P^- eigenvalues of the M parton state. They may be extracted from the partition function by writing the expansion

$$Z(M,N)=\sum_r e^{-NE_r(M)} \tag{4.2}$$

and duality gives a relation

$$\sum_r e^{-NE_r(M)}=\sum_r e^{-ME_r(N)} . \tag{4.3}$$

The limit $N\to\infty$ leaves only the ground-state contribution on the left-hand side whereas, in general, arbitrarily large eigenvalues contribute to the right-hand side.

The duality constraints become very powerful when there are $1/M$ excitations above the ground state. Let us suppose that

$$E_r(M)\underset{M\to\infty}{\sim}\alpha M+\frac{\gamma_r}{M}+O\left(\frac{1}{M^2}\right) . \tag{4.4}$$

Then (4.3) takes the form for both M and N large, M/N fixed,

$$\sum_r e^{-\gamma_r N/M}=\sum_r e^{-\gamma_r N/M}+O(e^{-M},e^{-N}) . \tag{4.5}$$

For N/M large, (4.5) relates the lowest value of γ_r, γ_0 to the distribution of large values of γ_r.

Let us consider first the $\Delta=0$ F-model partition function for large M and N. From (4.5), we need only consider the $O(1/M)$ excitations which have been discussed in Sec. III. For $\Delta=0$ these excitations are identical to the excitation of four sets of noninteracting fermion oscillators

$$b_m^{(1)}, b_m^{(2)}, d_m^{(1)}, d_m^{(2)}$$

with the anticommutation relations

$$\{b_m^i, b_n^{\dagger j}\} = \delta^{ij}\delta_{m,n},$$
$$\{d_m^i, d_n^{\dagger j}\} = \delta^{ij}\delta_{m,n},$$

all others zero. $b_m^\dagger$ flips a down spin up, $d_m^\dagger$ flips an up spin down. The excitation energies are different in even and odd fermion sectors. In the even fermion sector each $b_m^\dagger, d_m^\dagger$ raises the energy an amount $(2\pi/Ma)(m-\frac{1}{2})$, $m=1,2,\ldots$. In the odd fermion sector, each $b_m^\dagger d_m^\dagger$ raises the energy an amount $(2\pi/Ma)m$, $m=1,2,\ldots$ and in addition there are nondegenerate zero modes $b_0^\dagger$, $d_0^\dagger$ which do not change the energy at all.

The ground state has total spin zero, and the ground state in the odd fermion sector is doubly degenerate with spin ± 1, and has an energy $\pi/2Ma$ above the true ground state. To keep track of all these possibilities let us define ($x \equiv e^{-\pi N/M}$)

$$Z(M,N,\theta) \equiv \mathrm{Tr}[T(M)^N e^{i\theta S_z}]$$
$$\underset{M,N\to\infty}{\sim} e^{-E_0(M)N}\left\{\frac{1}{2}\left[\prod_{k=1}^{\infty}(1+x^{2k-1}e^{i\theta})^2(1+x^{2k-1}e^{-i\theta})^2 + (\theta\to\theta+\pi)\right]\right.$$
$$\left. + e^{-\pi N/2M}(\tfrac{1}{2})\left[(1+e^{i\theta})(1+e^{-i\theta})\prod_{k=1}^{\infty}(1+x^{2k}e^{i\theta})^2(1+x^{2k}e^{-i\theta})^2 - (\theta\to\theta+\pi)\right]\right\}. \quad (4.6)$$

If $Z(M,N,\theta)$ is expanded in a power series in x and $e^{i\theta}$, the coefficient of $x^n e^{im\theta}$ counts the number of states with excitation number n and total spin m. This expression can be simplified by the identity[18]

$$\prod_{k=1}^{\infty}(1+x^{2k-1}z)\left(1+x^{2k-1}\frac{1}{z}\right) = \prod_{k=1}^{\infty}\frac{1}{1-x^{2k}}\sum_{k=-\infty}^{\infty}x^{m^2}z^m. \quad (4.7)$$

By letting $\bar{z}=z/x$ in (4.7) we obtain

$$\left(1+\frac{1}{\bar{z}}\right)\prod_{k=1}^{\infty}(1+x^{2k}\bar{z})\left(1+x^{2k}\frac{1}{\bar{z}}\right) = \prod_{k=1}^{\infty}\frac{1}{1-x^{2k}}\sum_{m=-\infty}^{\infty}x^{m^2+m}\bar{z}^m. \quad (4.8)$$

Using (4.7) and (4.8) in (4.6), we find ($z=e^{i\theta}$)

$$Z(M,N,z) \sim e^{-E_0(M)N}\prod_{k=1}^{\infty}\frac{1}{(1-x^{2k})^2}\left\{\frac{1}{2}\left(\sum_{m=-\infty}^{\infty}x^{m^2}z^m\right)^2 + \frac{1}{2}\left(\sum_{m=-\infty}^{\infty}x^{m^2}(-z)^m\right)^2\right.$$
$$\left. + \frac{z}{2}\left[\left(\sum_{m=-\infty}^{\infty}x^{(m+1/2)^2}z^m\right)^2 + \left(\sum_{m=-\infty}^{\infty}x^{(m+1/2)^2}(-z)^m\right)^2\right]\right\}. \quad (4.9)$$

We simplify further by writing

$$\frac{1}{2}\left[\left(\sum_{m=-\infty}^{\infty}x^{m^2}z^m\right)^2 + (z\to -z)\right] = \sum_{n\text{ even}} z^n \sum_{m=-\infty}^{\infty}x^{m^2+(n-m)^2}$$
$$= \left(\sum_{n\text{ even}} z^n x^{n^2/2}\right)\left(\sum_{m=-\infty}^{\infty}x^{2m^2}\right)$$

and similarly

$$\frac{1}{2}\left[\left(\sum_{m=-\infty}^{\infty}x^{(m+1/2)^2}z^m\right)^2 + (z\to -z)\right] = \frac{1}{z}\left(\sum_{n\text{ odd}} z^n x^{n^2/2}\right)\left(\sum_{m=-\infty}^{\infty}x^{2m^2}\right),$$

so that

$$Z(M,N,z) \underset{M,N\to\infty}{\sim} e^{-E_0(M)N}\prod_{k=1}^{\infty}\frac{1}{(1-x^{2k})^2}\left(\sum_{m=-\infty}^{\infty}x^{2m^2}\right)\left(\sum_{n=-\infty}^{\infty}z^n x^{n^2/2}\right)$$
$$\sim e^{-\alpha MN}x^{\gamma_0/\pi}\left(\sum_{m=-\infty}^{\infty}x^{2m^2}\right)\left(\sum_{n=-\infty}^{\infty}z^n x^{n^2/2}\right)\prod_{k=1}^{\infty}\frac{1}{(1-x^{2k})^2}. \quad (4.10)$$

The duality transform $N \leftrightarrow M$ is just the Jacobi imaginary transformation $x \to q$ with

$$\ln q \ln x = \pi^2 ,$$

and duality requires

$$f(x) \equiv x^{\gamma^0/\pi}\left(\sum_{m=-\infty}^{\infty} x^{2m^2}\right)\left(\sum_{n=-\infty}^{\infty} x^{n^2/2}\right)\prod_{k=1}^{\infty}\frac{1}{(1-x^{2k})^2}$$

$$=f(q) , \tag{4.11}$$

and this relation requires that $\gamma^0 = -\pi/6$.[19]

The various factors in (4.10) can be seen to correspond to the different types of excitations described in Sec. III. $\prod_{k=1}^{\infty} 1/(1-x^{2k})^2$ accounts for the number of particle-hole excitations within a given charge sector and "chirality" sector. $(\sum_{m=-\infty}^{\infty} z^m x^{m^2/2})$ accounts for the energy shifts of the ground states within each charge sector, and $(\sum_{n=-\infty}^{\infty} x^{2n^2})$ accounts for the energy shifts of the ground states within each "chirality" sector. For $\Delta \neq 0$, but $-1<\Delta<1$, it is only these latter two energy shifts that are modified. From Sec. III, we infer, using $\gamma_0 = -\pi/6$, $\cos\mu = -\Delta$,

$$Z(M,N,z,\Delta) \underset{M,N\to\infty}{\sim} e^{-\alpha(\Delta)MN} x^{-1/6}\left(\sum_{m=-\infty}^{\infty} z^m x^{[(\pi-\mu)/\pi]m^2}\right)\left(\sum_{n=-\infty}^{\infty} x^{[\pi/(\pi-\mu)]n^2}\right)\prod_{k=1}^{\infty}\frac{1}{(1-x^{2k})^2} . \tag{4.12}$$

Note that we could have inferred the modification of the second factor given the modification of the other two from duality.

It is amusing to compare (4.12) to the partition function for the coordinate degrees of freedom of the $(D-2)$-dimensional closed string[3]:

$$Z_{\text{closed string}} = x^{-(D-2)/6}\left(\int d\vec{p}_\perp x^{\vec{p}_\perp^{\,2}/2\pi T_0}\right) V \times \prod_{k=1}^{\infty}\frac{1}{(1-x^{2k})^{2(D-2)}} . \tag{4.13}$$

The internal degrees of freedom behave like one coordinate degree of freedom except that the momentum is discretized,

$$\frac{\vec{p}_0^{\,2}}{2\pi T_0} \leftrightarrow \frac{\pi-\mu}{\pi} m^2 ,$$

and there is an extra discrete zero mode. In Sec. V we shall discuss the properties of the resulting dual model and the interpretation of the second zero mode.

The duality we have exploited here is similar but not identical to duality used in other statistical-mechanics problems.[20] We have seen that duality gives information about shape dependence, i.e., finite-size effects. In most other problems the infinite-size limit is taken and these effects are neglected. The fact that there are $1/M$ excitations is associated with the infinite-volume system being critical. This is because a correlation between edges in one dimension goes like

$$O(e^{-N/M})$$

so the correlation length

$$\xi \sim M \to \infty$$

for the infinite system.

As another application of duality, let us calculate the open-string partition function including the gap term from our knowledge of the complete closed-string propagator.[3] We observe that the open-string partition function is just the closed-string propagator evaluated between states of total momentum density equal to zero:

$$Z^{\text{open}}(N+1,M) = \int d\vec{q}_m' \, d\vec{q}_m \langle \vec{q}_m', N+1 | \vec{q}_m, 0\rangle$$

$$\underset{M,N\to\infty}{\sim} \frac{V}{\sqrt{M}} 2^{M/4} e^{-(N+1)/2\left[\frac{M}{2}\int_0^1 dx \ln 2(1+\sin\pi x) - \frac{\pi}{3M}\right]} \prod_{m=1}^{\infty} (1-e^{-4(N+1)m\pi/M})^{-1} ,$$

but now $(N+1)/M = P^+/TT_0$ and to extract the spectrum we must perform a Jacobi imaginary transformation. The result for all the $(N+1)/M$ dependence is the expected open-string partition function,[3] and the gap and bulk terms are trivial:

$$Z^{\text{open}}(N+1,M) \sim V\left(\frac{a}{T}\right)^{1/2} \exp\left[-\frac{P^+T}{4a^2T_0}\int_0^1 dx \ln 2(1+\sin\pi x)\right] \times \exp\left(\frac{\pi T T_0}{24P^+} + \frac{T}{4a}\ln 2\right)\prod_{m=1}^{\infty}\frac{1}{1-e^{-\pi T T_0 m/P^+}} . \tag{4.14}$$

Now including the coupling-constant factors $(-\lambda/16\pi^2)(T_0/2\pi)^{d/2-1}$ at each vertex, we obtain

$$P_G^{-\text{open}} - P_G^{-\text{closed}} = \frac{(D-2)\pi T_0}{6P^+} - \frac{(D-2)\pi T_0}{24P^+} + \frac{1}{2a}\left[\ln\frac{-\lambda(T_0/2\pi)^{d/2-1}}{16\pi^2} - \frac{D-2}{2}\ln 2\right], \tag{4.15}$$

which is finite only if

$$\lambda = \lambda_{\text{critical}} \equiv -32\pi^2\left(\frac{4\pi}{T_0}\right)^{d/2-1}. \tag{4.16}$$

The effective expansion parameter in the strong-coupling expansion is

$$\lambda_0 \equiv \frac{-\lambda}{16\pi^2}\left(\frac{T_0}{2\pi}\right)^{d/2-1} \xrightarrow[\lambda\to\lambda_{\text{critical}}]{} 2^{d/2} = 2^{12}$$

for $d = d_{\text{critical}}$.

V. BOSE-FERMI EQUIVALENCE AND THE DUAL RESONANCE MODELS CORRESPONDING TO F MODELS

The form of the partition function (4.12) for the F model makes transparent the fact that the low-lying excitations of the transfer matrix are precisely those of a set of boson harmonic oscillators, in spite of the underlying fermionic character of the degrees of freedom. This boson-fermion equivalence is much deeper than identical spectra and has been recognized in several contexts. It has been known for a long time that the massless Thirring model is equivalent to a noninteracting scalar field theory.[16,21] This sort of equivalence has also been exploited by workers in statistical mechanics in order to solve the critical behavior of one-dimensional quantum spin systems.[22,23]

The Hamiltonian for the Heisenberg spin system

$$H_{\text{Heis}} = -\tfrac{1}{2}\sum_i (\vec{\sigma}_i^{\perp}\cdot\sigma_{i+1}^{\perp} + \Delta\sigma_i^z\sigma_{i+1}^z), \tag{5.1}$$

after the Jordon-Wigner transformation

$$\sigma_i^+ \equiv \tfrac{1}{2}(\sigma_i^x + i\sigma_i^y) = b_i^\dagger \exp\left(i\pi\sum_{j=1}^{i-1} b_j^\dagger b_j\right),$$

$$\sigma_i^z = 2b_i^\dagger b_i - \tfrac{1}{2}$$

and the Fourier transform

$$b_k = \frac{1}{\sqrt{M}}\sum_j e^{ikj} b_j,$$

can be rewritten as[22]

$$H_{\text{Heis}} = -2\sum_k \cos k\left[b_k^\dagger b_k + \frac{\Delta}{M}\rho(k)\rho(-k)\right], \tag{5.2}$$

where

$$\rho(k) = \frac{1}{M}\sum_j b_j^\dagger b_j e^{ikj}.$$

The values of the k's will depend on boundary conditions but will be spaced by $2\pi/M$. The ground state has $M/2$ overturned spins of momenta k_i, $i = 1, 2, \ldots, M/2$ distributed symmetrically about zero:

$$-\frac{\pi}{2} \cong k_1 < k_2 < \cdots < k_{M/2} \approx \frac{\pi}{2}.$$

Luther and Peschel[22] explain how to obtain an effective continuum Hamiltonian which reproduces the dynamics of (5.2) for $k \approx \pm k_F = \pm\pi/2$ and also for $k \approx 2k_F \approx \pi$. The contributions of k far away from these points should only affect microscopic physics and be lumped into renormalization of bulk parameters. This continuum Hamiltonian can be rewritten completely as a quadratic form in boson operators which are themselves *local* bilinears in the Fermi variables. In this form the Hamiltonian can be diagonalized by a canonical transformation.

This whole discussion ignores boundary conditions which would determine the detailed spacings of the low-lying states. But we have this information by direct calculation. What we learn from Luther and Peschel and other workers is that the macroscopic continuum properties of the Heisenberg model wave functions are accurately given by a local free quantum scalar field defined in the τ, σ parameter space of the string. Knowing this, we can guess with complete confidence the dual model which has the internal degrees of freedom of the F model.

In addition to the $(D-2)$ transverse coordinates $x^i(\sigma,\tau)$ of the string, we introduce another field $\phi(\sigma,\tau)$ whose excitations will account for the internal degrees of freedom. ϕ is essentially the total fermion number density. Since the boson modes are noninteracting massless waves we take for an action

$$iW = -\frac{T_0}{2}\int_{\tau_1}^{\tau_2} d\tau \int_0^{P^+/T_0} d\sigma(\dot{\phi}^2 + \phi'^2). \tag{5.3}$$

To obtain the correct ground-state energy in each charge sector, we must have

$$\int_0^{P^+/T_0} d\sigma\, T_0\dot{\phi} = k[2(\pi-\mu)T_0]^{1/2}, \quad k = 0, \pm1, \pm2, \ldots. \tag{5.4}$$

This is enforced by requiring the wave functional to be unchanged under

$$\phi(\sigma,\tau) \to \phi(\sigma,\tau) + \frac{2\pi}{[2(\pi-\mu)T_0]^{1/2}}.$$

For the closed string we require

$$\phi\left(\frac{P^+}{T_0}, T\right) = \phi(0,\tau) + \frac{2\pi l}{[2(\pi-\mu)T_0]^{1/2}} \tag{5.5}$$

since we demand that $\sigma = P^+/T_0$ be physically equivalent to $\sigma=0$. l is a topological quantum number which counts the number of times ϕ winds around its space as σ goes around the string. When we calculate the propagator of the closed string, we must sum over all values of l. This we do by writing[24]

$$\phi_l(\sigma,\tau)=\tilde{\phi}(\sigma,\tau)+\frac{2\pi l}{[2(\pi-\mu)T_0]^{1/2}}\frac{\sigma T_0}{P^+}$$

with $\tilde{\phi}$ periodic, and changing variables from ϕ to $\tilde{\phi}$:

$$\mathfrak{D}^{\text{closed strings}}=\sum_l \int d\phi_l \exp\left[-\frac{T_0}{2}\int_0^T d\tau\, d\sigma(\dot{\phi}^2+\phi'^2)\right]$$
$$=\left\{\sum_{l=-\infty}^{\infty}\exp\left[-\frac{2\pi^2 T_0}{(\pi-\mu)2P^+}l^2 T\right]\right\}\int d\tilde{\phi}\exp\left[-\frac{T_0}{2}\int_0^T d\tau\, d\sigma(\dot{\tilde{\phi}}^2+\tilde{\phi}'^2)\right]. \qquad (5.6)$$

The functional integral over $\tilde{\phi}$ gives the same result as a transverse coordinate except that the "momentum" is discretized according to (5.4).

The left running waves of ϕ can be identified with the density of left running fermions, and similarly for the right running waves. The two conserved quantum numbers of the closed string are linear combinations of the two fermion numbers of each type of wave: The "momentum" (5.4) is the sum, the topological quantum number l is half the difference of the two fermion numbers, which we identify as chirality. For an open string there is no topological quantum number: The boundary conditions $\phi'=0$ rule out a linear term in σ. In the limit that the total charge becomes continuous ($\mu\to\pi$), all the states of nonzero topological quantum number get infinite energy and play no role.

We close this section with a description of the dual model which has the F model as an internal-symmetry dynamics. At the end of the discussion we shall specialize our results to the $\lambda(\phi^\dagger\phi)^2$ case, i.e., when the F model is the ice model.

We represent the excited states of the closed string on a Fock space of $D-2$ "coordinate" oscillators $\vec{a}_k \vec{a}'_k$, $k=\pm1,\pm2,\ldots$ with $\vec{a}_k^\dagger=\vec{a}_{-k}$. The unprimed variables correspond to right running waves, the primed to left running waves. For the internal-symmetry degrees of freedom we introduce a similar set b_k, b'_k. These oscillations have the commutation relations

$$[a_k^i, a_l^j]=k\delta_{k,-l}\delta^{ij}, \qquad [b_k,b_l]=k\delta_{k,-l}.$$

The ground state in each sector is labeled by continuous momenta $\vec{p}_0$, the total charge k, and the topological quantum number l. The Hamiltonian is then

$$P^-_{\text{closed}}=\frac{1}{2P^+}\left[\vec{p}_0^{\,2}+2(\pi-\mu)T_0k^2+\frac{2\pi^2T_0}{\pi-\mu}l^2-\frac{(D-1)2\pi T_0}{6}+4\pi T_0\sum_{k'=1}^{\infty}(\vec{a}_{-k'}\cdot\vec{a}_{k'}+\vec{a}'_{-k'}\cdot\vec{a}'_{k'}+b_{-k'}b_{k'}+b'_{-k'}b'_{k'})\right], \qquad (5.7)$$

where $k,l=0,\pm1,\pm2,\ldots$.

We have calculated the low-lying excitations of the F model corresponding to open-string boundary conditions for the special case $\Delta=0$. We quote the partition function for even and odd M in both their Fermi and Bose forms ($x=e^{-\pi N/M}$ and we delete bulk terms):

$$M \text{ even: } Z(M,N)=x^{\alpha_0}\prod_{k=1}^{\infty}(1+x^{k-1/2})^2 = x^{\alpha_0}\sum_{m=-\infty}^{\infty}x^{m^2/2}\prod_{k=1}^{\infty}\frac{1}{1-x^k},$$
$$M \text{ odd: } Z(M,N)=2x^{\beta_0}\prod_{k=1}^{\infty}(1+x^k)^2 = x^{\beta_0-1/8}\sum_{m=-\infty}^{\infty}x^{(1/2)(m+1/2)^2}\prod_{k=1}^{\infty}\frac{1}{1-x^k} \qquad (5.8)$$

In (5.8) α_0 and β_0 can be inferred by considering that interchange of the roles of N, M shows that either of these expressions is also the propagator of a closed-string evaluated between initial and final states of zero "momentum" (i.e., charge) density. This requires

$$\alpha_0=\beta_0-\tfrac{1}{8}=-\tfrac{1}{24}. \qquad (5.9)$$

The Bose description for the free end boundary condition will still be given by the action (5.3) where ϕ at $\sigma=0$, P^+/T_0 are varied independently. To get the correct quantization of charge we allow both periodic and antiperiodic wave functions under $\phi\to\phi+2\pi/[2T_0(\pi-\mu)]^{1/2}$:

$$\Psi\left(\phi(\sigma)+\frac{2\pi}{[2T_0(\pi-\mu)]^{1/2}}\right)=\pm\Psi(\phi(\sigma)), \qquad (5.10)$$

where the (+) corresponds to the "even" sector and the (−) corresponds to the "odd" sector. As in (5.7), we can summarize all excited states of the open string by writing

$$P^-_{\text{open}} = \frac{1}{2P^+}\left[\vec{p}_0^{\,2} + 2(\pi-\mu)T_0 k^2 - \frac{(D-1)2\pi T_0}{24} + 2\pi T_0 \sum_{k'=1}^{\infty}(\vec{a}_{-k'}\cdot\vec{a}_{k'} + b_{-k'}b_{k'})\right], \quad (5.11)$$

$k=0,\pm\frac{1}{2},\pm 1,\pm\frac{3}{2},\ldots$, where we have inferred the $\Delta \neq 0$ result from duality. We remind the reader that $\mu = 2\pi/3$ for field theory. Other values of μ give a consistent *planar* dual model, but do not correspond to a field theory. We shall describe these dual models more fully elsewhere.

ACKNOWLEDGMENTS

We would like to thank J. Goldstone, K. Johnson, and F. Low for helpful conversations. We are especially grateful to T. Chang, L. P. Kadanoff, and F. Y. Wu, who helped us learn about critical phenomena in general and the vertex models in particular. This work was supported in part through funds provided by the DOE under Contract No. EY-76-C-02-3069.*000. The work of C.B.T. was also supported in part by the Alfred P. Sloan Foundation.

APPENDIX: THE SPECTRUM OF LOW-LYING STATES IN THE F MODEL

In this appendix, we shall determine the spectrum of low-lying states in the F model for $0<|\Delta|<1$. The solution of the F model in the thermodynamic limit ($M\to\infty$; Q/M, P fixed) is much discussed in the solid-state literature.[9] In this limit, the energy density can be computed as a function of charge density and momentum by converting sums over discrete Bethe wave numbers, k_j, into integrals over a continuous spectrum of k's.

We want to compute the $1/M$ corrections to the ground-state energy and the excitation spectrum of states whose charges are fixed and whose momenta are within $O(1/M)$ of a multiple of π. We adapt the integral equation technique accordingly. The resulting expression for the energy of low-lying states is seen to be the leading term in an asymptotic series in $1/M$. Our analytic results have been checked by a computer analysis.

The equation for the momenta is

$$k_j = \frac{2\pi}{M}I_j - \frac{1}{M}\sum_{l=1}^{m}\theta(k_j,k_l). \quad (A1)$$

As discussed in Sec. III, different choices of I_j characterize the distinct excitations of the lattice.

Before proceeding to the $M\to\infty$, we first introduce the variables μ and α by

$$\Delta = -\cos(\mu), \quad (A2)$$

$$e^{ik} = \frac{e^{i\mu}-e^{\alpha}}{e^{i\mu+\alpha}-1}. \quad (A3)$$

For $-1<\Delta<1$, we have $0<\mu<\pi$. The range of momenta $-(\pi-\mu)<k<\pi-\mu$ is mapped onto $-\infty<\alpha<\infty$. Yang and Yang[14] have shown that this range of momenta corresponds to the range allowed by the lowest states with $Q\geq 0$. The following identities are consequences of Eqs. (A2)–(A3):

$$k(\alpha) = -k(-\alpha), \quad (A4)$$

$$\cos k = -\cos\mu + \frac{\sin^2\mu}{\cosh\alpha - \cos\mu}, \quad (A5)$$

$$\sin k = \frac{\sin\mu \sinh\alpha}{\cosh\alpha - \cos\mu}, \quad (A6)$$

$$\frac{dk}{d\alpha} = \frac{\sin\mu}{\cosh\alpha-\cos\mu}, \quad (A7)$$

$$\theta(k,k') = 2\tan^{-1}\left(\cot\mu \tanh\frac{\alpha'-\alpha}{2}\right) \equiv \theta(\alpha,\alpha'), \quad (A8)$$

$$K(\alpha,\beta) \equiv \frac{1}{2\pi}\frac{\partial}{\partial\beta}\theta(\alpha,\beta) = \frac{1}{2\pi}\frac{\sin 2\mu}{\cosh(\alpha-\beta)-\cos 2\mu}. \quad (A9)$$

Expressed in these variables,

$$k(\alpha_j) = \frac{2\pi}{M}I_j - \frac{1}{M}\sum_{l=1}^{m}\theta(\alpha_j-\alpha_l) \quad (A10)$$

and the energy is

$$E = -\ln\left[\prod_{l=1}^{m}\frac{(e^{-2i\mu}-e^{\alpha})e^{i\mu}}{e^{\alpha}-1} + (\alpha\to-\alpha)\right]. \quad (A11)$$

In the thermodynamic limit, we replace the discrete sum in Eq. (A10) by a continuous integral. For charge and chiral excitations, where the I_j are uniformly distributed in a closed interval, we may convert this equation into a linear integral equation. Defining

$$R(\alpha) \equiv \frac{2\pi}{M}\frac{dj}{d\alpha} \quad (A12)$$

we differentiate Eq. (A10) with respect to α to obtain

$$\frac{dk}{d\alpha} = R(\alpha) + \int_{\alpha_-}^{\alpha_+} d\beta\, K(\alpha-\beta)R(\beta). \quad (A13)$$

The end points of integration, α_+ and α_-, are determined by the charge density

$$\frac{1}{2}\left(1-\frac{Q}{M}\right)=\int_{\alpha_-}^{\alpha_+}\frac{d\alpha}{2\pi}R(\alpha) \tag{A14}$$

and the momentum density

$$\frac{P}{M}=\int_{\alpha_-}^{\alpha_+}\frac{d\alpha}{2\pi}k(\alpha)R(\alpha). \tag{A15}$$

Corrections to the replacement of sums by integrals in (A10)–(A13) can be obtained from the Euler-Maclaurin formula. These terms generate the $-\pi/6M$ correction to the ground-state energy, but do not affect the excitation energies of low-lying states to order $1/M$. We do not discuss the derivation of $-\pi/6$ in detail here. Indeed, the $-\pi/6M$ can be inferred from the spectrum of low-lying excitations and duality as discussed in Sec. IV.

For $\alpha_+\to\infty$, $\alpha_-\to-\infty$, Eq. (A13) may be solved by Fourier transformation. The result is

$$R_0(\alpha)=\frac{\pi}{2\mu\cosh\pi\alpha/2\mu} \tag{A16}$$

and

$$P=Q=0. \tag{A17}$$

This choice of $\alpha_\pm$ corresponds, therefore, to the ground state. The bulk energy E_B may be determined from

$$\frac{E_0}{M}=\int_{-\infty}^{\infty}\frac{d\alpha}{2\pi}\ln\frac{2(e^{-2i\mu}-e^{\alpha})e^{i\mu}}{e^{\alpha}-1}. \tag{A18}$$

To evaluate the charge and chiral excitation energies, we follow Yang and Yang,[14] and transform Eq. (A13) into an equation for $R(\alpha)$ for α outside the interval $[\alpha_+,\alpha_-]$. With the resolvant operator

$$J=-\frac{1}{1+K}K, \tag{A19}$$

we may use the identity

$$R_0=\frac{1}{1+K}\frac{dk}{d\alpha} \tag{A20}$$

to transform Eq. (A13) into

$$R_0(\alpha)=R(\alpha)+\int_{-\infty}^{\alpha_-}d\beta\, J(\alpha-\beta)R(\beta) + \int_{\alpha_+}^{\infty}d\beta\, J(\alpha-\beta)R(\beta). \tag{A21}$$

This equation may be used to obtain expressions for the free energy, charge, and momenta as integrals over $R(\alpha)$ for α outside the interval $[\alpha_+,\alpha_-]$. The techniques used in this transformation are discussed by Yang and Yang.[14] For $|\alpha_\pm|\gg 1$, the results are

$$\frac{E-E_0}{M}=\left(\int_{-\infty}^{\alpha_-}+\int_{\alpha_+}^{\infty}\right)\frac{d\alpha}{2\pi}\frac{R(\alpha)}{\cosh\pi\alpha/2\mu}, \tag{A22}$$

$$\frac{Q}{M}=\frac{\pi}{\pi-\mu}\left(\int_{-\infty}^{\alpha_-}+\int_{\alpha_+}^{\infty}\right)\frac{d\alpha}{2\pi}R(\alpha), \tag{A23}$$

and

$$\frac{P}{M}=-\frac{\pi}{2}\left(\int_{\alpha_+}^{\infty}-\int_{-\infty}^{\alpha_-}\right)\frac{d\alpha}{2\pi}R(\alpha). \tag{A24}$$

The charge excitations arise when $\alpha_+=-\alpha_-=\alpha_0$. For these excitations, $R(\alpha)=R(-\alpha)$, and

$$\frac{E-E_0}{M}=\frac{2}{\pi}\int_{\alpha_0}^{\infty}d\alpha\, e^{-\pi\alpha/2\mu}R(\alpha), \tag{A25}$$

$$\frac{Q}{M}=\frac{1}{\pi-\mu}\int_{\alpha_0}^{\infty}d\alpha\, R(\alpha), \tag{A26}$$

$$\frac{P}{M}=0 \tag{A27}$$

for $\alpha>\alpha_0\gg 1$, Eq. (A21) simplifies to

$$R_0(\alpha)=R(\alpha)+\int_{\alpha_0}^{\infty}d\beta\, J(\alpha-\beta)R(\beta). \tag{A28}$$

The solution to this equation is accomplished by Laplace transforming, and using the Wiener-Hopf factorization.[14] These techniques yield

$$\frac{E-E_0}{M}=\frac{\pi-\mu}{4}\frac{Q^2}{M^2}. \tag{A29}$$

To determine the energies of the lowest states in nonzero chirality sectors we consider $\alpha_+\neq-\alpha_-$, but both large. In this limit, it is not hard to verify that the end-point contributions to the energy are such that it is of the form

$$C_1Q^2+C_2P^2.$$

Since we already know C_1 from (A29), we may evaluate C_2 in the limit $\alpha_-\to-\infty$. In this case Eq. (A29) for $R(x)$ applies. The energy is precisely $\frac{1}{2}$ that of the case $\alpha_+=-\alpha_-=\alpha_0$. Using

$$P=-\frac{2}{\pi-\mu}Q, \tag{A30}$$

we find

$$\frac{E-E_0}{M}=\frac{\pi-\mu}{4}\frac{Q^2}{M^2}+\frac{1}{\pi-\mu}\frac{P^2}{M^2}. \tag{A31}$$

Finally, we determine the particle-hole excitation energies.[15] We shall consider the excitation energy in the charge zero, chirality zero sector. Since the evaluation of charge, chiral, and particle-hole excitations involves first-order perturbations of the ground-state wave function, the result derived here also applies to the nonzero charge and chirality sectors.

The particle-hole excitation we consider arises from choosing the set I_j as

$$I_j = \frac{j-1-m}{2}\,, \quad 1 \leq j \leq j_0 \tag{A32}$$

$$I_j = \frac{j-m}{2}\,, \quad j_0+1 \leq j \leq m+1\,. \tag{A33}$$

We evaluate the momenta $k(\alpha)$ for this choice of I_j by expanding around the $k(\alpha)$'s appropriate to the $Q=P=0$ ground state. With

$$\chi(\alpha) = M\delta k(\alpha) R(\alpha)\frac{d\alpha}{dk}\,, \tag{A34}$$

Eq. (A10) is

$$\chi(\alpha) + \int_{-\infty}^{\infty} d\beta\, K(\alpha-\beta)\chi(\beta) = 2\pi\theta(\alpha-\alpha_0)\,, \tag{A35}$$

where $\theta(x)$ is the step function. The parameter α_0 is determined from

$$P = \int_{-\infty}^{\infty} \frac{d\alpha}{2\pi}\chi(\alpha)\frac{dk}{d\alpha}\,. \tag{A36}$$

The change in the free energy is

$$\frac{E-E_0}{M} = -\frac{1}{2}\,\frac{\sin^2\mu}{\cos\mu+1} \times \int \frac{d\alpha}{2\pi}\chi(\alpha)\frac{\sinh\alpha}{\cosh\alpha-\cos\mu} \times \left(\frac{1}{\cosh\alpha-1} + \frac{2\cos\mu+1}{\cosh\alpha-\cos2\mu}\right). \tag{A37}$$

The singularity at $\alpha=0$ is treated by a principal-value prescription which arises from a careful treatment of Eq. (A11).

Equation (A36) is easily evaluated by Fourier transformation. The resulting $\chi(\alpha)$ is inserted into Eqs. (A36) and (A37) with the result

$$\frac{E-E_0}{M} = \frac{|P|}{M}\,. \tag{A38}$$

[1]C. B. Thorn, Phys. Lett. 70B, 85 (1977).
[2]For a collection of excellent reviews of dual string theory see, for example *Dual Theory*, edited by M. Jacob (North-Holland, Amsterdam, 1974).
[3]C. B. Thorn, Phys. Rev. D 17, 1073 (1978).
[4]L. Pauling, J. Am. Chem. Soc. 57, 2680 (1935).
[5]E. H. Lieb, Phys. Rev. Lett. 18, 692 (1967); Phys. Rev. 162, 162 (1967).
[6]F. Rys, Helv. Phys. Acta 36, 537 (1963).
[7]F. Bloch, Z. Phys. 61, 206 (1930).
[8]R. Giles and C. B. Thorn, Phys. Rev. D 16, 366 (1977).
[9]For an excellent review of these vertex models, see E. H. Lieb and F. Y. Wu, *Phase Transitions and Critical Phenomena*, edited by C. Domb and M. S. Green (Academic, New York, 1972), p. 331 and references contained therein.
[10]R. B. Potts, Proc. Camb. Philos. Soc. 48, 106 (1952).
[11]E. H. Lieb, Phys. Rev. Lett. 18, 1046 (1967); B. McCoy and T. T. Wu, Nuovo Cimento 56B, 311 (1968); B. Sutherland, J. Math. Phys. 11, 3183 (1970).
[12]J. D. Johnson, S. Krinsky, and B. McCoy, Phys. Rev. A 5, 2526 (1973).
[13]H. A. Bethe, Z. Phys. 61, 206 (1930).
[14]C. N. Yang and C. P. Yang, Phys. Rev. 150, 321 (1966); 150, 327 (1966).
[15]J. des Cloizeaux and M. Gaudin, J. Math. Phys. 7, 1389 (1966); F. Y. Wu (private communication).
[16]Some references on the massless Thirring model are W. Thirring, Ann. Phys. (N.Y.) 3, 91 (1958); V. Glaser, Nuovo Cimento 9, 990 (1958); K. Johnson, *ibid.* 20, 773 (1961); C. Sommerfield, Ann. Phys. (N.Y.) 26, 1 (1968); G. Dell'Antonio, Y. Frishman, and D. Zwanziger, Phys. Rev. D 6, 988 (1972).
[17]This symmetry is a lattice version of duality as discussed by L. Brink and H. B. Nielsen, Phys. Lett. 43B, 319 (1973); Nucl. Phys. B89, 18 (1975); W. Nahm, Nucl. Phys. B81, 164 (1974); B114, 174 (1976); B124, 121 (1977); B120, 125 (1977).
[18]*Higher Transcendental Functions* (Bateman Manuscript Project), edited by A. Erdélyi (McGraw-Hill, New York, 1953), Eq. 17.22.
[19]Reference 18, Eq. 13.22.
[20]H. N. V. Temperly and E. H. Lieb, Proc. R. Soc. London A322, 251 (1971); C. Fan, Phys. Lett. 39A, 136 (1972); F. J. Wegener, Physica 68, 570 (1973); F. Y. Wu and K. Y. Lin, J. Phys. C 7, 181 (1974); F. Y. Wu and Y. K. Wang, J. Math. Phys. 17, 439 (1976).
[21]S. Coleman, Phys. Rev. D 11, 2088 (1975); S. Mandelstam, *ibid.* 11, 3026 (1975).
[22]A. Luther and I. Peshel, Phys. Rev. B 12, 3908 (1975).
[23]A. Luther and V. J. Emery, Phys. Rev. Lett. 33, 589 (1974); D. C. Mattis, J. Math. Phys. 15, 609 (1974); A. Luther and I. Peshel, Phys. Rev. B 9, 2911 (1974); B. Schroer, lecture notes, 1975 (unpublished); A. Luther, Phys. Rev. B 14, 2153 (1976); I. Goldhirsh, E. Levich, and V. Yakhot, Weizmann Institute Report No. WIS77/22-Ph (unpublished).
[24]We thank J. Goldstone for helpful discussions on the quantum-mechanical description of topological quantum numbers.