Splitting in energy and splitting in angular momentum of the classical field of a radiating point charge

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A splitting of Maxwell's energy tensor of the retarded Liénard-Wiechert field into three dynamically independent parts is considered. Two of them are identified as representing the emitted and bound energy-momenta. The third one is a sourceless tensor, built from a part of the interference field, which gives no contribution whatsoever to the total emitted or bound energies. This splitting induces a similar decomposition of the angular momentum tensor into three parts which are also independently conserved outside the world line of the particle. One of them corresponds to the bound angular momentum, and the other two describe two kinds of radiated angular momenta. The part associated with the sourceless energy tensor gives rise to the radiation of intrinsic angular momentum. The remaining part accounts for the angular momentum carried by the emitted energy.

I. INTRODUCTION

The physical picture underlying the energy-momentum content of the retarded Liénard-Wiechert field of a point charge is clearly understood due to a splitting, introduced by Teitelboim,[1] of Maxwell's tensor into two dynamically independent parts,

$$T_{\mu\nu}=T^{(r)}_{\mu\nu}+T^{(b)}_{\mu\nu}, \tag{1.1}$$

which are conserved outside the world line of the particle,

$$\partial^\nu T^{(r)}_{\mu\nu}=0,\quad \partial^\nu T^{(b)}_{\mu\nu}=0. \tag{1.2}$$

Here $T^{(r)}_{\mu\nu}$ accounts for the emitted energy-momentum, and $T^{(b)}_{\mu\nu}$ describes the energy-momentum that remains bound to the particle.

Later, a similar decomposition was found[2,3] for the angular momentum density tensor,

$$M_{\lambda\mu\nu}=M^{(r)}_{\lambda\mu\nu}+M^{(b)}_{\lambda\mu\nu}. \tag{1.3}$$

The emitted ($M^{(r)}_{\lambda\mu\nu}$) and bound ($M^{(b)}_{\lambda\mu\nu}$) parts, which are also conserved off the world line, possess similar properties to the corresponding pieces of Maxwell's tensor. However, the separate parts in both splittings are not connected by the same formula relating the whole tensors, namely,

$$M_{\lambda\mu\nu}=2x_{[\lambda}T_{\mu]\nu}. \tag{1.4}$$

It so happens that the bound energy-momentum density tensor $T^{(b)}_{\mu\nu}$ contributes both to $M^{(b)}_{\lambda\mu\nu}$ and to $M^{(r)}_{\lambda\mu\nu}$. On the other hand, $T^{(r)}_{\mu\nu}$ gives rise only to a part of $M^{(r)}_{\lambda\mu\nu}$.

This intertwining among the energy and angular momentum contents of the bound and emitted fields is rather obscure and deserves to be better understood. It is precisely the object of the present paper to show that, in fact, they may be completely disentangled. Towards this purpose, we perform a new decomposition of the energy and angular momentum density tensors into three dynamically independent parts:

$$T_{\mu\nu}=T^{*(b)}_{\mu\nu}+T^{(f)}_{\mu\nu}+T^{(r)}_{\mu\nu}, \tag{1.5a}$$

$$M_{\lambda\mu\nu}=M^{*(b)}_{\lambda\mu\nu}+M^{(ri)}_{\lambda\mu\nu}+M^{(re)}_{\lambda\mu\nu}. \tag{1.5b}$$

Each term in (1.5a) and (1.5b) is independently conserved outside the world line. Moreover, both decompositions are related by formulas analogous to (1.4), i.e.,

$$M^{*(b)}_{\lambda\mu\nu}=2x_{[\lambda}T^{*(b)}_{\mu]\nu}, \tag{1.6a}$$

$$M^{(ri)}_{\lambda\mu\nu}=2x_{[\lambda}T^{(f)}_{\mu]\nu}, \tag{1.6b}$$

$$M^{(re)}_{\lambda\mu\nu}=2x_{[\lambda}T^{(r)}_{\mu]\nu}. \tag{1.6c}$$

The bound energy-momentum density tensor $T^{(b)}_{\mu\nu}$ has been split into a free (sourceless) tensor $T^{(f)}_{\mu\nu}$ and a second tensor $T^{*(b)}_{\mu\nu}$ which turns out to possess the same energy-momentum content as $T^{(b)}_{\mu\nu}$. Similarly, the associated bound angular momentum density tensor $M^{*(b)}_{\lambda\mu\nu}$, given by (1.6a), is physically equivalent to the old $M^{(b)}_{\lambda\mu\nu}$. On the other hand, the emitted energy tensor $T^{(r)}_{\mu\nu}$ gives rise to the external radiated angular momentum density $M^{(re)}_{\lambda\mu\nu}$, namely the tensor which contains the angular momentum carried by the emitted energy. The remaining intrinsic part of the radiated angular momentum density tensor $M^{(ri)}_{\lambda\mu\nu}$, as shown by (1.6b), arises from the free energy tensor $T^{(f)}_{\mu\nu}$. However, contrary to $T^{(f)}_{\mu\nu}$, the intrinsic term $M^{(ri)}_{\lambda\mu\nu}$ does have a source on the world line of the particle and gives

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a flux of angular momentum along the light cones. The reason for this peculiar behavior is the presence of x_λ in (1.6b), which contributes with an extra factor linear in the distance.

In Sec. II we discuss in detail the properties of the splitting (1.5a) of Maxwell's tensor, for the simple case of a point charge without internal structure. To accomplish this we make use of a description of the bound part in terms of a superpotential introduced by Van Weert. In Sec. III we make a similar analysis of the decomposition (1.5b) of the angular momentum tensor. In particular, we find the δ-function sources on the world line for the separate parts of the emitted angular momentum tensor. In Sec. IV we extend the present analysis to the general case of a point particle with an arbitrary multipolar electromagnetic structure.

II. THE FREE ENERGY-MOMENTUM TENSOR

Let us consider the simple case of a point charge without internal structure. As shown by Teitelboim,[1] the energy-momentum density tensor $T_{\mu\nu}$ of its Liénard-Wiechert field may be decomposed into two parts separately conserved outside the world line of the particle:

$$T^{(b)}_{\mu\nu} = T^{(-4)}_{\mu\nu} + T^{(-3)}_{\mu\nu}\,, \tag{2.1a}$$

$$T^{(r)}_{\mu\nu} = T^{(-2)}_{\mu\nu}\,. \tag{2.1b}$$

The notation $(-n)$ is introduced to indicate those terms in $T_{\mu\nu}$ which behave like the power $-n$ of the invariant distance κ. The bound part $T^{(b)}_{\mu\nu}$ has the interesting property, pointed out by Van Weert,[4] of being the divergence of a local "superpotential" $K_{\mu\nu\gamma}$:

$$T^{(b)}_{\mu\nu} = \partial^\gamma K_{\mu\nu\gamma}\,, \quad K_{\mu\nu\gamma} = K_{\mu\gamma\nu} \tag{2.2}$$

where $K_{\mu\nu\gamma}$ is given by

$$K_{\mu\nu\gamma} = K^{(-3)}_{\mu\nu\gamma} + K^{(-2)}_{\mu\nu\gamma}\,, \tag{2.3}$$

with

$$K^{(-3)}_{\mu\nu\gamma} = (4\pi)^{-1} e^2 \kappa^{-3}\left(-\tfrac{3}{2}\kappa^{-2} s_\mu v_{[\nu} s_{\gamma]} + \tfrac{1}{2}\kappa^{-1}\eta_{\mu[\nu} s_{\gamma]}\right), \tag{2.4a}$$

$$K^{(-2)}_{\mu\nu\gamma} = (2\pi)^{-1} e^2 \kappa^{-4} s_\mu (\kappa' v_{[\nu} s_{\lambda]} - \dot{v}_{[\nu} s_{\lambda]})\,. \tag{2.4b}$$

This splitting of the superpotential suggests a natural decomposition of the bound energy tensor[5]

$$T^{(b)}_{\mu\nu} = T^{*(b)}_{\mu\nu} + T^{(f)}_{\mu\nu}\,, \tag{2.5}$$

where

$$T^{*(b)}_{\mu\nu} = \partial^\gamma K^{(-3)}_{\mu\nu\gamma}\,, \tag{2.6a}$$

$$T^{(f)}_{\mu\nu} = \partial^\gamma K^{(-2)}_{\mu\nu\gamma}\,. \tag{2.6b}$$

By virtue of the antisymmetry of (2.4a) and (2.4b) in their last two indexes (ν, γ), these tensors are also conserved off the world line,

$$\partial^\nu T^{*(b)}_{\mu\nu} = 0, \quad \partial^\nu T^{(f)}_{\mu\nu} = 0. \tag{2.7}$$

The explicit expressions for both parts are readily obtained from (2.4) and (2.5):

$$T^{*(b)}_{\mu\nu} = T^{(-4)}_{\mu\nu} - (2\pi)^{-1} e^2 \kappa^{-6} \kappa' s_\mu s_\nu\,, \tag{2.8a}$$

$$T^{(f)}_{\mu\nu} = T^{(-3)}_{\mu\nu} + (2\pi)^{-1} e^2 \kappa^{-6} \kappa' s_\mu s_\nu\,. \tag{2.8b}$$

The sum of (2.8a) and (2.8b) correctly gives (2.1a). Note that both tensors are symmetric despite the fact that their superpotentials are not. We shall examine first the properties of the free energy tensor $T^{(f)}_{\mu\nu}$.

It is easily shown from (2.8b) that $T^{(f)}_{\mu\nu}$ satisfies the relation

$$T^{(f)}_{\mu\nu} s^\nu = 0, \tag{2.9}$$

i.e., it gives no flux through the light cones drawn from points on the world line of the charge. This relation allows us to find the source of this tensor. To this end, let us apply Gauss's theorem to $T^{(f)}_{\mu\nu}$ in the region depicted in Fig. 1. The integration domain is the volume in Minkowski space between two light cones with vertexes on the world line, separated by a proper-time interval $d\tau$, and cut

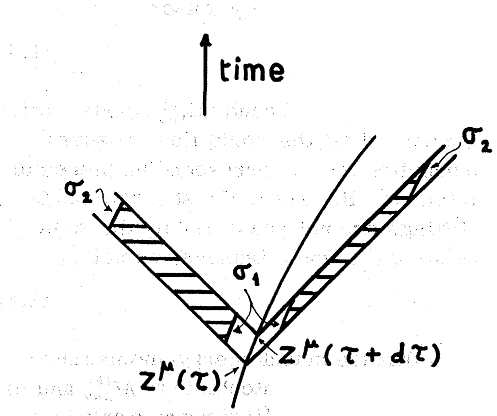

FIG. 1. Integration region considered in the evaluation of the emitted fluxes of the free Maxwell tensor $T^{(f)}_{\mu\nu}$ and the intrinsic radiated angular momentum tensor $M^{(ri)}_{\lambda\mu\nu}$. Gauss's integral theorem is applied to these tensors in the four-volume of Minkowski space enclosed by two light cones with vertexes at the point $z^\mu(\tau)$ and $z^\mu(\tau + d\tau)$, and two Bhabha tubes σ_1 and σ_2.

by two cylinders having a constant retarded radius κ_1 and κ_2 (Bhabha tubes). Taking into account relations (2.7) and (2.9), one obtains

$$T^{(f)}_{\mu\nu} d\sigma^\nu_1 = T^{(f)}_{\mu\nu} d\sigma^\nu_2 , \tag{2.10}$$

which shows that the flux of $T^{(f)}_{\mu\nu}$ is constant along the light cones. The surface element of these Bhabha tubes is

$$d\sigma^\nu = [\kappa v^\nu - (1-\kappa')s^\nu]\, d\Omega d\tau , \tag{2.11}$$

where $d\Omega$ is the element of solid angle in the instantaneous rest system of the charge. Hence (2.10) reduces to

$$T^{(f)}_{\mu\nu} d\sigma^\nu = \kappa^2 T^{(f)}_{\mu\nu} v^\nu d\Omega d\tau = O(\kappa^{-1}) , \tag{2.12}$$

which tends to zero in the limit $\kappa \to \infty$. Thus the constant flux (2.10) vanishes for all κ. This means that $T^{(f)}_{\mu\nu}$ has no source on the world line, so that

$$\partial^\nu T^{(f)}_{\mu\nu}(x) = 0 \quad \text{everywhere.} \tag{2.13}$$

It is a straightforward consequence of this result that the total energy-momentum content of $T^{(f)}_{\mu\nu}$ is zero. In fact, after integrating over the hyperplane $\sigma^0(\tau)$ orthogonal to the world line at the present position $z^\mu(\tau)$, we obtain

$$P^{(f)}_\mu(\tau) = \int_{\sigma^0(\tau)} T^{(f)}_{\mu\nu} d\sigma^\nu_0 \equiv 0, \tag{2.14}$$

since this integral may be decomposed into a series of vanishing contributions like (2.10) along the world line of the charge from the remote past up to the present proper time τ. From (2.14) we infer that the tensor

$$T^{*(b)}_{\mu\nu} = T^{(b)}_{\mu\nu} - T^{(f)}_{\mu\nu} \tag{2.15}$$

has the same energy-momentum content as $T^{(b)}_{\mu\nu}$, i.e.,

$$P^{(b)}_\mu(\tau) = \int_{\sigma_0(\tau)} T^{(b)}_{\mu\nu} d\sigma^\nu_0 = \int_{\sigma_0(\tau)} T^{*(b)}_{\mu\nu} d\sigma^\nu_0 . \tag{2.16}$$

Besides, as a consequence of (2.13), we know that both tensors possess the same source on the world line. However, as we show in the next section, only $T^{*(b)}_{\mu\nu}$ gives the correct angular momentum of the bound field.

III. THE THREE PIECES OF THE ANGULAR MOMENTUM TENSOR

As pointed out in the Introduction, the splitting (1.5a) of Maxwell's energy tensor gives rise to the related decomposition (1.5b) of the angular momentum tensor. We thus obtain three pieces independently conserved outside the world line,

$$\partial^\nu M^{(re)}_{\lambda\mu\nu} = 0, \quad \partial^\nu M^{(ri)}_{\lambda\mu\nu} = 0, \quad \partial^\nu M^{*(b)}_{\lambda\mu\nu} = 0. \tag{3.1}$$

These conservation laws are an immediate deduction from (1.6). We now study each part separately. Let us first compare the new definition (1.6a) of the bound angular momentum tensor with the previous one introduced in Ref. 2, namely

$$M^{(b)}_{\lambda\mu\nu} = 2z_{[\lambda} T^{(b)}_{\mu]\nu} + 2s_{[\lambda} T^{(-4)}_{\mu]\nu} . \tag{3.2}$$

For this purpose we insert (2.8a) into (1.6a) and arrive at

$$M^{*(b)}_{\lambda\mu\nu} = M^{(b)}_{\lambda\mu\nu} - 2z_{[\lambda} T^{(f)}_{\mu]\nu} . \tag{3.3}$$

It is a straightforward matter to verify that the second term in (3.3) is a divergenceless tensor giving no flux through the light cones starting on the world line. Moreover, it behaves like κ^{-3}, so that the same argument employed in the preceding section to the tensor $T^{(f)}_{\mu\nu}$ applies equally well here. Therefore $M^{*(b)}_{\lambda\mu\nu}$ has the same angular momentum content as $M^{(b)}_{\lambda\mu\nu}$. It also follows from (3.3), (1.3), and (1.5b) that

$$M^{*(r)}_{\lambda\mu\nu} \equiv M^{(ri)}_{\lambda\mu\nu} + M^{(re)}_{\lambda\mu\nu} = M^{(r)}_{\lambda\mu\nu} + 2z_{[\lambda} T^{(f)}_{\mu]\nu} , \tag{3.4}$$

i.e., the new radiation tensor also has the same content as the old one.

To understand the physical meaning of the separate parts of the emitted angular momentum (3.4), we consider more closely the intrinsic term. From (1.6b) and (2.9) one obtains

$$M^{(ri)}_{\lambda\mu\nu} s^\nu = 0. \tag{3.5}$$

Therefore $M^{(ri)}_{\lambda\mu\nu}$ does not give any flux through the light cones and is conserved outside the world line. Thus, via Gauss's theorem, we can look for the source of this tensor. Using the same domain of Fig. 1 we arrive at

$$M^{(ri)}_{\lambda\mu\nu} d\sigma^\nu_1 = M^{(ri)}_{\lambda\mu\nu} d\sigma^\nu_2 = \text{constant}. \tag{3.6}$$

The value of this constant flux along the light cones is

$$M^{(ri)}_{\lambda\mu\nu} d\sigma^\nu = \kappa^2 M^{(ri)}_{\lambda\mu\nu} v^\nu d\Omega d\tau . \tag{3.7}$$

To explicitly compute this expression we replace, in (1.6b), x_λ from the relation $s_\lambda = x_\lambda - z_\lambda(\tau)$, where $z_\lambda(\tau)$ is the retarded point on the world line corresponding to x_λ:

$$M^{(ri)}_{\lambda\mu\nu} = 2z_{[\lambda}T^{(f)}_{\mu]\nu} + 2s_{[\lambda}T^{(f)}_{\mu]\nu}. \qquad (3.8)$$

We have already seen that the first term here does not contribute because it behaves like κ^{-1}. On the other hand, the presence of s_λ in the second term gives the right κ^{-2} dependence. Referring to (2.8b), (3.7), and (3.8) we deduce that

$$M^{(ri)}_{\lambda\mu\nu}d\sigma^\nu = 2\kappa^2 s_{[\lambda}T^{(-3)}_{\mu]\nu}v^\nu d\Omega d\tau\,, \qquad (3.9)$$

or, introducing the explicit expression[2] for $T^{(-3)}_{\mu\nu}$,

$$M^{(ri)}_{\lambda\mu\nu}d\sigma^\nu = (4\pi)^{-1}e^2(-\kappa'\kappa^{-2}2s_{[\lambda}v_{\mu]} + \kappa^{-1}2s_{[\lambda}\dot{v}_{\mu]})d\Omega d\tau. \qquad (3.10)$$

The integration over $d\Omega$ is straightforward (see Ref. 2) giving

$$M^{(ri)}_{\lambda\mu\nu}d\sigma^\nu \equiv \mathring{M}^{(ri)}_{\lambda\mu}(\tau)d\tau = \tfrac{4}{3}e^2 v_{[\lambda}\dot{v}_{\mu]}d\tau\,. \qquad (3.11)$$

Here $\mathring{M}^{(ri)}_{\lambda\mu}(\tau)$ is the intrinsic angular momentum radiation rate. This result leads to the δ-function source of $M^{(ri)}_{\lambda\mu\nu}$ on the world line. Following the standard procedure,[1,2] we apply Gauss's theorem to the intrinsic angular momentum tensor in the region of Minkowski space inside a small Bhabha tube surrounding the world line, and cut by two light cones drawn from the points $z^\mu(\tau)$ and $z^\mu(\tau+d\tau)$ (see Fig. 2). Thus we have

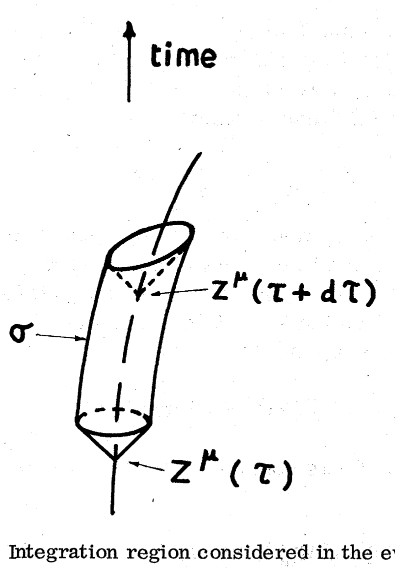

FIG. 2. Integration region considered in the evaluation of the δ-function source of the intrinsic radiated angular momentum tensor $M^{(ri)}_{\lambda\mu\nu}$. Gauss's integral theorem is applied to this tensor in the four-volume of Minkowski space inside a Bhabha tube σ surrounding a segment of the world line and cut by two light cones drawn from the points $z^\mu(\tau)$ and $z^\mu(\tau+d\tau)$.

$$\int_{\substack{\text{four-volume}\\ \text{inside tube}}} \partial^\nu M^{(ri)}_{\lambda\mu\nu}d^4x = \int_{\substack{\text{surface}\\ \text{of tube}}} M^{(ri)}_{\lambda\mu\nu}d\sigma^\nu. \qquad (3.12)$$

The flux on the right-hand side is obtained from (3.11), recalling that there is no flux through the light cones in both extremes. We thus obtain

$$\partial^\nu M^{(ri)}_{\lambda\mu\nu}d^3x dx^0 = \tfrac{4}{3}e^2 v_{[\lambda}\dot{v}_{\mu]}d\tau\,, \qquad (3.13)$$

whence[1,2]

$$\partial^\nu M^{(ri)}_{\lambda\mu\nu}(x) = \int_{-\infty}^{\infty} d\tau\,\delta^{(4)}(x - z(\tau))\tfrac{4}{3}e^2 v_{[\lambda}\dot{v}_{\mu]}. \qquad (3.14)$$

This expression shows that the intrinsic angular momentum tensor has an independent source on the world line of the charge; the intrinsic angular momentum is thus independently emitted by the charge and propagates along the light cones all the way up to infinity. This implies that the external angular momentum tensor $M^{(re)}_{\lambda\mu\nu}$ has as its source

$$\partial^\nu M^{(re)}_{\lambda\mu\nu}(x) = \int_{-\infty}^{\infty} d\tau\,\delta^{(4)}(x - z(\tau)) \times(-\tfrac{4}{3}e^2\dot{v}^2 z_{[\lambda}v_{\mu]}). \qquad (3.15)$$

We verify from this relation that the source of the external part depends on the coordinate z_λ and vanishes when the origin is chosen at the retarded point. On the other hand, as shown by (3.14), the source of $M^{(ri)}_{\lambda\mu\nu}(x)$ is independent of the origin.

A very interesting property of the intrinsic angular momentum emitted by the charge, which follows from (3.9), is that it only depends on the interference part $T^{(-3)}_{\mu\nu}$ of Maxwell's tensor.

On integrating (3.11) along the world line from the remote past up to the present position, we obtain

$$M^{(ri)}_{\lambda\mu}(\tau) = \int_{-\infty}^{\tau} \tfrac{4}{3}e^2 v_{[\lambda}(\tau')\dot{v}_{\mu]}(\tau')d\tau' \qquad (3.16)$$

and consequently (see Ref. 2),

$$M^{(re)}_{\lambda\mu}(\tau) = \int_{-\infty}^{\tau} 2z_{[\lambda}(\tau')\mathring{P}^{(r)}_{\mu]}(\tau')d\tau'. \qquad (3.17)$$

This formula means that $M^{(re)}_{\lambda\mu}(\tau)$ is the angular momentum carried by the emitted energy. On the other hand the intrinsic term $M^{(ri)}_{\lambda\mu}(\tau)$ is not associated to the emission of any kind of energy, because it is constructed from the sourceless energy tensor $T^{(f)}_{\mu\nu}$.

IV. POINT PARTICLE WITH ARBITRARY MULTIPOLAR MOMENTS

In this section we generalize the results obtained so far to the case of a point particle possessing an arbitrary multipolar electromagnetic internal structure. We shall use the results obtained by Van Weert. He has shown in Ref. 3 that the bound energy-momentum tensor is also the four-divergence of a third-rank tensor $K_{\mu\nu\gamma}$:

$$T^{(b)}_{\mu\nu} \equiv T^{(-3}_{\mu\nu} = \partial^\gamma K_{\mu\nu\gamma}. \qquad (4.1)$$

Here the opened parenthesis (−3 indicates all terms in $T_{\mu\nu}$ of third and higher order in the inverse retarded distance κ. The superpotential $K_{\mu\nu\gamma}$ may be split into a form analogous to (2.3), i.e.,

$$K_{\mu\nu\gamma} = K^{(-3}_{\mu\nu\gamma} + K^{(-2)}_{\mu\nu\gamma}, \qquad (4.2)$$

where both terms are antisymmetric in their last two indexes. Hence we may define

$$T^{(b)}_{\mu\nu} = T^{*(b)}_{\mu\nu} + T^{(f)}_{\mu\nu}, \qquad (4.3)$$

where

$$T^{*(b)}_{\mu\nu} = \partial^\gamma K^{(-3}_{\mu\nu\gamma}, \qquad (4.4a)$$

$$T^{(f)}_{\mu\nu} = \partial^\gamma K^{(-2)}_{\mu\nu\gamma}, \qquad (4.4b)$$

which are conserved off the world line,

$$\partial^\nu T^{*(b)}_{\mu\nu} = 0, \quad \partial^\nu T^{(f)}_{\mu\nu} = 0. \qquad (4.5)$$

To show that they are also symmetric, we make use of the relation [see Ref. 3, Eqs. (3.5) and (3.6)]

$$T^{(f)}_{\mu\nu} = T^{(-3)}_{\mu\nu} + s_\nu \partial^\gamma T^{(-3)}_{\mu\gamma}. \qquad (4.6)$$

The first term is symmetric by definition; the second is also symmetric as a consequence of the property [Ref. 6, Eq. (30)]

$$\partial^\gamma T^{(-3)}_{\mu\gamma} \propto s_\mu, \qquad (4.7)$$

It also follows from (4.6), (4.7), and the property [Ref. 6, Eq. (29)]

$$T^{(-3)}_{\mu\nu} s^\nu = 0, \qquad (4.8)$$

that $T^{(f)}_{\mu\nu}$ does not give any flux through the light cones drawn from points on the world line of the particle

$$T^{(f)}_{\mu\nu} s^\nu = 0. \qquad (4.9)$$

This property, combined with (4.5) and (4.6), allows us to show that $T^{(f)}_{\mu\nu}$ is sourceless. The proof which we worked out in Sec. II applies here also. We again consider the integration domain of Fig. 1 and obtain the following expression for the constant flux along the light cones:

$$T^{(f)}_{\mu\nu} d\sigma^\nu = \kappa^2 T^{(f)}_{\mu\nu} v^\nu d\Omega d\tau . \qquad (4.10)$$

The free energy tensor $T^{(f)}_{\mu\nu}$ behaves like κ^{-3} as is easily verified from (4.6). Therefore (4.10) vanishes and $T^{(f)}_{\mu\nu}$ obeys the conservation law

$$\partial^\nu T^{(f)}_{\mu\nu}(x) = 0 \quad \text{everywhere}, \qquad (4.11)$$

i.e., it is sourceless.

This result shows that there exists in this general case a splitting of Maxwell's tensor containing three terms with similar properties to the simple problem of a nonspinning charge. The same is true for the angular momentum tensor. Let us define the intrinsic radiated angular momentum part as

$$M^{(ri)}_{\lambda\mu\nu} \equiv 2x_{[\lambda} T^{(f)}_{\mu]\nu} = 2x_{[\lambda} \partial^\gamma K^{(-2)}_{\mu]\nu\gamma}, \qquad (4.12)$$

or, introducing

$$s_\lambda = x_\lambda - z_\lambda(\tau),$$

as

$$M^{(ri)}_{\lambda\mu\nu} = 2z_{[\lambda} T^{(f)}_{\mu]\nu} + 2s_{[\lambda} T^{(-3)}_{\mu]\nu}. \qquad (4.13)$$

The first term here is divergenceless and does not give any flux through the light cones drawn from the world line of the particle. Moreover, it behaves like κ^{-3}. Therefore the same argument leading to (4.10) states that it gives no flux along the light cones. We thus obtain from (4.13) the intrinsic angular momentum radiation rate

$$\dot{M}^{(ri)}_{\lambda\mu}(\tau) = \int 2s_{[\lambda} T^{(-3)}_{\mu]\nu} v^\nu \kappa^2 d\Omega, \qquad (4.14)$$

a result already found by Van Weert.[6] Thus, in practice, we need only consider the interference energy term $T^{(-3)}_{\mu\nu}$ for the evaluation of the intrinsic angular momentum emitted by the particle. However, it is $T^{(f)}_{\mu\nu}$ and not $T^{(-3)}_{\mu\nu}$ the tensor which possesses an independent dynamical existence.

ACKNOWLEDGMENT

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[1]C. Teitelboim, Phys. Rev. D 1, 1572 (1970); 2, 1763(E) (1970); 3, 297 (1971).
[2]C. A. López and D. Villarroel, Phys. Rev. D 11, 2724 (1975); 13, 1802 (1976). We use the same notation of these references.
[3]Ch. G. Van Weert, Physica (Utrecht) 76, 345 (1974).
[4]Ch. G. Van Weert, Phys. Rev. D 9, 339 (1974).
[5]This decomposition has recently been considered in a different context by E. G. Peter Rowe, University of Durham Report, 1977 (unpublished).
[6]Ch. G. Van Weert, Physica (Utrecht) 66, 79 (1973).