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Approximately relativistic Lagrangians for classical interacting point particles. III

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In part I of this series the general form to order c^{-2} of approximately relativistic Lagrangians following from classical Poincaré-invariant variational principles of the Fokker type was established for point particles with two-body (neutral) interactions. In this paper the relativistic action principles discussed in I are generalized to include interactions involving classical isospin and the general form to order c^{-2} of the associated approximate Lagrangians is obtained by the methods developed in I. The post-Newtonian interaction may contain up to six independent functions of the interparticle separation and of the isospin vectors of each particle, in addition to the Newtonian potential. Unlike the neutral interactions considered in I, nonsymmetry of the particles' variables in the relativistic interaction can be evident in order c^{-1} as well as c^{-2}. The form of the ten conservation laws following from Poincaré invariance and of the charge conservation law following from invariance under rotations about the three-direction in charge space as well as the corresponding approximate conservation laws are derived using Noether's theorem. Examples discussed include interactions allowing the definition of "adjunct fields" (including the scalar and vector meson interactions) and an interaction leading to correction terms in order c^{-1}.

I. INTRODUCTION

Relativistic equations of motion for interacting particles are generally multitime equations in configuration space, and little is known about their mathematical properties or their quantization. Although very special examples have been solved,[1] no standard methods exist for solving such equations. However, in some cases where a Newtonian description of a system of particles is not adequate, an exact relativistic description may not be necessary. Consequently, a single-time approximation to the relativistic description can be useful in gleaning some information about such a system of particles.

Those relativistic equations of motion that are derivable from a Fokker-type[2,3] variational principle have the advantage that the conservation laws[3] can be established by the use of Noether's theorem.[4,5] In the first paper of this series[6] (referred to as I) a method was developed for directly making a single-time approximation of Poincaré-invariant Fokker-type variational principles describing N point particles with two-body interactions depending on the particle's four-separations and four-velocities. To second order in c^{-1}, this method was used to determine the general form of those approximately relativistic Lagrangians that have a static Newtonian limit [i.e., a potential $V_{ij}(r_{ij})$ for particles i and j separated by a distance r_{ij}]. It is noteworthy that it does not contain any nontrivial correction terms of order c^{-1}. It includes as special cases those Lagrangians studied earlier[7] which arose from slow-motion approximations to particle interactions via various special and general relativistic fields. The general form of the approximate Lagrangian contains three functions of r_{ij} which may in principle be independent of the Newtonian potential V_{ij}. One of these functions demonstrates that the effects of a nonsymmetric Poincaré-invariant interaction can be evident in order c^{-2}.

Recently, the exact Bakamjian-Thomas[8] theory was approximated to order c^{-2}, classically by Pauri and Prosperi[9] and quantum mechanically by Coester and Havas.[10] For spinless particles, the approximate Hamiltonian obtained by Pauri and Prosperi and the classical limit of the one obtained by Coester and Havas are equivalent to the Hamiltonian obtained in I.

The advantage of deriving an approximately relativistic dynamics either from a Poincaré-invariant variational principle or from the exact canonical Bakamjian-Thomas theory is that the form obtained to a given order is determined by the assumptions of the exact theory; if, instead, it is computed according to what could be consistent with certain assumptions to a given order, then this does not ensure the existence of nontrivial results in higher orders.[11,12]

One way of obtaining approximately relativistic Hamiltonians is to start from a realization of the Lie algebra of the Galilei group and to add correction terms of various orders of c^{-1} to obtain realizations of the Lie algebra of the Poincaré group correct to the order desired. This procedure was carried out recently to order c^{-2}, classically by Stachel and Havas,[11] and quantum mechanically by Foldy and Krajcik.[13] If the results of these two investigations are required to be consistent with the expansion of an exact theory, then for particles without spin or isospin they reduce to those of I and Ref. 10.

In I it was noted that the approximation procedure developed there could also be applied to exact action principles describing particles with intrinsic attributes such as spin and isospin. Here we determine, to order c^{-2}, the general form of the classical approximate Lagrangian for point particles with isospin.

A treatment of classical isospin was first introduced by Fierz[14] to describe neutrons interacting through charged meson fields. Patterned after the quantum-theoretical treatment by Kemmer[15] and Møller and Rosenfeld[16] of charge-symmetric meson fields—in which both charged and neutral meson fields are combined in such a way as to preserve charge independence for the field sources (i.e., the heavy particles)—Le Couteur[17] gave a classical theory of retarded charge-symmetric vector meson fields which was extended by Havas[18] to half-retarded-half-advanced fields. In addition, it has been shown[18,19] that a consistent theory of action at a distance derivable from a variational principle is possible for particles interacting through both scalar and vector charge-symmetric adjunct meson fields.

Here, a generalization of the Poincaré-invariant action principles considered in Ref. 3 and further discussed in I is presented in Sec. II for directly interacting point particles with any value of hypercharge and isospin. The two-body interactions are assumed to depend at most on the four-velocities and isospin and as in I are not assumed to be symmetric in the particles' variables. Only interactions possessing a static nonrelativistic limit are considered. Since the isospin vector is a dynamical variable, the action principle yields equations describing the variation in time of the charge of the particles, in addition to the translational equations of motion.

The conserved quantities which follow from invariances of the action principle are derived in Sec. III. Invariance under the full ten-parameter Poincaré group leads to the usual ten conserved quantities. Invariance under the three-parameter rotation group in isospin space leads to a conserved total isospin three-vector, the third component of which determines the conserved total electric charge. Since this is the only conservation law required from physical considerations, it is sufficient to require only invariance under rotations about the three-direction in isospin space.

Using the approximation method developed in I, the approximate Lagrangian is derived in Sec. IV and in the Appendix. While, as shown in I, the absence of terms of order c^{-1} is a necessary feature of the relativistic variational principles describing point particles without spin or isospin if the interactions possess a static Newtonian limit, here it is found that this is no longer the case for point particles with isospin.

The approximate conservation laws following from the invariances of the approximate Lagrangian are also determined, via Noether's theorem, in Sec. V. Special cases of approximate Lagrangians are given in Sec. VI, and Sec. VII contains a discussion of results.

II. THE EXACT VARIATIONAL PRINCIPLE

We use essentially the same notation as in I, except that λ_i represents an arbitrary parameter along the world line z_i^μ of particle i, which can be chosen to be the proper time τ_i or the coordinate time t_i. The quantity

$$\mathfrak{v}_i^\mu \equiv \frac{dz_i^\mu}{d\lambda_i}, \quad \mathfrak{v}_i \equiv (\mathfrak{v}_i^\mu \mathfrak{v}_{i\mu})^{1/2} , \tag{1}$$

is related to the four-velocity

$$v_i^\mu \equiv \frac{dz_i^\mu}{d\tau_i}, \quad v_i^\mu v_{i\mu} = 1 , \tag{2}$$

by

$$v_i^\mu = \mathfrak{v}_i^\mu / \mathfrak{v}_i , \tag{3}$$

independent of the choice of parameter.

To describe the electric charge on particle i, a vector $\underline{\tau}_i(\tau_i)$ in an abstract three-dimensional "charge space" is introduced[14,17] with the property that its magnitude is unity:

$$\underline{\tau}_i \circ \underline{\tau}_i = 1 , \tag{4a}$$

and consequently

$$\underline{\tau}_i \cdot d\underline{\tau}_i / d\tau_i = 0 , \tag{4b}$$

where the center dot denotes a scalar product and $\wedge$ a vector product in charge space. Vectors in this space are represented by letters with a straight line below them (an arrow above indicates a vector in ordinary three-space) and their components with respect to a fixed orthonormal basis in charge space are labeled by lower-case latin letters taking the values 1, 2, 3. The usual summation convention applies to these as well as to the four-space indices.

A generalized expression for the charge of particle i used in studies of classical charge-symmetric field theories[17] is given by (in the notation of Ref. 20)

$$q_i(\tau_i) = e[\tfrac{1}{2}Y_i + I_i \tau_{i3}(\tau_i) - \tfrac{1}{2}L_i \underline{\gamma} \cdot \underline{\tau}_i \wedge d\underline{\tau}_i / d\tau_i] , \tag{5}$$

where e, Y_i, I_i, and L_i are arbitrary constants and $\underline{\gamma}$ is a unit isovector oriented in the three-direction in charge space. For $L_i = 0$, this agrees with the standard description of the charge of a particle with isospin I and hypercharge Y in elementary particle physics (in units with $\hbar=1$). The form (5) is compatible with linear fields and therefore with equations of motion of the type we shall consider, which are derivable from a variational principle containing only two-body interactions. For $L_i\neq 0$, it is not compatible with equations of motion for particles interacting via nonlinear fields,[20] but we shall not consider direct-particle interactions corresponding to such theories here.

In Sec. III a conserved total charge is defined for a system of N particles whose interactions are invariant under rotations about the three-axis in charge space. If the interactions are invariant under arbitrary rotations in charge space, the forces between particles are charge independent. Interactions that are invariant only under rotations about the three-axis are charge dependent. Interactions that do not depend on isospin are neutral.

Very general Poincaré-invariant variational principles yielding equations of motion for point particles with neutral two-body interactions were introduced in Ref. 3 and further described in I. The principal new feature for point particles with isospin is the necessity of deriving equations which describe the variation in time of the isospin variables $\underline{\tau}_i$. Such equations must be consistent with the constraints (4). For the theory of action at a distance corresponding to charge-symmetric scalar and vector meson fields, a variational formalism using spinors[21] was employed in Ref. 18. As noted there, an alternate method is afforded by the use of quasicoordinates,[22,23,19] which we shall employ here. Still another approach, which treats the isospin variables themselves as dynamic variables in a variational principle, was introduced in Ref. 24.

Thus, generalizing Ref. 3 and I, we consider Poincaré-invariant equations of motion for point particles with isospin interacting through two-body forces which may be obtained from a variational principle

$$\delta\mathcal{I}=0,\quad \mathcal{I}\equiv\mathcal{I}_1+\mathcal{I}_2+\mathcal{I}_3\,, \tag{6a}$$

where

$$\mathcal{I}_1=-\sum\sum_{i<j}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}c\,d\lambda_i d\lambda_j \mathfrak{v}_i\mathfrak{v}_j \times U_{ij}\left(s_{ij}^{\mu},\frac{\mathfrak{v}_i^{\mu}}{\mathfrak{v}_i},\frac{\mathfrak{v}_j^{\mu}}{\mathfrak{v}_j};\underline{\tau}_i,\underline{\tau}_j\right), \tag{6b}$$

$$\mathcal{I}_2=-\sum_i c^2\int_{-\infty}^{\infty}d\lambda_i m_i\mathfrak{v}_i\,, \tag{6c}$$

$$\mathcal{I}_3=\sum_i\int_{-\infty}^{\infty}d\lambda_i\left(I_i\underline{w}_i\cdot\underline{\tau}_i-\tfrac{1}{4}L_i\frac{\dot{\underline{\tau}}_i^{\,2}}{\mathfrak{v}_i}\right), \tag{6d}$$

and

$$s_{ij}^{\mu}\equiv z_i^{\mu}(\lambda_i)-z_j^{\mu}(\lambda_j),\quad \dot{\underline{\tau}}_i\equiv d\underline{\tau}_i/d\lambda_i\,. \tag{6e}$$

The isovectors $\underline{w}_i$ are the angular velocities of the $\underline{\tau}_i$'s. Since the $\underline{\tau}_i$'s have fixed magnitude, we obtain

$$\dot{\underline{\tau}}_i=\underline{w}_i\wedge\underline{\tau}_i\,. \tag{7}$$

The action principle (6) differs from (I34) only in the inclusion of isospin dependence in U_{ij} (where the coupling constants in I are here absorbed into U_{ij}) and in the presence of the term $\mathcal{I}_3$, which is responsible for producing dynamic equations for the $\underline{\tau}_i$'s which are consistent with Eq. (5). It can be specialized to the action-at-a-distance principles of scalar and vector mesodynamics.[18] It cannot, however, describe the complete nonlinear theory of combined mesic and electromagnetic interactions,[20] because for any theory possessing adjunct fields a description in terms of two-body forces implies linearity of these fields.

The $\frac{1}{2}N(N-1)$ possibly distinct functions U_{ij} describing the particle interactions may contain a sum of several different types of interactions with appropriate coupling constants for each type; some of these interactions may be neutral. Each U_{ij} is assumed to be Poincaré-invariant and to depend only on the positions and velocities in Minkowski space, as well as on the isospin variables $\underline{\tau}_i$; a more general form of U_{ij} depending also on $\dot{\underline{\tau}}_i$ and $\dot{\underline{\tau}}_j$ is considered in Ref. 24. In addition, U_{ij} is assumed to be invariant under rotations about the three-direction in charge space to assure a conserved total charge for the system. The Poincaré invariance will later be made manifest by using a complete set of two-body invariants in Minkowski space. Two-body invariants in charge space will also be discussed at that time.

As discussed in I, in a parameter-invariant formulation the m_i appearing in Eq. (6c) are constants interpreted as the inertial masses, which facilitates the approximation of the variational principle. An alternative approach is to use the proper time as a parameter, using freely the relation $v_i^{\mu}v_{i\mu}=1$, and to introduce Lagrange multipliers $M_i(\tau_i)$ (which are not constants) to maintain $v_{i\mu}\delta v_i^{\mu}=0$.

Variation of $\mathcal{I}_1$ and $\mathcal{I}_2$ for arbitrary $\delta z_i^{\mu}(\lambda_i)$ which vanish at $\lambda_i=\pm\infty$ yields (for details see I)

$$\delta\mathcal{I}_1=-\sum_i\int_{-\infty}^{\infty}d\lambda_i\delta z_i^{\mu}\mathfrak{L}_{i\mu}(\mathfrak{v}_iV_i) \tag{8}$$

and

$$\delta\mathcal{I}_2=\sum_i c^2\int_{-\infty}^{\infty} d\lambda_i \delta z_i^\mu \frac{d}{d\lambda_i}\left(m_i\frac{\mathfrak{v}_{i\mu}}{\mathfrak{v}_i}\right), \tag{9}$$

respectively, where $\mathfrak{L}_{i\mu}$ is the Lagrangian derivative

$$\mathfrak{L}_{i\mu}\equiv\frac{\partial}{\partial z_i^\mu}-\frac{d}{d\lambda_i}\frac{\partial}{\partial \mathfrak{v}_i^\mu}, \tag{10}$$

and the "generalized potential" V_i is defined by

$$V_i\left(z_i^\mu, \frac{\mathfrak{v}_i^\mu}{\mathfrak{v}_i}; \underline{\tau}_i\right)\equiv\sum_{j>i}\int_{-\infty}^{\infty} c\,d\lambda_j \mathfrak{v}_j U_{ij} + \sum_{j<i}\int_{-\infty}^{\infty} c\,d\lambda_j \mathfrak{v}_j U_{ji}. \tag{11}$$

The variation of $\mathcal{I}_3$ for $z_i^\mu \to z_i^\mu + \delta z_i^\mu$ gives

$$\delta\mathcal{I}_3=\sum_i\int_{-\infty}^{\infty} d\lambda_i\left(\tfrac{1}{4}L_i\dot{\underline{\tau}}_i^{\,2}\frac{\mathfrak{v}_{i\mu}\delta\mathfrak{v}_i^\mu}{\mathfrak{v}_i^3}\right).$$

An integration by parts turns this into

$$\delta\mathcal{I}_3=-\sum_i\int_{-\infty}^{\infty} d\lambda_i \delta z_i^\mu \frac{d}{d\lambda_i}\left(\tfrac{1}{4}L_i\dot{\underline{\tau}}_i^{\,2}\frac{\mathfrak{v}_{i\mu}}{\mathfrak{v}_i^3}\right). \tag{12}$$

The translational equations of motion following from the vanishing of the sum of Eqs. (8), (9), and (12) are

$$\frac{d}{d\lambda_i}\left[\left(m_i-\frac{1}{4c^2}L_i\frac{\dot{\underline{\tau}}_i^{\,2}}{\mathfrak{v}_i^2}\right)\frac{\mathfrak{v}_{i\mu}}{\mathfrak{v}_i}\right]=\frac{1}{c^2}\mathfrak{L}_{i\mu}(\mathfrak{v}_iV_i). \tag{13}$$

To obtain the dynamic equations for $\underline{\tau}_i$ from the variational principle, we treat the components of the angular velocity $\underline{w}_i$ as derivatives of quasi-coordinates $\underline{\theta}_i$ defined by (for details see Ref. 23)

$$d\underline{\theta}_i/d\lambda_i=\underline{w}_i. \tag{14}$$

The true coordinates are the Euler angles, with $\underline{\tau}_i$ playing the role of the usual three-direction in a hypothetical "body" set of axes. The relation

$$d\underline{\tau}_i=d\underline{\theta}_i\wedge\underline{\tau}_i \tag{15}$$

is then an identity, as is Eq. (7). The isospin equations of motion result from the variation $\underline{\theta}_i \to \underline{\theta}_i+\delta\underline{\theta}_i$ which induces

$$\delta\underline{\tau}_i=\delta\underline{\theta}_i\wedge\underline{\tau}_i. \tag{16}$$

Performing the variation on $\mathcal{I}_1$, we obtain

$$\delta\mathcal{I}_1=-\sum_i\int_{-\infty}^{\infty} d\lambda_i\mathfrak{v}_i\delta\underline{\tau}_i\cdot\frac{\partial V_i}{\partial\underline{\tau}_i},$$

and using Eq. (16) gives

$$\delta\mathcal{I}_1=\sum_i\int_{-\infty}^{\infty} d\lambda_i\mathfrak{v}_i\delta\underline{\theta}_i\cdot\frac{\partial V_i}{\partial\underline{\tau}_i}\wedge\underline{\tau}_i. \tag{17}$$

The variation of $\mathcal{I}_2$ is identically zero, while variation of $\mathcal{I}_3$ for arbitrary $\delta\underline{\theta}_i$ yields

$$\delta\mathcal{I}_3=\sum_i\int_{-\infty}^{\infty} d\lambda_i\left(I_i\delta\underline{w}_i\cdot\underline{\tau}_i+I_i\underline{w}_i\cdot\delta\underline{\tau}_i-\tfrac{1}{2}L_i\dot{\underline{\tau}}_i\cdot\frac{\delta\dot{\underline{\tau}}_i}{\mathfrak{v}_i}\right). \tag{18}$$

The operators d and δ do not commute when acting on the quasicoordinates $\underline{\theta}_i$, but satisfy

$$\delta d\underline{\theta}_i-d\delta\underline{\theta}_i=\delta\underline{\theta}_i\wedge d\underline{\theta}_i. \tag{19}$$

Using this relation in Eq. (18), and then integrating by parts gives

$$\delta\mathcal{I}_3=\sum_i\int_{-\infty}^{\infty} d\lambda_i\left[-I_i\dot{\underline{\tau}}_i\cdot\delta\underline{\theta}_i+I_i\delta\underline{\theta}_i\wedge\underline{w}_i\cdot\underline{\tau}_i+I_i\underline{w}_i\cdot\delta\underline{\tau}_i+\tfrac{1}{2}L_i\frac{d}{d\lambda_i}\left(\frac{\dot{\underline{\tau}}_i}{\mathfrak{v}_i}\right)\cdot\delta\underline{\tau}_i\right],$$

which reduces to

$$\delta\mathcal{I}_3=\sum_i\int_{-\infty}^{\infty} d\lambda_i\delta\underline{\theta}_i\cdot\left[I_i\dot{\underline{\tau}}_i+\tfrac{1}{2}L_i\frac{d}{d\lambda_i}\left(\frac{\dot{\underline{\tau}}_i}{\mathfrak{v}_i}\right)\wedge\underline{\tau}_i\right] \tag{20}$$

by virtue of Eq. (16). The dynamic equations for $\underline{\tau}_i$ following from the vanishing of the sum of Eqs. (18) and (20) are

$$I_i\frac{\dot{\underline{\tau}}_i}{\mathfrak{v}_i}+\tfrac{1}{2}L_i\frac{1}{\mathfrak{v}_i}\frac{d}{d\lambda_i}\left(\frac{\dot{\underline{\tau}}_i}{\mathfrak{v}_i}\right)\wedge\underline{\tau}_i=\frac{\partial V_i}{\partial\underline{\tau}_i}\wedge\underline{\tau}_i. \tag{21}$$

The parameter-invariant equations (13) and (21) govern the dynamics of the system of N particles. Using Eq. (10) together with

$$\frac{\partial(\mathfrak{v}_iV_i)}{\partial\mathfrak{v}_i^\mu}=\frac{\mathfrak{v}_{i\mu}}{\mathfrak{v}_i}V_i+\mathfrak{v}_i\frac{\partial V_i}{\partial(\mathfrak{v}_i^\rho/\mathfrak{v}_i)}\left(\frac{\delta_\mu^\rho}{\mathfrak{v}_i}-\frac{\mathfrak{v}_i^\rho\mathfrak{v}_{i\mu}}{\mathfrak{v}_i^3}\right) \tag{22}$$

in Eq. (13), and subsequently choosing the arbitrary parameter λ_i to be the proper time τ_i [and thus $\mathfrak{v}_i\to(v_i^\mu v_{i\mu})^{1/2}=1$] in both Eq. (13) and (21), we obtain the following for them:

$$\frac{d}{d\tau_i}\left[\left(m_i-\frac{1}{4c^2}L_i\frac{d\underline{\tau}_i}{d\tau_i}\cdot\frac{d\underline{\tau}_i}{d\tau_i}\right)v_{i\mu}\right]=\frac{1}{c^2}\left\{\frac{\partial V_i}{\partial z_i^\mu}-\frac{d}{d\tau_i}\left[\frac{\partial V_i}{\partial v_i^\mu}+v_{i\mu}\left(V_i-v_i^\rho\frac{\partial V_i}{\partial v_i^\rho}\right)\right]\right\}, \tag{23}$$

and

$$I_i\frac{d\underline{\tau}_i}{d\tau_i}+\tfrac{1}{2}L_i\frac{d^2\underline{\tau}_i}{d\tau_i^2}\wedge\underline{\tau}_i=\frac{\partial V_i}{\partial\underline{\tau}_i}\wedge\underline{\tau}_i, \tag{24}$$

respectively.

III. THE EXACT CONSERVATION LAWS

Invariance of the variational principle (6) under the ten-parameter Poincaré group leads to ten conservation laws. One of these corresponds to the conservation of energy, three to the conservation of linear momentum, three to the conservation of angular momentum, and three express the uniform motion of the center of mass.

Invariance under the three-parameter group of isospin rotations leads to three additional conserved quantities interpreted as the components of a total isospin vector $\underline{T}$. The conservation of T_3 corresponds to conservation of charge. Since this is the only conservation law required from physical considerations, it is sufficient to require only invariance under rotations about the three-direction in charge space.

The Poincaré invariance can be made manifest through the use of the following independent invariants[3,6]:

$$s_{ij}^2 \equiv \eta_{\mu\nu} s_{ij}^\mu s_{ij}^\nu, \quad \omega_{ij} \equiv v_i^\mu v_{j\mu}, \qquad \kappa_i \equiv v_{i\mu} s_{ji}^\mu, \quad \kappa_j \equiv v_{j\mu} s_{ij}^\mu. \tag{25}$$

These are a complete set of independent invariants formed from the four-dimensional separation s_{ij}^μ and the corresponding four-velocities v_i^μ and v_j^μ.

Invariance under rotations about the three-direction in charge space can be built into the variational principle by considering U_{ij} to depend on isospin only through the independent quantities

$$\tau_{ij} \equiv \underline{\tau}_i \cdot \underline{\tau}_j, \quad \Gamma_i \equiv \underline{\gamma} \cdot \underline{\tau}_i, \qquad \Gamma_j \equiv \underline{\gamma} \cdot \underline{\tau}_j, \quad \Upsilon_{ij} \equiv \underline{\gamma} \cdot \underline{\tau}_i \wedge \underline{\tau}_j. \tag{26}$$

For neutral interactions, the conservation laws corresponding to Fokker-type variational principles invariant up to a divergence under either the Poincaré group or the Galilei group and not necessarily symmetric in the particles' variables were derived in Ref. 3 using a method similar to the one introduced by Dettman and Schild[25] for electrodynamics. Here we obtain the conservation laws using Noether's theorem, which has not previously been formulated in a way appropriate for application to Fokker-type action principles. Such a formulation will be discussed elsewhere.[5]

We consider the principle (6) modified in the form

$$\mathcal{I}_*^{**} \equiv \sum_i \int_{\lambda_i^*}^{\lambda_i^{**}} d\lambda_i \left(I_i \underline{w}_i \cdot \underline{\tau}_i - \tfrac{1}{4} L_i \frac{\dot{\underline{\tau}}_i^2}{\mathfrak{v}_i} - m_i c^2 \mathfrak{v}_i - \tfrac{1}{2} \mathfrak{v}_i V_i \right), \tag{27}$$

where λ_i^* and λ_i^{**} are arbitrary, and V_i is defined in Eq (11). The variations leading to the equations of motion are performed with the limits $\lambda_i^* \to -\infty$ and $\lambda_i^{**} \to \infty$, for all i. Then for those infinitesimal variations

$$z_i^\mu \to z_i^\mu + \delta z_i^\mu, \quad \underline{\theta}_i \to \underline{\theta}_i + \delta \underline{\theta}_i \tag{28}$$

that leave $\mathcal{I}_*^{**}$ invariant up to a divergence, i.e.,

$$\delta \mathcal{I}_*^{**} = -\sum_i \int_{\lambda_i^*}^{\lambda_i^{**}} d\lambda_i \frac{dC_i}{d\lambda_i}, \tag{29}$$

it follows that[5]

$$\frac{d}{d\lambda_i}\left\{ -\sum_i C_i - \sum_i \delta\underline{\theta}_i \cdot \left(I_i \underline{\tau}_i + \tfrac{1}{2} L_i \frac{\dot{\underline{\tau}}_i}{\mathfrak{v}_i} \wedge \underline{\tau}_i \right) + \sum_i \delta z_i^\mu \left[\left(m_i c^2 - \tfrac{1}{4} L_i \frac{\dot{\underline{\tau}}_i^2}{\mathfrak{v}_i^2} \right) \frac{\mathfrak{v}_{i\mu}}{\mathfrak{v}_i} + \frac{\partial}{\partial \mathfrak{v}_i^\mu} (\mathfrak{v}_i V_i) \right] \right.$$
$$\left. + \tfrac{1}{2} \sum_{i<j} \sum \left(\int_{\lambda_i}^{\infty} \int_{-\infty}^{\lambda_j} - \int_{-\infty}^{\lambda_i} \int_{\lambda_j}^{\infty} \right) c\, d\lambda_i d\lambda_j [\delta_i(\mathfrak{v}_i \mathfrak{v}_j U_{ij}) - \delta_j(\mathfrak{v}_i \mathfrak{v}_j U_{ij})] \right\}$$
$$= -\delta\underline{\theta}_i \cdot \left[I_i \dot{\underline{\tau}}_i + \tfrac{1}{2} L_i \frac{d}{d\lambda_i}\left(\frac{\dot{\underline{\tau}}_i}{\mathfrak{v}_i} \right) \wedge \underline{\tau}_i - \frac{\partial(\mathfrak{v}_i V_i)}{\partial \underline{\tau}_i} \wedge \underline{\tau}_i \right] - \delta z_i^\mu \left\{ \frac{d}{d\lambda_i} \left[\left(m_i c^2 - \tfrac{1}{4} L_i \frac{\dot{\underline{\tau}}_i^2}{\mathfrak{v}_i^2} \right) \frac{\mathfrak{v}_{i\mu}}{\mathfrak{v}_i} \right] - \mathcal{L}_{i\mu}(\mathfrak{v}_i V_i) \right\}, \tag{30}$$

where

$$\delta_i \equiv \delta z_i^\mu \frac{\partial}{\partial z_i^\mu} + \delta \mathfrak{v}_i^\mu \frac{\partial}{\partial \mathfrak{v}_i^\mu} + \delta \underline{\theta}_i \cdot \underline{\tau}_i \wedge \frac{\partial}{\partial \underline{\tau}_i}, \tag{31}$$

and $\mathcal{L}_{i\mu}$ is defined in Eq. (10). The conservation laws follow from Eq. (30) when the equations of motion (13) and (21) are satisfied.

Equation (30) together with the constant infinitesimal space-time translations

$$\delta z_i^\mu = \epsilon^\mu, \quad \delta \underline{\theta}_i = 0, \tag{32}$$

for which $\delta \mathcal{I}_*^{**}$ vanishes, gives the law of conservation of energy-momentum

$$\frac{d}{d\tau_i}P_\mu(\tau_1,\tau_2,\ldots,\tau_N)=0,\quad i=1,\ldots,N,$$

$$P_\mu\equiv\sum_i\left(m_i-\frac{1}{4c^2}L_i\frac{d\underline{\tau}_i}{d\tau_i}\cdot\frac{d\underline{\tau}_i}{d\tau_i}\right)v_{i\mu}+\frac{1}{c^2}\sum_i\left[\frac{\partial V_i}{\partial v_i^\mu}+v_{i\mu}\left(V_i-v_i^\rho\frac{\partial V_i}{\partial v_i^\rho}\right)\right]$$
$$+\frac{1}{c^2}\sum_{i<j}\sum\left(\int_{\tau_i}^{\infty}\int_{-\infty}^{\tau_j}-\int_{-\infty}^{\tau_i}\int_{\tau_j}^{\infty}\right)c\,d\tau_i d\tau_j\frac{\partial U_{ij}}{\partial s_{ij}^\mu},\tag{33}$$

where we have used Eq. (22) and subsequently chosen proper-time parametrization. The law of conservation of angular momentum and the center-of-mass theorem follow from the vanishing of $\delta\mathcal{I}_*^*$ due to the invariance of $\mathcal{I}$ under the infinitesimal four-dimensional rotation

$$\delta z_i^\mu=\eta^{\mu\nu}\epsilon_{\nu\rho}z_i^\rho/c^2,\quad \epsilon_{\nu\rho}=-\epsilon_{\rho\nu},\quad \delta\underline{\theta}_i=0,\tag{34}$$

where the $\epsilon_{\nu\rho}$ are arbitrary constants. For proper-time parametrization Eqs. (30) and (34) yield

$$\frac{d}{d\tau_i}L^{\mu\nu}(\tau_1,\tau_2,\ldots,\tau_N)=0,\quad i=1,\ldots,N,$$

$$L^{\mu\nu}=\sum_i\left(m_i-\frac{1}{4c^2}L_i\frac{d\underline{\tau}_i}{d\tau_i}\cdot\frac{d\underline{\tau}_i}{d\tau_i}\right)(v_i^\mu z_i^\nu-v_i^\nu z_i^\mu)+\frac{1}{c^2}\sum_i\left[\frac{\partial V_i}{\partial v_i^\rho}(\eta^{\rho\mu}z_i^\nu-\eta^{\rho\nu}z_i^\mu)+\left(V_i-v_i^\rho\frac{\partial V_i}{\partial v_i^\rho}\right)(v_i^\mu z_i^\nu-v_i^\nu z_i^\mu)\right]$$
$$+\frac{1}{2c^2}\sum_{i<j}\sum\left(\int_{\tau_i}^{\infty}\int_{-\infty}^{\tau_j}-\int_{-\infty}^{\tau_i}\int_{\tau_j}^{\infty}\right)c\,d\tau_i d\tau_j\left\{\frac{\partial U_{ij}}{\partial s_{ij}^\rho}[\eta^{\rho\mu}(z_i^\nu+z_j^\nu)-\eta^{\rho\nu}(z_i^\mu+z_j^\mu)]\right.$$
$$\left.+\frac{\partial U_{ij}}{\partial v_i^\rho}(\eta^{\rho\mu}v_i^\nu-\eta^{\rho\nu}v_i^\mu)-\frac{\partial U_{ij}}{\partial v_j^\rho}(\eta^{\rho\mu}v_j^\nu-\eta^{\rho\nu}v_j^\mu)\right\},\tag{35}$$

where, again, Eq. (22) has been used before letting $\lambda_i\to\tau_i$.

For the case of charge independence we have

$$\partial U_{ij}/\partial\Gamma_i=\partial U_{ij}/\partial\Gamma_j=\partial U_{ij}/\partial\Upsilon_{ij}=0,\tag{36}$$

and the variational principle is invariant under the infinitesimal transformations

$$\delta z_i^\mu=0,\quad \delta\underline{\theta}_i=\underline{\Omega},\tag{37}$$

where $\underline{\Omega}$ is an arbitrary constant isovector. The last of these equations implies the infinitesimal rotation

$$\delta\underline{\tau}_i=\underline{\Omega}\wedge\underline{\tau}_i.\tag{38}$$

Using Eqs. (30) and (36), we obtain the law of conservation of total isospin:

$$\frac{d}{d\tau_i}\underline{T}(\tau_1,\tau_2,\ldots,\tau_N)=0,\quad i=1,\ldots,N,$$

$$\underline{T}\equiv\sum_i\left(I_i\underline{\tau}_i+\tfrac{1}{2}L_i\frac{d\underline{\tau}_i}{d\tau_i}\wedge\underline{\tau}_i\right)+\tfrac{1}{2}\sum_{i<j}\sum\left(\int_{\tau_i}^{\infty}\int_{-\infty}^{\tau_j}-\int_{-\infty}^{\tau_i}\int_{\tau_j}^{\infty}\right)c\,d\tau_i d\tau_j\left(\frac{\partial U_{ij}}{\partial\underline{\tau}_i}\wedge\underline{\tau}_i-\frac{\partial U_{ij}}{\partial\underline{\tau}_j}\wedge\underline{\tau}_j\right),\tag{39}$$

in proper-time parametrization.

We now define the total charge to be

$$Q(\tau_1,\tau_2,\ldots,\tau_N)\equiv e\left[\sum_i\tfrac{1}{2}Y_i+T_3(\tau_1,\tau_2,\ldots,\tau_N)\right],\tag{40}$$

which by Eq. (39) is conserved. By virtue of the definition (5) of the charge on each particle, we interpret Eq. (40) together with (39) to mean

$$Q=\sum_i q_i+\text{“charge in transit”},\tag{41}$$

in analogy to meson field theories where the interaction is mediated by a field which carries charge.

Interactions that are invariant only under rotations about the three-direction in charge space (in addition to Poincaré invariance) imply that $\underline{\Omega}=\Omega\underline{\gamma}$ in Eq. (37), and Eqs. (36) no longer hold. Then only T_3 is conserved, which is sufficient for conservation of the total charge.

IV. THE APPROXIMATELY RELATIVISTIC LAGRANGIAN

We take the nonrelativistic limit of the action principle (6) to mean a variational principle

$$\delta \mathcal{I}=0,\quad \mathcal{I}\equiv \int_{-\infty}^{\infty} dt\, L[\vec{r}_i(t),\vec{v}_i(t);\underline{\tau}_i(t),\underline{w}_i(t)]\,,\qquad i=1,\ldots,N,$$

$$\begin{aligned}
&L\equiv T+A-V,\quad T\equiv\sum_i \tfrac{1}{2}m_i\vec{v}_i^{\,2},\\
&\vec{v}_i\equiv\frac{d\vec{r}_i}{dt},\quad A\equiv\sum_i(I_i\underline{w}_i\cdot\underline{\tau}_i-\tfrac{1}{4}L_i\dot{\underline{\tau}}_i^{2}),\\
&V\equiv\sum_{i<j}\sum V_{ij}(r_{ij};\tau_{ij},\Gamma_i,\Gamma_j,\Upsilon_{ij}),\\
&r_{ij}\equiv|\vec{r}_i-\vec{r}_j|\,.
\end{aligned}\tag{42}$$

Just like the corresponding limit of the neutral case discussed in I, this is not the only possible limit of variational principles of the form (6).

Equation (42) generalizes the traditional Newtonian variational principle by the inclusion of isospin, but it still leads to forces which are static and central. The two-body potential energy V_{ij} need not necessarily by symmetric in $\underline{\tau}_i$ and $\underline{\tau}_j$; analogous asymmetries in the neutral case considered in I appear first in order c^{-2}.

Owing to differing space-time concepts, substantial differences exist between the relativistic variational principle (6) and the nonrelativistic principle (42). The relativistic principle involves $2N$ independent variables in charge space and $4N$ coordinates and N parameters in a four-dimensional space-time, while the nonrelativistic principle involves the same number of charge-space variables, but $3N$ coordinates and a single time parameter in a Euclidean three-space. The equations of motion following from the relativistic variational principle (6) are given by Eqs. (23) and (24), while the nonrelativistic principle (42) yields the dynamical equations

$$m_i\vec{a}_i=-\sum_{j>i}\frac{1}{r_{ij}}\frac{\partial V_{ij}}{\partial r_{ij}}\vec{r}_{ij}-\sum_{j<i}\frac{1}{r_{ji}}\frac{\partial V_{ji}}{\partial r_{ji}}\vec{r}_{ij},\qquad \vec{a}_i(t)\equiv\frac{d^2\vec{r}_i(t)}{dt^2}\,,\tag{43a}$$

and

$$I_i\frac{d\underline{\tau}_i}{dt}+\tfrac{1}{2}L_i\frac{d^2\underline{\tau}_i}{dt^2}\wedge\underline{\tau}_i=\left(\sum_{j>i}\frac{\partial V_{ij}}{\partial\underline{\tau}_i}+\sum_{j<i}\frac{\partial V_{ji}}{\partial\underline{\tau}_i}\right)\wedge\underline{\tau}_i\,.\tag{43b}$$

Equation (43b) implies that the magnitudes of the $\underline{\tau}_i$'s are constants which we choose to be equal to 1 for agreement with Eq. (4). The system of N particles described by Eqs. (43) thus requires $6N+4N$ initial data; for the case $L_i=0$, $6N+2N$ initial data are required. On the other hand, the number of initial data associated with particle systems described by relativistic equations of motion containing N independent parameters has not been established even for neutral interactions.[26]

To apply the approximation method of I, we first choose the N arbitrary parameters λ_i in the relativistic action principle (6) to be the coordinate times t_i, thus relating the four-dimensional formulation to a "three-plus-one" formulation; this is appropriate because, as in I, the nonrelativistic limit has been chosen in three-plus-one form. Thus

$$z_i^\mu(t_i)=(t_i,\vec{r}_i(t_i))\,,\tag{44}$$

and

$$\begin{aligned}
&\mathfrak{v}_i^\mu\to\frac{dz_i^\mu}{dt_i}=(1,\vec{v}_i(t_i)),\\
&\mathfrak{v}_i\to\frac{d\tau_i}{dt_i}=\gamma_i^{-1}\equiv(1-\vec{v}_i^{\,2}/c^2)^{1/2},\\
&v_i^\mu=\mathfrak{v}_i^\mu/\mathfrak{v}_i=\gamma_i(1,\vec{v}_i)\,.
\end{aligned}\tag{45}$$

Following I, we now build the restriction of the existence of a static limit directly into the variational principle (6). As shown in I, although not all of the invariants in the set (25) possess a static limit, an equivalent set which does have this property is

$$\begin{aligned}
&\sigma_{ij}\equiv c^2s_{ij}^{\,2}=c^2(t_i-t_j)^2-|\vec{r}_i(t_i)-\vec{r}_j(t_j)|^2\\
&\qquad\equiv c^2t_{ij}^{\,2}-\vec{r}_{ij}^{\,2},\\
&\omega_{ij}=\gamma_i\gamma_j\left[1-\frac{\vec{v}_i(t_i)\cdot\vec{v}_j(t_j)}{c^2}\right],\\
&\chi_{ij}\equiv c\kappa_i=-\gamma_i[ct_{ij}-c^{-1}\vec{v}_i(t_i)\cdot\vec{r}_{ij}(t_i,t_j)],\\
&\zeta_{ij}\equiv c\kappa_j=\gamma_j[ct_{ij}-c^{-1}\vec{v}_j(t_j)\cdot\vec{r}_{ij}(t_i,t_j)],
\end{aligned}\tag{46}$$

where use has been made of Eq. (45).

Here, as in I, the approximation will be carried out only to order c^{-2}. $\mathcal{I}_2$ in Eq. (6c) can be easily approximated using Eq. (45) and then expanding in a Taylor series in c^{-1} to obtain

$$\mathcal{I}_2\approx\sum_i\int_{-\infty}^{\infty}dt\left[-m_ic^2+\tfrac{1}{2}m_i\vec{v}_i^{\,2}(t)+\tfrac{1}{8}m_i\vec{v}_i^{\,2}(t)\frac{\vec{v}_i^{\,2}(t)}{c^2}\right],\tag{47}$$

where the particle label on t_i has been deleted, since it becomes superfluous once all two-body interaction terms in $\mathcal{I}_1$ are written in terms of a single time. Similarly, the approximate form of $\mathcal{I}_3$ is obtained by using Eq. (45) in (6d) and then expanding in c^{-1}. This yields

$$\mathcal{I}_3 \approx \sum_i \int_{-\infty}^{\infty} dt \left[I_i \underline{w}_i \cdot \underline{\tau}_i - \tfrac{1}{4} L_i \dot{\underline{\tau}}_i^2 \left(1 + \tfrac{1}{2} \frac{\vec{v}_i^2}{c^2}\right)\right]. \quad (48)$$

The approximation of $\mathcal{I}_1$ can be accomplished by a straightforward application of the method developed in I. The details, which are quite lengthy, are given in the Appendix.

The approximately relativistic Lagrangian can be obtained from Eqs. (47), (48), and (A19) as

$$L = L_2 + A_2 - V + \mathcal{I}_{\text{PN}}, \quad (49a)$$

where

$$L_2 = \sum_i \left(-m_i c^2 + \tfrac{1}{2} m_i \vec{v}_i^2 + \tfrac{1}{8} m_i \frac{\vec{v}_i^4}{c^2}\right), \quad (49b)$$

$$A_2 = \sum_i \left[I_i \underline{w}_i \cdot \underline{\tau}_i - \tfrac{1}{4} L_i \dot{\underline{\tau}}_i^2 \left(1 + \tfrac{1}{2} \frac{\vec{v}_i^2}{c^2}\right)\right]. \quad (49c)$$

V is the nonrelativistic potential energy given in Eq. (42), and the post-Newtonian interaction is given by

$$\begin{aligned}
\mathcal{I}_{\text{PN}} = &-\frac{1}{c} \sum_{i<j}\sum \left[(\vec{v}_i - \vec{v}_j) \cdot \vec{r}_{ij} A_{ij} + \dot{\tau}_{ia} \frac{\partial B_{ij}}{\partial \tau_{ia}} - \dot{\tau}_{ja} \frac{\partial B_{ij}}{\partial \tau_{ja}}\right] \\
&+ \frac{1}{2c^2} \sum_{i<j}\sum \Bigg[\vec{v}_i \cdot \vec{v}_j V_{ij} - \vec{v}_i \cdot \vec{r}_{ij} \vec{v}_j \cdot \vec{r}_{ij} \frac{1}{r_{ij}} \frac{\partial V_{ij}}{\partial r_{ij}} + (\vec{v}_i - \vec{v}_j)^2 (V_{ij} + X_{ij}) + [(\vec{v}_i - \vec{v}_j) \cdot \vec{r}_{ij}]^2 Y_{ij} \\
&\quad + (\vec{v}_i^2 - \vec{v}_i \cdot \vec{v}_j) W_{ij} - [(\vec{v}_i \cdot \vec{r}_{ij})^2 - \vec{v}_i \cdot \vec{r}_{ij} \vec{v}_j \cdot \vec{r}_{ij}] \frac{1}{r_{ij}} \frac{\partial W_{ij}}{\partial r_{ij}} + \vec{v}_i \cdot \vec{r}_{ij} \dot{\tau}_{ja} \left(\frac{\partial V_{ij}}{\partial \tau_{ja}} + \frac{\partial W_{ij}}{\partial \tau_{ja}}\right) \\
&\quad - \vec{v}_j \cdot \vec{r}_{ij} \dot{\tau}_{ia} \left(\frac{\partial V_{ij}}{\partial \tau_{ia}} - \frac{\partial W_{ij}}{\partial \tau_{ia}}\right) + \dot{\tau}_{ia} \dot{\tau}_{jb} \frac{\partial^2 Z_{ij}}{\partial \tau_{ia} \partial \tau_{jb}}\Bigg]. \quad (49d)
\end{aligned}$$

The seven functions of r_{ij}, $\underline{\tau}_i$, and $\underline{\tau}_j$ appearing in Eq. (49d) are determined by the relativistic interaction kernel U_{ij} as described in Eqs. (A17) and (A20). If U_{ij} is symmetric upon interchange of the space-time variables of particles i and j, then both B_{ij} and W_{ij} vanish.

V. THE APPROXIMATE CONSERVATION LAWS

The approximate Lagrangian (49) is invariant under the seven-parameter group of time and space translations and spatial rotations, and either the one-parameter group or three-parameter group of charge-space rotations for charge-dependent and charge-independent interactions, respectively. The associated conservation laws established by Noether's theorem are

$$\frac{dE}{dt} = 0, \quad E \equiv \sum_i (\vec{p}_i \cdot \vec{v}_i + I_i \underline{T}_i \cdot \underline{w}_i) - L,$$

$$\vec{p}_i \equiv \frac{\partial L}{\partial \vec{v}_i}, \quad \underline{T}_i \equiv I_i^{-1} \frac{\partial L}{\partial \underline{w}_i}, \quad (50a)$$

$$\frac{d\vec{P}}{dt} = 0, \quad \vec{P} \equiv \sum_i \vec{p}_i, \quad (50b)$$

$$\frac{d\vec{J}}{dt} = 0, \quad \vec{J} \equiv \sum_i \vec{r}_i \times \vec{p}_i, \quad (50c)$$

$$\frac{d\underline{T}}{dt} = 0, \quad \underline{T} \equiv \sum_i I_i \underline{T}_i, \quad (51)$$

corresponding to the conservation of energy, of linear and angular momentum, and of total isospin (for arbitrary charge-space rotations).

To order c^{-2}, the charge on particle i is

$$q_i(t) = e\left[\tfrac{1}{2} Y_i + I_i \tau_{i3} + \tfrac{1}{2} L_i \underline{\gamma} \cdot \dot{\underline{\tau}}_i \wedge \underline{\tau}_i \left(1 + \tfrac{1}{2} \frac{\vec{v}_i^2}{c^2}\right)\right], \quad (52)$$

which results from using Eq. (45) in (5) and expanding to order c^{-2}. The total charge is given by

$$Q = e\left(\sum_i \tfrac{1}{2} Y_i + T_3\right), \quad (53)$$

which, by Eq. (51), is conserved both for charge-dependent and charge-independent interactions.

The conservation laws (50) and (51) are valid for both Newtonian and approximately relativistic Lagrangians, since they result from invariance properties which apply to both Newtonian and relativistic variational principles, and thus to approximately relativistic principles. However, the center-of-mass theorem for Galilei-invariant and for Poincaré-invariant particle systems are generated by different transformations relating inertial frames in relative motion. These do not correspond to exact invariances of the approximate Lagrangian.

Instead, the approximately relativistic Lagrangian is invariant under another three-parameter set of infinitesimal transformations[7,11]:

$$\delta t = \frac{\vec{\epsilon}\cdot\vec{R}(t)}{c^2}, \quad \delta\vec{r}_i = \vec{\epsilon}\, t, \quad \delta\underline{\theta}_i = 0, \tag{54a}$$

which leads to a center-of-mass theorem

$$\frac{d\vec{G}}{dt} = 0, \quad \vec{G} = \frac{E}{c^2}\vec{R} - \vec{P}t, \tag{54b}$$

where it is sufficient to calculate $\vec{R}$ from

$$\vec{R}(t) = \frac{c^2}{E}\int^t d\bar{t}\,\vec{P}(\vec{r}_i(\bar{t}), \vec{v}_i(\bar{t}); \underline{\tau}_i(\bar{t}), \underline{w}_i(\bar{t})) \tag{54c}$$

with the help of the equations of motion.

The conserved quantities can be expressed in terms of the canonical momenta $\vec{p}_i$. We have

$$\vec{p}_i \equiv \frac{\partial L}{\partial \vec{v}_i} = \left(m_i - \frac{1}{4c^2}L_i\dot{\underline{\tau}}_i^2\right)\vec{v}_i + \frac{1}{2c^2}(m_i\vec{v}_i^2)\vec{v}_i - \frac{1}{c}\left(\sum_{j>i}A_{ij}\vec{r}_{ij} - \sum_{j<i}A_{ji}\vec{r}_{ji}\right)$$
$$+\frac{1}{2c^2}\sum_{j>i}\Big[\vec{v}_jV_{ij} - \vec{r}_{ij}(\vec{v}_j\cdot\vec{r}_{ij})\frac{1}{r_{ij}}\frac{\partial V_{ij}}{\partial r_{ij}} + 2(\vec{v}_i-\vec{v}_j)(V_{ij}+X_{ij})$$
$$+2\vec{r}_{ij}(\vec{v}_i-\vec{v}_j)\cdot\vec{r}_{ij}Y_{ij} + (2\vec{v}_i-\vec{v}_j)W_{ij} + \vec{r}_{ij}(\vec{v}_j\cdot\vec{r}_{ij})\frac{1}{r_{ij}}\frac{\partial W_{ij}}{\partial r_{ij}}\Big]$$
$$+\frac{1}{2c^2}\sum_{j<i}\Big[\vec{v}_jV_{ji} - \vec{r}_{ji}(\vec{v}_j\cdot\vec{r}_{ji})\frac{1}{r_{ji}}\frac{\partial V_{ji}}{\partial r_{ji}} - 2(\vec{v}_j-\vec{v}_i)(V_{ji}+X_{ji})$$
$$-2\vec{r}_{ji}(\vec{v}_j-\vec{v}_i)\cdot\vec{r}_{ji}Y_{ji} - \vec{v}_jW_{ji} - \vec{r}_{ji}(2\vec{v}_i\cdot\vec{r}_{ji} - \vec{v}_j\cdot\vec{r}_{ji})\frac{1}{r_{ji}}\frac{\partial W_{ji}}{\partial r_{ji}}\Big]$$
$$+\frac{1}{2c^2}\left[\sum_{j>i}\vec{r}_{ij}\dot{\tau}_{ja}\left(\frac{\partial W_{ij}}{\partial\tau_{ja}} + \frac{\partial V_{ij}}{\partial\tau_{ja}}\right) + \sum_{j<i}\vec{r}_{ji}\dot{\tau}_{ja}\left(\frac{\partial W_{ji}}{\partial\tau_{ja}} - \frac{\partial V_{ji}}{\partial\tau_{ja}}\right)\right]. \tag{55}$$

Similarly the generalized canonical isomomentum $\underline{T}_i$ equals

$$\underline{T}_i \equiv I_i^{-1}\frac{\partial L}{\partial\underline{w}_i} = \underline{\tau}_i + \tfrac{1}{2}I_i^{-1}L_i\dot{\underline{\tau}}_i\wedge\underline{\tau}_i\left(1+\tfrac{1}{2}\frac{\vec{v}_i^2}{c^2}\right) + \frac{I_i^{-1}}{c}\left(\sum_{j>i}\frac{\partial B_{ij}}{\partial\underline{\tau}_i}\wedge\underline{\tau}_i - \sum_{j<i}\frac{\partial B_{ji}}{\partial\underline{\tau}_i}\wedge\underline{\tau}_i\right)$$
$$+\frac{I_i^{-1}}{2c^2}\sum_{j>i}\left[\vec{v}_j\cdot\vec{r}_{ij}\left(\frac{\partial V_{ij}}{\partial\underline{\tau}_i} - \frac{\partial W_{ij}}{\partial\underline{\tau}_i}\right)\wedge\underline{\tau}_i - \dot{\tau}_{jb}\frac{\partial^2 Z_{ij}}{\partial\tau_{jb}\partial\underline{\tau}_i}\wedge\underline{\tau}_i\right]$$
$$-\frac{I_i^{-1}}{2c^2}\sum_{j<i}\left[\vec{v}_j\cdot\vec{r}_{ji}\left(\frac{\partial V_{ji}}{\partial\underline{\tau}_i} + \frac{\partial W_{ji}}{\partial\underline{\tau}_i}\right)\wedge\underline{\tau}_i + \dot{\tau}_{jb}\frac{\partial^2 Z_{ji}}{\partial\tau_{jb}\partial\underline{\tau}_i}\wedge\underline{\tau}_i\right]. \tag{56}$$

Using Eq. (55) in (50b) and simplifying gives

$$\vec{P} \equiv \sum_i \vec{p}_i = \vec{P}_V + \vec{P}_W, \tag{57a}$$

where

$$\vec{P}_V \equiv \sum_i\left(m_i - \frac{1}{4c^2}L_i\dot{\underline{\tau}}_i^2 + \frac{1}{2c^2}m_i\vec{v}_i^2\right)\vec{v}_i + \frac{1}{2c^2}\sum_{i<j}\sum\Big[(\vec{v}_i+\vec{v}_j)V_{ij} - \vec{r}_{ij}(\vec{v}_i+\vec{v}_j)\cdot\vec{r}_{ij}\frac{1}{r_{ij}}\frac{\partial V_{ij}}{\partial r_{ij}}$$
$$+\vec{r}_{ij}\left(-\dot{\tau}_{ia}\frac{\partial V_{ij}}{\partial\tau_{ia}} + \dot{\tau}_{ja}\frac{\partial V_{ij}}{\partial\tau_{ja}}\right)\Big], \tag{57b}$$

$$\vec{P}_W \equiv \frac{1}{2c^2}\sum_{i<j}\sum\left[(\vec{v}_i-\vec{v}_j)W_{ij} + \vec{r}_{ij}(\vec{v}_i-\vec{v}_j)\cdot\vec{r}_{ij}\frac{1}{r_{ij}}\frac{\partial W_{ij}}{\partial r_{ij}} + \vec{r}_{ij}\left(\dot{\tau}_{ia}\frac{\partial W_{ij}}{\partial\tau_{ia}} + \dot{\tau}_{ja}\frac{\partial W_{ij}}{\partial\tau_{ja}}\right)\right]. \tag{57c}$$

Thus $\vec{P}$ depends only on V_{ij} and W_{ij}, just as in the neutral case treated in I. Using Eq. (55) in (50c) gives the total angular momentum

$$\vec{J} \equiv \sum_i \vec{r}_i \times \vec{p}_i = \sum_i \left(m_i - \frac{1}{4c^2} L_i \dot{\underline{\tau}}_i{}^2 + \frac{1}{2c^2} m_i \vec{v}_i{}^2\right) \vec{r}_i \times \vec{v}_i$$

$$+ \frac{1}{2c^2} \sum_{i<j}\sum \left\{ (\vec{r}_i \times \vec{v}_j + \vec{r}_j \times \vec{v}_i) V_{ij} + \vec{r}_i \times \vec{r}_j (\vec{v}_i + \vec{v}_j) \cdot \vec{r}_{ij} \frac{1}{r_{ij}} \frac{\partial V_{ij}}{\partial r_{ij}} \right.$$

$$+ 2\vec{r}_{ij} \times (\vec{v}_i - \vec{v}_j)(V_{ij} + X_{ij}) + [\vec{r}_i \times (\vec{v}_i - \vec{v}_j) + \vec{r}_{ij} \times \vec{v}_i] W_{ij}$$

$$\left. - \vec{r}_i \times \vec{r}_j (\vec{v}_i - \vec{v}_j) \cdot \vec{r}_{ij} \frac{1}{r_{ij}} \frac{\partial W_{ij}}{\partial r_{ij}} \right\}$$

$$- \frac{1}{2c^2} \sum_{i<j}\sum \vec{r}_i \times \vec{r}_j \left[\dot{\tau}_{ia} \left(\frac{\partial W_{ij}}{\partial \tau_{ia}} - \frac{\partial V_{ij}}{\partial \tau_{ia}} \right) + \dot{\tau}_{ja} \left(\frac{\partial W_{ij}}{\partial \tau_{ja}} + \frac{\partial V_{ij}}{\partial \tau_{ja}} \right) \right], \tag{58}$$

which again depends only on the same functions that the corresponding $\vec{J}$ depended upon in I.

To obtain the total isospin we insert Eq. (56) into (51). This yields

$$\underline{T} \equiv \sum_i I_i \underline{T}_i = \sum_i \left[I_i \underline{\tau}_i + \tfrac{1}{2} L_i \dot{\underline{\tau}}_i \wedge \underline{\tau}_i \left(1 + \tfrac{1}{2} \frac{\vec{v}_i{}^2}{c^2}\right)\right] + \frac{1}{c} \sum_{i<j}\sum \left(\frac{\partial B_{ij}}{\partial \underline{\tau}_i} \wedge \underline{\tau}_i - \frac{\partial B_{ij}}{\partial \underline{\tau}_j} \wedge \underline{\tau}_j \right)$$

$$+ \frac{1}{2c^2} \sum_{i<j}\sum \left[\vec{v}_j \cdot \vec{r}_{ij} \left(\frac{\partial V_{ij}}{\partial \underline{\tau}_i} - \frac{\partial W_{ij}}{\partial \underline{\tau}_i} \right) \wedge \underline{\tau}_i - \vec{v}_i \cdot \vec{r}_{ij} \left(\frac{\partial V_{ij}}{\partial \underline{\tau}_j} + \frac{\partial W_{ij}}{\partial \underline{\tau}_j} \right) \wedge \underline{\tau}_j \right.$$

$$\left. - \dot{\tau}_{jb} \frac{\partial^2 Z_{ij}}{\partial \tau_{jb} \partial \underline{\tau}_i} \wedge \underline{\tau}_i - \dot{\tau}_{ib} \frac{\partial^2 Z_{ij}}{\partial \tau_{ib} \partial \underline{\tau}_j} \wedge \underline{\tau}_j \right], \tag{59}$$

which does not depend on A_{ij}, X_{ij}, or Y_{ij}.

The total energy is obtained by inserting Eqs. (49), (55), and (56) into (50a). After collecting terms we obtain

$$E \equiv \sum_i (\vec{p}_i \cdot \vec{v}_i + I_i \underline{T}_i \cdot \underline{w}_i) - L$$

$$= \sum_i \left[m_i c^2 + \tfrac{1}{2}\left(m_i - \frac{1}{4c^2} L_i \dot{\underline{\tau}}_i{}^2\right) \vec{v}_i{}^2 + \tfrac{3}{8} m_i \frac{\vec{v}_i{}^4}{c^2} - \tfrac{1}{4} L_i \dot{\underline{\tau}}_i{}^2 \left(1 + \frac{\vec{v}_i{}^2}{c^2}\right)\right]$$

$$+ \frac{1}{2c^2} \sum_{i<j}\sum \left[\vec{v}_i \cdot \vec{v}_j V_{ij} - \vec{v}_i \cdot \vec{r}_{ij} \vec{v}_j \cdot \vec{r}_{ij} \frac{1}{r_{ij}} \frac{\partial V_{ij}}{\partial r_{ij}} + (\vec{v}_i - \vec{v}_j)^2 (V_{ij} + X_{ij}) + [(\vec{v}_i - \vec{v}_j) \cdot \vec{r}_{ij}]^2 Y_{ij} \right.$$

$$+ (\vec{v}_i{}^2 - \vec{v}_i \cdot \vec{v}_j) W_{ij} - [(\vec{v}_i \cdot \vec{r}_{ij})^2 - \vec{v}_i \cdot \vec{r}_{ij} \vec{v}_j \cdot \vec{r}_{ij}] \frac{1}{r_{ij}} \frac{\partial W_{ij}}{\partial r_{ij}} + \vec{v}_i \cdot \vec{r}_{ij} \dot{\tau}_{ja} \left(\frac{\partial V_{ij}}{\partial \tau_{ja}} + \frac{\partial W_{ij}}{\partial \tau_{ja}} \right)$$

$$\left. - \vec{v}_j \cdot \vec{r}_{ij} \dot{\tau}_{ia} \left(\frac{\partial V_{ij}}{\partial \tau_{ia}} - \frac{\partial W_{ij}}{\partial \tau_{ia}} \right) + \dot{\tau}_{ia} \dot{\tau}_{jb} \frac{\partial^2 Z_{ij}}{\partial \tau_{ia} \partial \tau_{jb}} \right], \tag{60}$$

which does not contain any terms of order c^{-1}.

For the conserved center-of-mass quantity (54b), we seek an expression for $(E/c^2)\vec{R}$ such that its time derivative yields $\vec{P}$ given by Eq. (57), as required by Eq. (54c). The appropriate expression is

$$\frac{E}{c^2} \vec{R} = \sum_i \left(m_i - \frac{1}{4c^2} L_i \dot{\underline{\tau}}_i{}^2 + \frac{1}{2c^2} m_i \vec{v}_i{}^2\right) \vec{r}_i$$

$$+ \frac{1}{2c^2} \sum_{i<j}\sum [(\vec{r}_i + \vec{r}_j) V_{ij} + \vec{r}_{ij} W_{ij}], \tag{61}$$

which for $L_i = 0$ has the same form as for the neutral case considered in I. It is clear by inspection of Eq. (57) that $\vec{P}_W$ is the time derivative of

$$\frac{1}{2c^2} \sum_{i<j}\sum \vec{r}_{ij} W_{ij}(r_{ij}, \underline{\tau}_i, \underline{\tau}_j). \tag{62}$$

The time derivative of the remaining terms in Eq. (62) can be shown to equal $\vec{P}_V$ by using the nonrelativistic equations of motion (43) to eliminate the resulting $\ddot{\underline{\tau}}_i$'s and $\vec{a}_i$'s which appear in order c^{-2}; this is facilitated by noting that Eq. (43b) implies that

$$\frac{1}{c^2}\dot{\underline{\tau}}_i\cdot\left[\tfrac{1}{2}L_i\ddot{\underline{\tau}}_i-\left(\sum_{j>i}\frac{\partial V_{ij}}{\partial \underline{\tau}_i}+\sum_{j<i}\frac{\partial V_{ji}}{\partial \underline{\tau}_i}\right)\right]=0. \quad (63)$$

Equation (62) can also be written in the form

$$\frac{E}{c^2}\vec{R}=\sum_i M_i\vec{r}_i\,,$$

$$M_i\equiv m_i-\frac{1}{4c^2}L_i\dot{\underline{\tau}}_i^{\,2}+\frac{1}{2c^2}m_i\vec{v}_i^{\,2} \quad (64)$$

$$+\frac{1}{2c^2}\left[\sum_{j>i}(V_{ij}+W_{ij})+\sum_{j<i}(V_{ji}-W_{ji})\right],$$

which exhibits the same general structure as in the Newtonian case, i.e., a sum of terms each multiplied by a single position vector.

VI. SPECIAL CASES OF APPROXIMATE INTERACTIONS

We shall consider a class of exact interactions which allow the definition of adjunct fields,[2,3,6,27] and give two examples of this class corresponding to charge-symmetric scalar and vector adjunct meson fields[18]; as is well known, the nonrelativistic limit of these two interactions is proportional to the Yukawa potential.

As another example, we shall consider a special case which does not possess adjunct fields, which also implies the Yukawa potential in the limit $c^{-1}\to 0$, but which, unlike the previous examples, leads to correction terms of order c^{-1}.

As discussed in Ref. 3 and in I, the special case of theories possessing adjunct fields can be obtained from variational principles of the form (6) provided the generalized potential V_i given by Eq. (11) can be separated into a sum of terms which are products of two factors, one of which involves only the coordinates z_i^μ of particle i. Then this factor, with the dependence on z_i^μ replaced by x^μ, can be considered as an adjunct potential defined at all points in space and determined by sources which involve all particles other than i. If all particles are to act as if they were field sources that are similar except for the strength of their coupling, then U_{ij} must be symmetric in the variables of particles i and j.

Generalizing the form of U_{ij} given in I, the desired separation is possible when U_{ij} consists of a sum of separable terms each having the form

$$U_{ij}^{(lmnpqs)}=g_ig_jf(\Gamma_i)f(\Gamma_j)\tau_{ij}^p(\tau_{ij}-\Gamma_i\Gamma_j)^q\Upsilon_{ij}^{2s}$$

$$\times O_{i\mu\dots\nu}O_j^{\mu\cdots\nu}\omega_{ij}^l\chi_{ij}^m\zeta_{ij}^m\phi_{ij}^{(lmn)}(\sigma_{ij})\,, \quad (65)$$

where l,m,n,p,q,s are non-negative integers, g_i and g_j are coupling constants, f is an arbitrary function and $O_{i\mu\dots\nu}$ (of rank n) may consist of a sum of terms each built up only from v_i^μ and $\partial/\partial z_i^\mu$. The quantity $\tau_{ij}-\Gamma_i\Gamma_j$ is separable because it can be written in the form

$$\tau_{ij}-\Gamma_i\Gamma_j=\tau_{i1}\tau_{j1}+\tau_{i2}\tau_{j2}$$

$$=\Delta_{ab}\tau_{ia}\tau_{jb}\,, \quad (66a)$$

where the Cartesian tensor

$$\Delta_{ab}\equiv\mathrm{diag}(1,1,0) \quad (66b)$$

is invariant under rotations about the three-axis in charge space. The quantity Υ_{ij} may appear only in even powers because we require U_{ij} to be symmetric in the variables of particles i and j, whereas $\Upsilon_{ij}=-\Upsilon_{ji}$ by Eq. (26).

To facilitate displaying of the separated form of $V_i^{(lmnpqs)}$ corresponding to Eq. (65), we define

$$U_{ji}^{(lmnpqs)}=U_{ij}^{(lmnpqs)}\,, \quad (67)$$

since $U_{ij}^{(lmnpqs)}$ is symmetric, and introduce the notation

$$[\tau_{ia}]_p\equiv\tau_{ia_1}\tau_{ia_2}\cdots\tau_{ia_p}\,,$$

$$[\Delta_{bc}]_q\equiv\Delta_{b_1c_1}\Delta_{b_2c_2}\cdots\Delta_{b_qc_q}\,, \quad (68)$$

$$[v_{i\alpha}]_l\equiv v_{i\alpha_1}v_{i\alpha_2}\cdots v_{i\alpha_l}\,,$$

$$\cdots.$$

Then choosing the arbitrary parameter λ_i to be the proper time τ_i [and thus $\mathfrak{v}_i\to(v_i^\mu v_{i\mu})^{1/2}=1$], inserting Eq. (65) into (11), and using the definitions (46) [with (25)] and (26), we obtain

$$V_i^{(lmnpqs)}=g_ic^mf(\Gamma_i)[\tau_{ia}]_p[\tau_{ib}]_q[(\underline{\gamma}\wedge\underline{\tau}_i)_d]_s[\tau_{ie}]_sO_{i\mu\cdots\nu}[v_{i\alpha}]_l[v_{i\beta}]_m\phi_{i\,a_1\cdots a_p,b_1\cdots b_q,d_1\cdots d_s,e_1\cdots e_s}^{\mu\cdots\nu,\alpha_1\cdots\alpha_l,\beta_1\cdots\beta_m}\,,$$

$$\phi_{i\,a_1\cdots a_p,b_1\cdots b_q,d_1\cdots d_s,e_1\cdots e_s}^{\mu\cdots\nu,\alpha_1\cdots\alpha_l,\beta_1\cdots\beta_m}(z_i^\rho)=\sum_{j\neq i}g_j\int_{-\infty}^{\infty}cd\tau_jf(\Gamma_j)[\tau_{ja}]_p[\Delta_{bc}\tau_{jc}]_q[(\underline{\tau}_j\wedge\underline{\gamma})_e]_s[\tau_{jd}]_s \quad (69)$$

$$\times O_j^{\mu\cdots\nu}[v_j^\alpha]_l[s_{ji}^\beta]_m\zeta_{ij}^m\phi_{ij}^{(lmn)}(\sigma_{ij})\,.$$

While the question of the existence of suitable equations determining the $\phi_i^{\vdots\vdots\vdots}(x^\rho)$ directly from the source distribution is of no concern for a theory based on a variational principle of the form (6), it is possible to establish a connection with known field equations in some cases. As in I, this is possible if $m=0$ and if the $\phi_{ij}^{(l0n)}$, considered as functions G_j of $x^\rho - z_j^\rho$ rather than s_{ij}^ρ, are Green functions satisfying some linear partial differential equations

$$LG_j[c^2\eta_{\mu\nu}(x^\mu - z_j^\mu)(x^\nu - z_j^\nu)] = \frac{4\pi}{c}\delta^4(x^\rho - z_j^\rho) , \tag{70}$$

where L is a linear differential operator and δ^4 is a four-fold product of Dirac δ functions. Then we obtain

$$\begin{gathered} L\phi_{i a_1 \cdots a_p, b_1 \cdots b_q, d_1 \cdots d_s, e_1 \cdots e_s}^{\mu \cdots \nu, \alpha_1 \cdots \alpha_l}(x^\rho) = 4\pi j_{i a_1 \cdots a_p, b_1 \cdots b_q, d_1 \cdots d_s, e_1 \cdots e_s}^{\mu \cdots \nu, \alpha_1 \cdots \alpha_l} , \\ j_{i a_1 \cdots, b_1 \cdots, d_1 \cdots, e_1 \cdots}^{\mu \cdots, \alpha_1 \cdots}(x^\rho) \equiv \sum_{j \neq i} g_j \int_{-\infty}^{\infty} d\tau_j f(\Gamma_j)[\tau_{ja}]_p[\Delta_{bc}\tau_{jc}]_q[(\underline{\tau}_j \wedge \underline{\gamma})_e]_s[\tau_{jd}]_s O_j^{\mu \cdots \nu}[v_j^\alpha]_l \delta^4(x^\rho - z_j^\rho(\tau_j)) , \end{gathered} \tag{71}$$

where $j_i^{\cdots}(x^\rho)$ is the source density of the adjunct field of the ith particle. Special cases of neutral ($f=1$, $p=q=s=0$) theories of direct particle interaction which allow such association with known field theories are discussed in I. The variational principles of charge-symmetric scalar and vector meson theories[18] are discussed later in this section.

Here we shall only consider the approximate Lagrangian corresponding to field-theory related interactions which can be described by a sum of terms of the form (65) with $n=0$, i.e., $O_i = 1$. This case, however, also allows the consideration of interactions for which $l=m=0$ but for which the application of $O_{i\mu \cdots \nu} O_j^{\mu \cdots \nu}$ to $\phi_{ij}^{(00n)}$ leads to a sum of n terms each of the form (65) but with $n=0$, $l=n-r$, and $m=r$,

$$U_{ij}^{(00npqs)} = \sum_{r=1}^{n} U_{ij}^{(n-r, r, 0, pqs)} ; \tag{72}$$

the interaction corresponding to the charge-symmetric vector meson theory, considered later, is of this form.

Thus we consider

$$U_{ij}^{(lm0pqs)} = g_i g_j F_{ij}^{(pqs)} \omega_{ij}^l \chi_{ij}^m \zeta_{ij}^m \phi_{ij}^{(lm0)} , \tag{73}$$

where

$$F_{ij}^{(pqs)} \equiv f(\Gamma_i) f(\Gamma_j) \tau_{ij}^p (\tau_{ij} - \Gamma_i \Gamma_j)^q \Upsilon_{ij}^{2s} .$$

The calculation of the functions defined in Eq. (A17) for U_{ij} given by (73) proceeds exactly as in I and yields

$$\begin{aligned} V_{ij}^{(lm0pqs)} &= F_{ij}^{(pqs)} \mathcal{V}_{ij}^{(lm0)}(r_{ij}) , \\ W_{ij}^{(lm0pqs)} &= 0 , \\ X_{ij}^{(lm0pqs)} &= -(l+m) F_{ij}^{(pqs)} \mathcal{V}_{ij}^{(lm0)} , \\ Y_{ij}^{(lm0pqs)} &= -\frac{m(m-1)}{2m-1} F_{ij}^{(pqs)} \frac{1}{r_{ij}} \frac{d\mathcal{V}_{ij}^{(lm0)}}{dr_{ij}} , \end{aligned} \tag{74a}$$

where

$$\mathcal{V}_{ij}^{(lm0)}(r_{ij}) \equiv g_i g_j (-1)^m \int_{-\infty}^{\infty} d\zeta\, \zeta^{2m} \phi_{ij}^{(lm0)}(\zeta^2 - r^2) . \tag{74b}$$

The functions defined in Eq. (A20) reduce to

$$\begin{aligned} A_{ij} &= B_{ij} = 0 , \\ Z_{ij}^{(lm0pqs)} &= (2m+1) F_{ij}^{(pqs)} \int^{r_{ij}} r_{ij} dr_{ij} \mathcal{V}_{ij}^{(lm0)}(r_{ij}) . \end{aligned} \tag{74c}$$

The relation between Z_{ij} and V_{ij} follows by differentiating Eq. (A20c) with U_{ij} given by (73) and comparing the result with (74b); this relation is valid provided that

$$\left\{ \int^r r dr [\zeta^{2m+1} \phi_{ij}(\zeta^2 - r^2)] \right\} \Bigg|_{\zeta=-\infty}^{\infty} = 0 . \tag{75}$$

Inserting Eqs. (74) into (49d) gives

$$\begin{aligned} \mathcal{G}_{\rm PN}^{(lm0pqs)} = \frac{1}{2c^2} \sum_{i<j}\sum F_{ij}^{(pqs)} \Big[&[(1-l-m)(\vec{v}_i - \vec{v}_j)^2 + \vec{v}_i \cdot \vec{v}_j] \mathcal{V}_{ij}^{(lm0)}(r_{ij}) \\ &- \left(\vec{v}_i \cdot \vec{r}_{ij} \vec{v}_j \cdot \vec{r}_{ij} + \frac{m(m-1)}{2m-1} [(\vec{v}_i - \vec{v}_j) \cdot \vec{r}_{ij}]^2 \right) \frac{1}{r_{ij}} \frac{d\mathcal{V}_{ij}^{(lm0)}}{dr_{ij}} \Big] \\ + \frac{1}{2c^2} \sum_{i<j}\sum \Big[&\left(\vec{v}_i \cdot \vec{r}_{ij} \dot{\tau}_{ja} \frac{\partial F_{ij}^{(pqs)}}{\partial \tau_{ja}} - \vec{v}_j \cdot \vec{r}_{ij} \dot{\tau}_{ia} \frac{\partial F_{ij}^{(pqs)}}{\partial \tau_{ia}} \right) \mathcal{V}_{ij}^{(lm0)} \\ &+ (2m+1) \dot{\tau}_{ia} \dot{\tau}_{jb} \frac{\partial^2 F_{ij}^{(pqs)}}{\partial \tau_{ia} \partial \tau_{jb}} \int^{r_{ij}} r_{ij} dr_{ij} \mathcal{V}_{ij}^{(lm0)} \Big] , \end{aligned} \tag{76a}$$

where

$$V=\sum_{i<j}\sum F_{ij}^{(pqs)}v_{ij}^{(lm0)}(r_{ij}) \tag{76b}$$

is the nonrelativistic potential. $\mathcal{G}_{PN}$ contains no correction terms of order c^{-1} for relativistic interactions of the form (73), but this does not necessarily follow for the more general interactions (65) which allow the definition of adjunct potentials because operators $O_{i\mu\cdots\nu}$ can be constructed such that U_{ij} is not time-reversal invariant.

The action-at-a-distance variational principle of charge-symmetric scalar mesodynamics[18] is given by Eq. (6) with

$$U_{ij}=-g_ig_j\underline{\tau}_i\circ\underline{\tau}_jG_{ij}(\sigma_{ij}) ,$$
$$G_{ij}(\sigma_{ij})=\delta(\sigma_{ij})-\tfrac{1}{2}\theta(\sigma_{ij})\frac{\kappa}{\sigma_{ij}^{1/2}}J_1(\kappa\sigma_{ij}^{1/2}) , \tag{77}$$

where κ is a constant having dimensions of reciprocal length (corresponding to $\mu c/\hbar$ in quantum theory, with μ being the meson mass); J_1 is the Bessel function of order 1, and

$$\theta(\sigma_{ij})\equiv\begin{cases}1, & \sigma_{ij}>0\\ 0, & \sigma_{ij}<0 .\end{cases} \tag{78}$$

Equation (77) is a special case of (73) with $f=1, p=1, q=s=l=m=0$. To obtain the approximate Lagrangian we need only

$$\overline{U}=-g_ig_j\underline{\tau}_i\circ\underline{\tau}_jG(\zeta^2-r^2) . \tag{79}$$

Then the only nonvanishing functions in $\mathcal{G}_{PN}$ are

$$V_{ij}=-g_ig_j\underline{\tau}_i\circ\underline{\tau}_j\int_{-\infty}^{\infty}d\zeta\, G(\zeta^2-r^2) , \tag{80a}$$

$$Z_{ij}=-g_ig_j\underline{\tau}_i\cdot\underline{\tau}_j\int_{-\infty}^{\infty}d\zeta\,\zeta^2G(\zeta^2-r^2) . \tag{80b}$$

These integrals are evaluated in Ref. 24. Equation (80a) reduces to the Yukawa potential

$$V_{ij}=-g_ig_j\underline{\tau}_i\cdot\underline{\tau}_j\frac{e^{-\kappa r_{ij}}}{r_{ij}} , \tag{81a}$$

and Eq. (80b) integrates to

$$Z_{ij}=g_ig_j\underline{\tau}_i\cdot\underline{\tau}_j\kappa^{-1}e^{-\kappa r_{ij}} . \tag{81b}$$

Thus, since Eqs. (81) represent the only nonvanishing functions in $\mathcal{G}_{PN}$, Eq. (49d) takes the form

$$\mathcal{G}_{PN}=\frac{1}{2c^2}\sum_{i<j}\sum g_ig_j\Big\{\underline{\tau}_i\cdot\underline{\tau}_j\Big[-[(\vec{v}_i-\vec{v}_j)^2+\vec{v}_i\cdot\vec{v}_j]\frac{e^{-\kappa r_{ij}}}{r_{ij}}-\vec{v}_i\cdot\vec{r}_{ij}\vec{v}_j\cdot\vec{r}_{ij}\frac{e^{-\kappa r_{ij}}}{r_{ij}^2}\Big(\frac{1}{r_{ij}}+\kappa\Big)\Big]$$
$$-(\vec{v}_i\cdot\vec{r}_{ij}\underline{\tau}_i\cdot\dot{\underline{\tau}}_j-\vec{v}_j\cdot\vec{r}_{ij}\dot{\underline{\tau}}_i\cdot\underline{\tau}_j)\frac{e^{-\kappa r_{ij}}}{r_{ij}}+\dot{\underline{\tau}}_i\cdot\dot{\underline{\tau}}_j\frac{e^{-\kappa r_{ij}}}{\kappa}\Big\} , \tag{82}$$

which is of the form (76) with $l=m=p=s=0, p=1$, and

$$V_{ij}^{(000)}=-g_ig_j\frac{e^{-\kappa r_{ij}}}{r_{ij}} , \quad F_{ij}^{(100)}=\underline{\tau}_i\cdot\underline{\tau}_j . \tag{83}$$

The action-at-a-distance variational principle of charge-symmetric vector mesodynamics[18] is given by Eq. (6) with (choosing proper-time parametrization)

$$U_{ij}=g_ig_j\underline{\tau}_i\circ\underline{\tau}_j\left(v_i^\nu+\frac{1}{\kappa^2c^2}v_i^\rho\frac{\partial^2}{\partial z_i^\rho\partial z_{i\nu}}\right)\left(v_{j\nu}+\frac{1}{\kappa^2c^2}v_j^\lambda\frac{\partial^2}{\partial z_j^\lambda\partial z_j^\nu}\right)G_{ij} , \tag{84}$$

where $G_{ij}(\sigma_{ij})$ is given by Eq. (77). This is of the form (65) with $f=1,\ p=1,\ q=s=l=m=0$, and

$$O_i^\mu=v_i^\mu+\frac{1}{\kappa^2c^2}v_i^\rho\frac{\partial^2}{\partial z_i^\rho\partial z_{i\mu}} . \tag{85}$$

Carrying out the indicated differentiations in Eq. (84) using

$$\frac{\partial\sigma_{ij}}{\partial z_i^\nu}=\frac{\partial\sigma_{ij}}{\partial s_{ij}^\nu}=2c^2s_{ij\nu}=-\frac{\partial\sigma_{ij}}{\partial z_j^\nu} , \tag{86}$$

and subsequently employing definitions (46) with (25) yields

$$U_{ij}=g_ig_j\underline{\tau}_i\cdot\underline{\tau}_j\omega_{ij}\left\{G(\sigma)+\frac{4}{\kappa^2}\left[G+\frac{1}{\kappa^2}(4G_\sigma+2\sigma G_{\sigma\sigma})\right]_\sigma\right\}-\frac{8}{\kappa^2}g_ig_j\underline{\tau}_i\cdot\underline{\tau}_j\chi_{ij}\zeta_{ij}\left[G+\frac{1}{\kappa^2}(4G_\sigma+2\sigma G_{\sigma\sigma})\right]_{\sigma\sigma} , \tag{87}$$

which is a sum of two terms, each of the form (73).

The integrals in Eqs. (A17) and (A20) for U_{ij} given by (87) together with (77) are evaluated in Ref. 24. The results are

$$V_{ij}=g_i g_j \underline{\tau}_i \cdot \underline{\tau}_j \frac{e^{-\kappa r_{ij}}}{r_{ij}}, \quad X_{ij}=-V_{ij} ,$$
$$W_{ij}=Y_{ij}=A_{ij}=B_{ij}=0 , \tag{88}$$
$$Z_{ij}=-g_i g_j \underline{\tau}_i \cdot \underline{\tau}_j \kappa^{-1} e^{-\kappa r_{ij}}+2\kappa^{-2}V_{ij} ,$$

for which Eq. (49d) takes the form

$$\mathcal{G}_{PN}=\frac{1}{2c^2}\sum_{i<j}\sum g_i g_j \underline{\tau}_i \cdot \underline{\tau}_j\left[\vec{v}_i\cdot\vec{v}_j\frac{e^{-\kappa r_{ij}}}{r_{ij}}+\vec{v}_i\cdot\vec{r}_{ij}\,\vec{v}_j\cdot\vec{r}_{ij}\frac{e^{-\kappa r_{ij}}}{r_{ij}^2}\left(\frac{1}{r_{ij}}+\kappa\right)\right]$$
$$+\frac{1}{2c^2}\sum_{i<j}\sum g_i g_j\left[(\vec{v}_i\cdot\vec{r}_{ij}\underline{\tau}_i\cdot\dot{\underline{\tau}}_j-\vec{v}_j\cdot\vec{r}_{ij}\dot{\underline{\tau}}_i\cdot\underline{\tau}_j)\frac{e^{-\kappa r_{ij}}}{r_{ij}}+\dot{\underline{\tau}}_i\cdot\dot{\underline{\tau}}_j\kappa^{-2}e^{-\kappa r_{ij}}\left(\frac{2}{r_{ij}}-\kappa\right)\right]. \tag{89}$$

The interaction kernels of the charged scalar and vector meson theories can be obtained from the charge-symmetric ones given by Eqs. (77) and (84), respectively, by replacing $\underline{\tau}_i \cdot \underline{\tau}_j$ by $\tau_{ij}-\Gamma_i\Gamma_j$ [see Eq. (66a)]. The corresponding approximate Lagrangians can be obtained by employing this substitution in Eqs. (81) for scalar mesons and in Eqs. (88) for vector mesons.

An example of a relativistic interaction which does not allow definition of adjunct fields is given by

$$U_{ij}=g_i g_j \underline{\tau}_i \cdot \underline{\tau}_j \omega_{ij}^l[e^{-a\zeta_{ij}}\theta(\zeta_{ij}) + e^{-a\chi_{ij}}\theta(\chi_{ij})]\delta(\sigma_{ij}) , \tag{90}$$

where a is a constant having dimensions of reciprocal length and the θ function is defined in Eq. (78). This interaction is symmetric and lightlike. From Eqs. (46) it can be seen that ζ_{ij} is proportional to the time difference $t_i - t_j$ in the rest frame of the jth particle and that χ_{ij} is proportional to $t_j - t_i$ in the rest frame of the ith particle. Consequently, since the sign of t_{ij} is an invariant for null separations, Eq. (90) is equivalent to

$$U_{ij}=g_i g_j \underline{\tau}_i \cdot \underline{\tau}_j \omega_{ij}^l \times \begin{cases} e^{-a\zeta_{ij}}\delta(\sigma_{ij}) , & t_i > t_j \\ e^{-a\chi_{ij}}\delta(\sigma_{ij}) , & t_j > t_i . \end{cases} \tag{91}$$

To obtain the approximate Lagrangian corresponding to this interaction, we need

$$\overline{U}=g_i g_j \underline{\tau}_i \cdot \underline{\tau}_j[e^{-a\zeta}\theta(\zeta)+e^{a\zeta}\theta(-\zeta)]\delta(\zeta^2-r^2) ,$$
$$\overline{U}_\zeta=g_i g_j \underline{\tau}_i \cdot \underline{\tau}_j(-a)e^{-a\zeta}\theta(\zeta)\delta(\zeta^2-r^2) ,$$
$$\overline{U}_\chi=g_i g_j \underline{\tau}_i \cdot \underline{\tau}_j(-a)e^{a\zeta}\theta(-\zeta)\delta(\zeta^2-r^2) , \tag{92}$$
$$\overline{U}_\omega=l\overline{U} ,$$
$$\overline{U}_{\chi\chi}=g_i g_j \underline{\tau}_i \cdot \underline{\tau}_j a^2 e^{a\zeta}\theta(-\zeta)\delta(\zeta^2-r^2) ;$$

the terms proportional to $\delta(\zeta)=d\theta(\zeta)/d\zeta$ have been omitted because the arguments of $\delta(\zeta)$ and $\delta(\zeta^2-r^2)$ cannot be zero simultaneously. Inserting Eqs. (92) into (A17) and (A20) yields

$$V_{ij}=g_i g_j \underline{\tau}_i \cdot \underline{\tau}_j \frac{e^{-ar_{ij}}}{r_{ij}} ,$$
$$X_{ij}=(\tfrac{1}{2}ar_{ij}-l)V_{ij} , \tag{93a}$$
$$Y_{ij}=\tfrac{1}{2}a^2V_{ij} , \quad W_{ij}=0 ,$$

and

$$A_{ij}=-\tfrac{1}{2}aV_{ij} , \quad B_{ij}=0 , \quad Z_{ij}=r_{ij}^2V_{ij} , \tag{93b}$$

respectively. The post-Newtonian interaction for this case is given by Eq. (49d) together with Eqs. (93). This example serves to emphasize that requiring U_{ij} to be symmetric in the particle variables is not sufficient to exclude the possibility of nontrivial correction terms of order c^{-1}.

VII. DISCUSSION

In Sec. II of this paper the neutral Poincaré-invariant variational principles of Ref. 3 have been generalized to describe point particles with isospin. The method developed in I for approximating such principles was then employed in Sec. IV to obtain the general form, to order c^{-2}, of the classical approximately relativistic Lagrangian describing point particles with two-body interactions depending on isospin as well as the particles' velocities and interparticle separations. Only those relativistic interactions that possess a static nonrelativistic limit have been considered. The charge on the ith particle is determined by Eq. (5), which reduces to the standard relation for the charge on a particle when $L_i=0$. The application of this approximation procedure is straightforward and results in Eq. (49) with the charge given by Eq. (52).

Unlike the neutral case, effects of order c^{-1} are not excluded for point particles with isospin if their relativistic interaction is not time-reversal invariant, a fact which may be of interest in nuclear and elementary particle physics. In the neutral case, as shown in I, terms of order c^{-1} in the approximate Lagrangian constitute a total time derivative and thus can be omitted. For interactions involving isospin, this is the case only for special classes of interactions; however, these do include the time-reversal invariant interactions corresponding to charge-symmetric scalar and vector meson theories,[18] for which the post-Newtonian interactions are given in Sec. VI. Relativistic interactions that are symmetric in the particles' variables imply $B_{ij}=W_{ij}=0$ in Eq. (49d), but do not exclude effects of order c^{-1}; an example of such an interaction is included in Sec. VI. The appearance of c^{-1} terms is also not restricted to interactions of a non-field-theoretical character because operators $O_{i\mu\cdots\nu}$ can be constructed so that Eq. (65) is not time-reversal invariant.

The form given in Sec. VI of relativistic interactions for which adjunct fields can be defined is a generalization beyond the inclusion of isospin of the form given in I. In special cases, this more general form reduces to a sum of terms each of the form considered in I, multiplied by the appropriate isospin factors; the approximate Lagrangian has been obtained here only for this type of field-theory related interaction.

The exact conservation laws following from the invariance properties of the relativistic variational principle (6) are given in Sec. III. Invariance under tne Poincaré group leads to the usual ten conservation laws. In addition, invariance under rotations about the three-direction in charge space leads to a conserved total charge, while invariance under arbitrary rotations leads also to a conserved total isospin vector, whose third component determines the total charge as in elementary particle physics. The approximate conservation laws obtained in Sec. V follow from the invariance properties of the approximate Lagrangian (49). As in the neutral case, the center-of-mass theorem derived by the method of Ref. 7 contains a center-of-mass coordinate which is a sum of terms each proportional to a single position vector. These approximate conservation laws have also been derived in Ref. 24 by direct approximation of the exact conservation laws.

The approximately relativistic system of N particles with interactions possessing a static nonrelativistic limit is described by $6N+4N$ initial data, while the nonrelativistic system is described by $6N+4N$ initial data for $L_i\neq 0$ and $6N+2N$ initial data for $L_i=0$ (the standard description of charge). Thus, unlike the neutral case, the specification of initial data to order c^{-2} for the approximately relativistic system differs from that of the nonrelativistic system when $L_i=0$, under the assumption of a static nonrelativistic limit. The construction of the Hamiltonian for the case $L_i=0$ must be treated separately from the case $L_i\neq 0$, and the canonical formalism associated with the approximate Lagrangian (49), discussed in Ref. 24, will be presented elsewhere.

APPENDIX

To apply the approximation method developed in I, we use Eqs. (45) in (6b), with the space-time dependence of U_{ij} restricted to the Poincaré invariants (46), and then make a monotonic change of of variable from t_i to ζ_{ij} as in I to obtain

$$\mathcal{G}_1=-\sum\sum_{i<j}\int_{-\infty}^{\infty}dt_j\gamma_j^{-1}\int_{-\infty}^{\infty}d\zeta_{ij}\,\frac{U_{ij}(\sigma_{ij},\omega_{ij},\chi_{ij},\zeta_{ij};\underline{\tau}_i,\underline{\tau}_j)}{\omega_{ij}}\bigg|_+ , \tag{A1}$$

where $|_+$ means that the expression is to be evaluated with

$$t_i=t_j+\frac{1}{c}\left[\zeta_{ij}\gamma_j^{-1}+\frac{\vec{v}_j\cdot\vec{r}_{ij}(t_i,t_j)}{c}\right], \tag{A2}$$

following from the last of Eqs. (46). Equation (A2) is an implicit relation for t_i, which is handled by employing a Lagrange expansion as follows:

$$f(t_i,t_j)=f(t_j,t_j)+\sum_{n=1}^{\infty}\frac{1}{c^n n!}\frac{\partial^{n-1}}{\partial t_i^{n-1}}\left\{\left[\zeta\gamma_j^{-1}+\frac{\vec{v}_j\cdot\vec{r}(t_i,t_j)}{c}\right]^n\frac{\partial f(t_i,t_j)}{\partial t_i}\right\}\bigg|_{t_i=t_j} \tag{A3}$$

for any infinitely differentiable function $f(t_i,\,t_j)$. Here

$$\frac{\partial}{\partial t_i}=\left(\frac{d}{dt_i}\right)_{\zeta,\,t_j\,\text{constant}}. \tag{A4}$$

We shall follow the convention used in I of omitting particle subscripts on quantities where confusion is not likely to arise. The integrand in Eq. (A1) depends explicitly on c^{-1} through $\sigma|_+$, $\omega|_+$, and $\chi|_+$ [see (I43)], and implicitly on c^{-1} through Eq. (A2). Since an approximation is desired only to order c^{-2}, this integrand is first expanded in a Taylor series to second order followed by a Lagrange expansion to the same order.

Because the isospin dependence of U_{ij} embodies no explicit factors of c^{-1}, the Taylor expansion of $U_{ij}/\omega_{ij}|_+$ proceeds exactly as in I and results in (I54), which has the form

$$\begin{aligned}\mathcal{g}_1 \approx -\sum_{i<j}\sum \int_{-\infty}^{\infty} dt_j\left[1-\frac{1}{2c^2}(\vec{v}_i-\vec{v}_j)^2-\frac{1}{2c^2}\vec{v}_j^{\,2}\right]\\ \times\int_{-\infty}^{\infty} d\zeta\Big({}^0U(\zeta)+\frac{1}{c}[2\zeta\,{}^0U_\sigma\vec{v}_j\cdot\vec{r}+{}^0U_\chi(\vec{v}_i-\vec{v}_j)\cdot\vec{r}]\\ +\frac{1}{2c^2}\{2\,{}^0U_\sigma(\vec{v}_j\cdot\vec{r})^2-2\zeta^2\,{}^0U_\sigma\vec{v}_j^{\,2}+4\zeta^2\,{}^0U_{\sigma\chi}[\vec{v}_i\cdot\vec{r}\,\vec{v}_j\cdot\vec{r}-(\vec{v}_j\cdot\vec{r})^2]\\ +{}^0U_\omega(\vec{v}_i-\vec{v}_j)^2+\zeta\,{}^0U_\chi(\vec{v}_j^{\,2}-\vec{v}_i^{\,2})+{}^0U_{\chi\chi}[(\vec{v}_i-\vec{v}_j)\cdot\vec{r}]^2\}\Big)\Big|_+,\end{aligned} \tag{A5}$$

where

$$U_\sigma\equiv\frac{\partial U}{\partial\sigma},\quad U_\omega\equiv\frac{\partial U}{\partial\omega},\quad U_\chi\equiv\frac{\partial U}{\partial\chi}. \tag{A6}$$

The superscript zero represents explicit $c^{-1}\to 0$, so that

$${}^0U(\zeta)\equiv U(\zeta^2-\vec{r}^{\,2}(t_i,t_j),1,-\zeta,\zeta;\underline{\tau}_i(t_i),\underline{\tau}_j(t_j))=U({}^0\sigma,{}^0\omega,{}^0\chi,\zeta;\tau_{ij},\Gamma_i,\Gamma_j,\Upsilon_{ij}), \tag{A7}$$

which differs from (I49) only in the dependence on isospin.

The Lagrange expansion of the term of order c^0 is

$${}^0U(\zeta)|_+\approx\overline{U}(\zeta,r;\underline{\tau}_i,\underline{\tau}_j)+\frac{1}{c}\left[\left(\zeta+\frac{1}{c}\vec{v}_j\cdot\vec{r}\right)\frac{\partial\,{}^0U(\zeta)}{\partial t_i}\right]\Bigg|_{t_i=t_j=t}+\frac{1}{2c^2}\left[\zeta^2\frac{\partial^2\,{}^0U(\zeta)}{\partial t_i^{\,2}}\right]\Bigg|_{t_i=t_j=t}, \tag{A8}$$

where

$$\overline{U}(\zeta,r;\underline{\tau}_i,\underline{\tau}_j)\equiv{}^0U(\zeta)|_{t_i=t_j=t}=U(\zeta^2-\vec{r}^{\,2}(t,t),1,-\zeta,\zeta;\underline{\tau}_i(t),\underline{\tau}_j(t)). \tag{A9}$$

To evaluate Eq. (A8), we need

$$\begin{aligned}\frac{\partial\,{}^0U(\zeta)}{\partial t_i}&={}^0U_\sigma(-2\vec{v}_i\cdot\vec{r})+\frac{\partial\,{}^0U}{\partial\tau_{ia}}\dot{\tau}_{ia},\\ \frac{\partial^2\,{}^0U(\zeta)}{\partial t_i^{\,2}}&=-2[{}^0U_\sigma(\vec{a}_i\cdot\vec{r}+\vec{v}_i^{\,2})+\vec{v}_i\cdot\vec{r}(-2\vec{v}_i\cdot\vec{r}){}^0U_{\sigma\sigma}]-4\frac{\partial\,{}^0U_\sigma}{\partial\tau_{ia}}\vec{v}_i\cdot\vec{r}\,\dot{\tau}_{ia}+\frac{\partial^2\,{}^0U}{\partial\tau_{ia}\partial\tau_{ib}}\dot{\tau}_{ia}\dot{\tau}_{ib}+\frac{\partial\,{}^0U}{\partial\tau_{ia}}\ddot{\tau}_{ia},\end{aligned} \tag{A10}$$

whereby Eq. (A8) takes the form

$$\begin{aligned}{}^0U(\zeta)|_+\approx\overline{U}(\zeta,r;\underline{\tau}_i,\underline{\tau}_j)+\frac{1}{c}\left(\zeta+\frac{1}{c}\vec{v}_j\cdot\vec{r}\right)\left(-2\overline{U}_\sigma\vec{v}_i\cdot\vec{r}+\frac{\partial\overline{U}}{\partial\tau_{ia}}\dot{\tau}_{ia}\right)\\ +\frac{1}{2c^2}\left\{-2\zeta^2[\overline{U}_\sigma(\vec{a}_i\cdot\vec{r}+\vec{v}_i^{\,2})-2(\vec{v}_i\cdot\vec{r})^2\overline{U}_{\sigma\sigma}]-4\zeta^2\frac{\partial\overline{U}_\sigma}{\partial\tau_{ia}}\dot{\tau}_{ia}\vec{v}_i\cdot\vec{r}+\zeta^2\frac{\partial^2\overline{U}}{\partial\tau_{ia}\partial\tau_{ib}}\dot{\tau}_{ia}\dot{\tau}_{ib}+\zeta^2\frac{\partial\overline{U}}{\partial\tau_{ia}}\ddot{\tau}_{ia}\right\};\end{aligned} \tag{A11}$$

it should be understood that all expressions which are coefficients of a barred quantity are evaluated at $t_i=t_j\equiv t$. Similarly, the Lagrange expansion of the terms of order c^{-1} in Eq. (A5) yields

$$\begin{aligned}\frac{1}{c}[2\zeta\vec{v}_j\cdot\vec{r}\,\overline{U}_\sigma+(\vec{v}_i-\vec{v}_j)\cdot\vec{r}\,\overline{U}_\chi]\\ +\frac{\zeta}{2c^2}\Big\{4\zeta\vec{v}_i\cdot\vec{v}_j\overline{U}_\sigma-8\zeta\vec{v}_i\cdot\vec{r}\,\vec{v}_j\cdot\vec{r}\,\overline{U}_{\sigma\sigma}-4[(\vec{v}_i\cdot\vec{r})^2-\vec{v}_i\cdot\vec{r}\,\vec{v}_j\cdot\vec{r}]\overline{U}_{\chi\sigma}+2(\vec{a}_i\cdot\vec{r}+\vec{v}_i^{\,2}-\vec{v}_i\cdot\vec{v}_j)\overline{U}_\chi\\ +4\zeta\vec{v}_j\cdot\vec{r}\frac{\partial\overline{U}_\sigma}{\partial\tau_{ia}}\dot{\tau}_{ia}+2(\vec{v}_i-\vec{v}_j)\cdot\vec{r}\frac{\partial\overline{U}_\chi}{\partial\tau_{ia}}\dot{\tau}_{ia}\Big\}.\end{aligned} \tag{A12}$$

The Lagrange expansion of the terms that are already of order c^{-2} requires only the substitution $t_i \to t_j = t$. Thus, inserting Eqs. (A11) and (A12) in (A5) and simplifying, we obtain

$$\begin{aligned}\mathcal{G}_1 \approx -\sum_{i<j}\sum \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} d\zeta \Big\{ &\overline{U}(\zeta, r; \underline{\tau}_i, \underline{\tau}_j) + \frac{1}{c}\left[(\vec{v}_i - \vec{v}_j)\cdot\vec{r}(\overline{U}_\chi - 2\zeta\overline{U}_\sigma) + \zeta\frac{\partial\overline{U}}{\partial\tau_{ia}}\dot{\tau}_{ia}\right] \\ &+ \frac{1}{2c^2}\Big[(\vec{v}_i - \vec{v}_j)^2(\overline{U}_\omega - \overline{U} + \zeta\overline{U}_\chi - 2\zeta^2\overline{U}_\sigma) - \vec{v}_j^{\,2}\overline{U} + 2(\vec{v}_j\cdot\vec{r})^2\overline{U}_\sigma - 4\vec{v}_i\cdot\vec{r}\,\vec{v}_j\cdot\vec{r}\,\overline{U}_\sigma \\ &+ [(\vec{v}_i - \vec{v}_j)\cdot\vec{r}]^2(\overline{U}_{\chi\chi} - 4\zeta\overline{U}_{\chi\sigma} + 4\zeta^2\overline{U}_{\sigma\sigma}) + \vec{a}_i\cdot\vec{r}(2\zeta\overline{U}_\chi - 2\zeta^2\overline{U}_\sigma) \\ &+ 2(\vec{v}_i - \vec{v}_j)\cdot\vec{r}\,\dot{\tau}_{ia}\left(\zeta\frac{\partial\overline{U}_\chi}{\partial\tau_{ia}} - 2\zeta^2\frac{\partial\overline{U}_\sigma}{\partial\tau_{ia}}\right) \\ &+ 2\vec{v}_j\cdot\vec{r}\,\dot{\tau}_{ia}\frac{\partial\overline{U}}{\partial\tau_{ia}} + \zeta^2\frac{\partial^2\overline{U}}{\partial\tau_{ia}\tau_{ib}}\dot{\tau}_{ia}\dot{\tau}_{ib} + \zeta^2\frac{\partial\overline{U}}{\partial\tau_{ia}}\ddot{\tau}_{ia}\Big]\Big\}. \end{aligned} \tag{A13}$$

In this expression, unlike the neutral case of I, the terms of order c^{-1} do not constitute a total time derivative and thus can not be omitted. We can write them in a form which also includes $\dot{\underline{\tau}}_j$ by adding to the time integral of Eq. (A13) the total time derivative

$$-\frac{d}{dt}\left(\tfrac{1}{2}\int_{-\infty}^{\infty} d\zeta\,\zeta\,\overline{U}\right) = -\tfrac{1}{2}\int_{-\infty}^{\infty} d\zeta\left[-2\zeta\,\overline{U}_\sigma(\vec{v}_i - \vec{v}_j)\cdot\vec{r} + \zeta\frac{\partial\overline{U}}{\partial\tau_{ia}}\dot{\tau}_{ia} + \zeta\frac{\partial\overline{U}}{\partial\tau_{ja}}\dot{\tau}_{ja}\right], \tag{A14}$$

which is irrelevant for the variational principle. The acceleration and $\ddot{\underline{\tau}}_i$ dependence of Eq. (A13) can be removed by integrating by parts and omitting the irrelevant integrated term. We have

$$\begin{aligned}\int_{-\infty}^{\infty} dt&\left[\vec{a}_i\cdot\vec{r}(2\zeta\overline{U}_\chi - 2\zeta^2\overline{U}_\sigma) + \zeta^2\frac{\partial\overline{U}}{\partial\tau_{ia}}\ddot{\tau}_{ia}\right] \\ &= \int_{-\infty}^{\infty} dt\Big\{(-\vec{v}_i^{\,2} + \vec{v}_i\cdot\vec{v}_j)(2\zeta\overline{U}_\chi - 2\zeta^2\overline{U}_\sigma) + [-(\vec{v}_i\cdot\vec{r})^2 + \vec{v}_i\cdot\vec{r}\,\vec{v}_j\cdot\vec{r}]\frac{1}{r}\frac{\partial}{\partial r}(2\zeta\overline{U}_\chi - 2\zeta^2\overline{U}_\sigma) \\ &\quad - \vec{v}_i\cdot\vec{r}\,\dot{\tau}_{ia}\frac{\partial}{\partial\tau_{ia}}(2\zeta\overline{U}_\chi - 2\zeta^2\overline{U}_\sigma) - \vec{v}_i\cdot\vec{r}\,\dot{\tau}_{ja}\frac{\partial}{\partial\tau_{ja}}(2\zeta\overline{U}_\chi - 2\zeta^2\overline{U}_\sigma) + 2(\vec{v}_i - \vec{v}_j)\cdot\vec{r}\,\dot{\tau}_{ia}\zeta^2\frac{\partial\overline{U}_\sigma}{\partial\tau_{ia}} \\ &\quad - \zeta^2\frac{\partial^2\overline{U}}{\partial\tau_{ia}\partial\tau_{ib}}\dot{\tau}_{ia}\dot{\tau}_{ib} - \zeta^2\frac{\partial^2\overline{U}}{\partial\tau_{ia}\partial\tau_{jb}}\dot{\tau}_{ia}\dot{\tau}_{jb}\Big\}. \end{aligned} \tag{A15}$$

Using Eqs. (A14) and (A15) in (A13), we obtain

$$\begin{aligned}\mathcal{G}_1 \approx -\sum_{i<j}\sum \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} d\zeta \Big\{ &\overline{U}(\zeta, r; \underline{\tau}_i, \underline{\tau}_j) + \frac{1}{c}\left[(\vec{v}_i - \vec{v}_j)\cdot\vec{r}(\overline{U}_\chi - \zeta\overline{U}_\sigma) + \tfrac{1}{2}\zeta\frac{\partial\overline{U}}{\partial\tau_{ia}}\dot{\tau}_{ia} - \tfrac{1}{2}\zeta\frac{\partial\overline{U}}{\partial\tau_{ja}}\dot{\tau}_{ja}\right] \\ &- \frac{1}{2c^2}\Big[\vec{v}_j^{\,2}\overline{U} + (\vec{v}_i - \vec{v}_j)^2(\overline{U} - \overline{U}_\omega - \zeta\overline{U}_\chi + 2\zeta^2\overline{U}_\sigma) + (\vec{v}_i^{\,2} - \vec{v}_i\cdot\vec{v}_j)(2\zeta\overline{U}_\chi - 2\zeta^2\overline{U}_\sigma) \\ &+ 2[(\vec{v}_i - \vec{v}_j)\cdot\vec{r}]^2(2\zeta\overline{U}_{\chi\sigma} - 2\zeta^2\overline{U}_{\sigma\sigma}) + [(\vec{v}_i\cdot\vec{r})^2 - \vec{v}_i\cdot\vec{r}\,\vec{v}_j\cdot\vec{r}]\frac{1}{r}\frac{\partial}{\partial r}(2\zeta\overline{U}_\chi - 2\zeta^2\overline{U}_\sigma) \\ &+ 2[2\vec{v}_i\cdot\vec{r}\,\vec{v}_j\cdot\vec{r} - (\vec{v}_j\cdot\vec{r})^2]\overline{U}_\sigma - [(\vec{v}_i - \vec{v}_j)\cdot\vec{r}]^2\overline{U}_{\chi\chi} \\ &+ \vec{v}_j\cdot\vec{r}\,\dot{\tau}_{ia}\frac{\partial}{\partial\tau_{ia}}(2\zeta\overline{U}_\chi - 2\zeta^2\overline{U}_\sigma - 2\overline{U}) + \vec{v}_i\cdot\vec{r}\,\dot{\tau}_{ja}\frac{\partial}{\partial\tau_{ja}}(2\zeta\overline{U}_\chi - 2\zeta^2\overline{U}_\sigma) \\ &+ \zeta^2\frac{\partial^2\overline{U}}{\partial\tau_{ia}\partial\tau_{jb}}\dot{\tau}_{ia}\dot{\tau}_{jb}\Big]\Big\}. \end{aligned} \tag{A16}$$

The terms of order c^0 and c^{-2} that do not contain $\dot{\tau}_i$ or $\dot{\tau}_j$ have the same form as in (I63). Consequently, they can be simplified just as in I, so the nonrelativistic two-body potential energy is given by

$$V_{ij}(r; \underline{\tau}_i, \underline{\tau}_j) \equiv \int_{-\infty}^{\infty} d\zeta\,\overline{U}(\zeta, r; \underline{\tau}_i, \underline{\tau}_j), \tag{A17a}$$

and we define the functions

$$W_{ij}(r;\underline{\tau}_i,\underline{\tau}_j)\equiv\int_{-\infty}^{\infty}d\zeta\,\zeta(\overline{U}_\chi+\overline{U}_\zeta)\ , \tag{A17b}$$

$$X_{ij}(r;\underline{\tau}_i,\underline{\tau}_j)\equiv-\int_{-\infty}^{\infty}d\zeta\,(\overline{U}_\omega+\zeta\,\overline{U}_\zeta), \tag{A17c}$$

$$Y_{ij}(r;\underline{\tau}_i,\underline{\tau}_j)\equiv-\int_{-\infty}^{\infty}d\zeta\,\overline{U}_{\chi\chi}, \tag{A17d}$$

as the natural generalization of the corresponding neutral functions introduced in I. The identity

$$\int_{-\infty}^{\infty}d\zeta\,(2\zeta\,\overline{U}_\chi-2\zeta^2\,\overline{U}_\sigma)=V_{ij}+W_{ij}\,, \tag{A18}$$

proved following (I67), is also valid here provided U and its derivatives vanish sufficiently rapidly as $\zeta\to\pm\infty$. Then, also using Eq. (A18) to simplify the terms linear in $\dot{\tau}_{ia}$ and $\dot{\tau}_{ja}$, Eq. (A16) reduces to

$$\begin{aligned}\mathcal{G}_1\approx-\sum_{i<j}\sum\int_{-\infty}^{\infty}dt\bigg[&V_{ij}(r;\underline{\tau}_i,\underline{\tau}_j)+\frac{1}{c}\left((\vec{v}_i-\vec{v}_j)\cdot\vec{r}A_{ij}+\dot{\tau}_{ia}\frac{\partial B_{ij}}{\partial\tau_{ia}}-\dot{\tau}_{ja}\frac{\partial B_{ij}}{\partial\tau_{ja}}\right)\\&-\frac{1}{2c^2}\bigg(\vec{v}_i\cdot\vec{v}_jV_{ij}-\vec{v}_i\cdot\vec{r}\,\vec{v}_j\cdot\vec{r}\,\frac{1}{r}\frac{\partial V}{\partial r}+(\vec{v}_i-\vec{v}_j)^2(V_{ij}+X_{ij})\\&+[(\vec{v}_i-\vec{v}_j)\cdot\vec{r}]^2Y_{ij}+(\vec{v}_i^{\,2}-\vec{v}_i\cdot\vec{v}_j)W_{ij}-[(\vec{v}_i\cdot\vec{r})^2-\vec{v}_i\cdot\vec{r}\,\vec{v}_j\cdot\vec{r}]\frac{1}{r}\frac{\partial W_{ij}}{\partial r}\\&+\vec{v}_j\cdot\vec{r}\,\dot{\tau}_{ia}\frac{\partial}{\partial\tau_{ia}}(W_{ij}-V_{ij})+\vec{v}_i\cdot\vec{r}\,\dot{\tau}_{ja}\frac{\partial}{\partial\tau_{ja}}(W_{ij}+V_{ij})+\dot{\tau}_{ia}\dot{\tau}_{jb}\frac{\partial^2Z_{ij}}{\partial\tau_{ia}\partial\tau_{jb}}\bigg)\bigg]\,,\end{aligned} \tag{A19}$$

where

$$A_{ij}(r;\underline{\tau}_i,\underline{\tau}_j)\equiv\tfrac{1}{2}\int_{-\infty}^{\infty}d\zeta\,(\overline{U}_\chi+\overline{U}_\zeta)\ , \tag{A20a}$$

$$B_{ij}(r;\underline{\tau}_i,\underline{\tau}_j)\equiv\tfrac{1}{2}\int_{-\infty}^{\infty}d\zeta\,\zeta\,\overline{U}, \tag{A20b}$$

$$Z_{ij}(r;\underline{\tau}_i,\underline{\tau}_j)\equiv\int_{-\infty}^{\infty}d\zeta\,\zeta^2\,\overline{U}. \tag{A20c}$$

A_{ij} is identical to the coefficient of $(\vec{v}_i-\vec{v}_j)\cdot\vec{r}$ in order c^{-1} in Eq. (A16) because

$$\int_{-\infty}^{\infty}d\zeta\,(\overline{U}_\chi-\zeta\,\overline{U}_\sigma)=\tfrac{1}{2}\int_{-\infty}^{\infty}d\zeta(\overline{U}_\chi+\overline{U}_\zeta), \tag{A21}$$

which follows from

$$\frac{d\overline{U}}{d\zeta}=2\zeta\,\overline{U}_\sigma-\overline{U}_\chi+\overline{U}_\zeta\ , \tag{A22}$$

and the assumption that $\overline{U}\to0$ as $\zeta\to\pm\infty$.

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