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Radiation gauge in general relativity

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A new radiation gauge for general relativity is presented. In analogy to the radiation (or Coulomb) gauge for electrodynamics, it leads to spatially covariant elliptic equations for the four gauge variables $g^{0\mu}$ of the spacetime metric. The radiation gauge conditions include as special cases maximal or hyperboloid time slicing and minimal distortion shift vectors. For the linearized theory, our gauge conditions reduce to those of Arnowitt, Deser, and Misner and of Dirac; however, our gauge is equally useful for strong time-dependent gravitational fields. An extensive discussion of appropriate asymptotic boundary conditions is given. Finally, our gauge is compared to and contrasted with the de Donder or harmonic gauge.

I. INTRODUCTION

In any dynamical theory possessing a gauge invariance, such as classical electrodynamics or general relativity, there are nondynamical variables present; the field equations naturally divide into constraint equations and evolution equations. The constraint equations can be used to remove certain longitudinal variables from an initial set. However, it is still necessary to impose gauge conditions on the remaining variables to specify the evolution uniquely. It is with the latter problem that this paper is concerned.

For electrodyanmics these issues were settled long ago. There are two preferred gauge choices: the radiation or Coulomb gauge and the Lorentz gauge. We shall review these gauges in Sec. II. In general relativity, the proper gauge conditions and their relation to the constraint equations were matters of intense interest in the late 1950's and early 1960's. Many investigators[1] were trying to unravel the canonical formalism of general relativity so they could use it to quantize the theory. They were primarily interested in obtaining the "true" Hamiltonian and identifying the dynamical variables in the theory. This required the introduction of a radiation gauge. Various attempts[2] at formulating this gauge were made, but none of them was completely satisfactory.

A new approach, presented here, received its impetus from an entirely different direction. During the last decade, the program[3] to construct numerically solutions to Einstein's field equations has made enormous progress. In this program, it is crucial to use gauge conditions which are adapted to strong fields and time-dependent situations. From this point of view, we returned to the questions of the 1960's, but with a fresh perspective.[4] The result of our investigation is a set of gauge conditions which are covariant under three-dimensional coordinate transformations, which possess simple geometric and physical interpretation, and which stand in almost exact parallel with the traditional radiation gauge of electrodynamics.

The gauge is presented in Sec. III. It is explicitly worked out in Sec. IV for the linearized theory of gravity, so that it can be compared with the earlier gauge conditions of Dirac and of Arnowitt, Deser, and Misner (ADM). Since our approach yields a set of four elliptic equations on a spacelike hypersurface, boundary conditions are of essential importance. We treat these in Sec. V. Having settled the radiation gauge, we briefly review the de Donder or harmonic gauge condition in Sec. VI. This gauge is to general relativity what the Lorentz gauge is to electrodynamics. Our discussion of gauge conditions is summarized in Sec. VII. Two appendices contain some mathematical details.

II. RADIATION GAUGE IN ELECTRODYNAMICS

As a model for our discussion of gauge conditions, we briefly review the situation for electrodynamics in Minkowski spacetime. The decomposition into space plus time is automatically performed by the standard time slices. We allow arbitrary spatial coordinates on these slices. The 4-vector potential splits[5] into a scalar potential φ and a 3-vector potential A_i:

$$A_\mu = (-\varphi, A_i). \tag{2.1}$$

In addition, we introduce as an independent variable the electric field E_i. Because of the gauge invariance of electrodynamics, not all components of E_i are freely specifiable. In particular, the longitudinal values of E_i are constrained by

$$D^i E_i = 4\pi\rho\,, \tag{2.2}$$

where D^i is the spatial covariant derivative. The charge density ρ and the charge current j_i are components of the 4 current

$$J_\mu = (-\rho, j_i). \tag{2.3}$$

The *initial-value problem* of electrodynamics is to give (A_i, E_i, ρ, j_i) at time $t=0$, subject to the constraint (2.2).

The *evolution problem* is to solve for these variables at $t=dt$. The evolution equations are

$$\partial_t A_i = -E_i - D_i\varphi, \tag{2.4}$$

$$\partial_t E_i = R_i - j_i, \tag{2.5}$$

where the R_i contains the second spatial derivatives of A_i:

$$R_i \equiv D^j D_j A_i - D_i D^j A_j. \tag{2.6}$$

These are supplemented by the matter evolution equations,

$$\nabla^\mu J_\mu = \partial_t \rho + D^i j_i = 0. \tag{2.7}$$

Note that j_i does not possess an independent evolution equation.

To use the evolution equations (2.4) and (2.5), we must specify the *gauge variable* φ at time $t=0$. Although it can be specified in a variety of ways, a natural way is to use φ to remove the longitudinal components of A_i, i.e., to require

$$D^i A_i = 0. \tag{2.8}$$

The transversality condition given at $t=0$ is not enough to specify φ. However, if we demand that (2.8) is maintained in time,

$$\partial_t (D^i A_i) = D^i(\partial_t A_i) = 0, \tag{2.9}$$

then using (2.4) we find

$$D^i D_i \varphi = -D^i E_i. \tag{2.10}$$

Substituting from the constraint equation (2.2), this becomes

$$\Delta\varphi = -4\pi\rho, \tag{2.11}$$

where Δ is the spatial covariant Laplacian. This is the *radiation gauge*.

The evolution equations in this gauge are the hyperbolic equations (2.4), (2.5), and (2.7), together with the spatially covariant elliptic gauge equation (2.11). By iterating the above procedure we have an algorithm for (numerically) solving electrodynamics.

III. RADIATION GAUGE IN GENERAL RELATIVITY

We now present our version of the radiation gauge in general relativity. The metric of spacetime is decomposed[6] on some set of spacelike hypersurfaces, called time slices and labeled by $t=$ constant:

$$g_{\mu\nu} = \begin{pmatrix} -\alpha^2 + \beta_m\beta_n\gamma^{mn} & \beta_j \\ \beta_i & \gamma_{ij} \end{pmatrix}. \tag{3.1}$$

The spatial metric γ_{ij} corresponds to A_i, while the set (α, β_i) contains the gauge variables corresponding to φ. For the field corresponding to E_i, we use K_{ij}, the extrinsic curvature tensor of the time slice. Finally, the 4-current J_μ, which serves as source for electrodynamics, is replaced by the energy-momentum tensor $T_{\mu\nu}$, containing the matter density ρ, the matter current j_i, and the spatial stress S_{ij} (Ref. 7):

$$\begin{aligned} \rho &= \alpha^{-2}(T_{00} - 2T_{0i}\beta_j\gamma^{ij} + T_{ij}\beta_k\beta_l\gamma^{ik}\gamma^{jl}), \\ j_i &= -\alpha^{-1}(T_{0i} - T_{ij}\beta_k\gamma^{jk}), \quad S_{ij} = T_{ij}. \end{aligned} \tag{3.2}$$

Because gravity is spin 2, its field variables carry two indices. As a result, we have trace terms or determinant terms in Einstein's equations which do not occur in a vector theory. Also since the gauge group of diffeomorphisms is more complicated than the gauge group of electrodynamics, there are four gauge variables and four constraints in general relativity instead of the one in electrodynamics. The constraints split[8] into a 3-vector piece,

$$D^i(K_{im} - \gamma_{im}K) = j_m, \tag{3.3}$$

corresponding to (2.2) and into a new scalar piece,

$$R + K^2 - K_{ij}K^{ij} = 2\rho, \tag{3.4}$$

where R is the scalar curvature of γ_{ij} and $K = \gamma_{ij}K^{ij}$. As explained in an earlier paper,[9] these can be used to eliminate the longitudinal components W^i of K_{ij} and the determinant γ or the conformal factor ψ of γ_{ij}. This is the initial-value problem of general relativity.

The evolution problem is to solve for $(\gamma_{ij}, K_{ij}, \rho, j_i, S_{ij})$ at a time dt to the future of the initial slice. The Einstein evolution equations are[10]

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + \mathcal{L}_\beta \gamma_{ij}, \tag{3.5}$$

$$\partial_t K_{ij} = \alpha\mathcal{R}_{ij} - \alpha\mathcal{S}_{ij} - \alpha\mathcal{K}_{ij} - D_iD_j\alpha + \mathcal{L}_\beta K_{ij}, \tag{3.6}$$

where

$$\mathcal{R}_{ij} \equiv R_{ij} - \tfrac{1}{4}R\gamma_{ij}, \tag{3.7}$$

$$\mathcal{S}_{ij} \equiv S_{ij} - \tfrac{1}{2}S\gamma_{ij}, \tag{3.8}$$

$$\mathcal{K}_{ij} = 2K_{mi}K^m_j - KK_{ij} + \tfrac{1}{4}(K^2 - K_{mn}K^{mn})\gamma_{ij}, \tag{3.9}$$

$$\mathcal{L}_\beta \gamma_{ij} = D_i\beta_j + D_j\beta_i, \tag{3.10}$$

$$\mathcal{L}_\beta K_{ij} = \beta^m D_m K_{ij} + K_{im}D_j\beta^m + K_{mj}D_i\beta^m. \tag{3.11}$$

Note that (3.5) and the first two terms on the right-hand side of (3.6) are formally analogous to the evolution equations of electrodynamics (2.4) and (2.5). Here R_{ij} is the Ricci tensor formed from first and second spatial derivatives of γ_{ij}, D_i

is the spatial covariant derivative formed from γ_{ij}, and $\mathcal{L}_\beta$ is the Lie derivative along β^i. The last three terms in (3.6) appear in general relativity, but nothing analogous occurs in electrodynamics. The first, $\mathcal{K}_{ij}$, is a nonlinear term, while the last two are extra gauge terms.

The evolution equations (3.5) and (3.6) must be supplemented by the matter evolution equations (∇^μ = spacetime covariant derivative):

$$\nabla^\mu T_{\mu\nu}=0\,. \tag{3.12}$$

Again, we note that the matter terms appearing in the constraints (3.3) and (3.4) are the ones which are evolved by (3.12), while the matter terms appearing in the evolution equations (3.6) are found from constitutive relations.

As in electrodynamics, the gauge variables (α, β^i) must be specified in order to perform an evolution. Mimicking (2.8) and (2.9), we can attempt to impose the transversality condition on γ^{ij} multiplied by some power of the determinant γ:

$$D_i(\gamma^N\gamma^{ij})=0 \tag{3.13}$$

since we have already solved for γ using (3.4).[9] Unfortunately, this expression vanishes identically.

The method used by Dirac[2] and others was to change D_i to the ordinary derivative

$$\partial_i(\gamma^N\gamma^{ij})=0 \tag{3.14}$$

and to require that this condition be preserved in time,

$$\partial_t\partial_i(\gamma^N\gamma^{ij})=0\,. \tag{3.15}$$

If $N=\frac{1}{2}$, then (3.14) states that the spatial coordinates in the initial slice are three-harmonic. Dirac considered this possibility, but chose instead $N=\frac{1}{3}$, to which we return in Sec. IV. In any case, it is true that (3.15) leads to three elliptic equations on β^i. In particular, for $N=\frac{1}{2}$, it has been shown[11] that (3.15) implies

$$\partial_j\{\gamma^{1/2}[(D^i\beta^j+D^j\beta^i-\gamma^{ij}D^k\beta_k)-2\alpha(K^{ij}-\gamma^{ij}K)]\}=0\,. \tag{3.16}$$

The problem with using (3.14) and (3.15) is that the gauge condition is *not* covariant under three-dimensional coordinate transformations as is the electrodynamic radiation gauge (2.9). In fact, (3.14) is used to *define* the spatial coordinates in the initial slice. Equation (3.16) then guarantees that the spatial coordinates in each successive slice are, for instance, three-harmonic [satisfy (3.15) for $N=\frac{1}{2}$]. Our viewpoint is very different. We believe one should be able to use *any* spatial coordinates to label points in the initial spacelike hypersurface. The β^i should be used solely to determine how these coordinates *propagate*[12] off the surface.

Therefore, we propose to replace (3.15) by

$$D_j\partial_t(\gamma^N\gamma^{ij})=0\,, \tag{3.17}$$

which, using (3.5), leads to the following equation for β^i, analogous to (2.10):

$$D_j[D^i\beta^j+D^j\beta^i-2N\gamma^{ij}D^k\beta_k-2\alpha(K^{ij}-N\gamma^{ij}K)]=0\,. \tag{3.18}$$

The advantages of this prescription are that (1) it is three-covariant and can be used with any initial choice of spatial coordinates, (2) it is closer in form to the electrodynamics radiation gauge condition (2.9) now that D^i is used instead of ∂^i, and (3) for certain values of N it has a simple physical and geometric interpretation. Just as in electrodynamics, the constraint equation (3.3) can be used to cast (3.18) into a form parallel to (2.11):

$$\begin{aligned}\Delta\beta^i+(1-2N)D^i(D\cdot\beta)+R^i_l\beta^l\\ =2\alpha j^i+2(K^{il}-N\gamma^{il}K)D_l\alpha+2\alpha(1-N)D^iK\,.\end{aligned} \tag{3.19}$$

Before continuing our discussion of (3.19), in particular with respect to a natural choice of N, we briefly turn to the other gauge variable α. Recall that the constraint (3.4) was used to eliminate the "scalar" part (γ) of γ_{ij} while β^i fixed its longitudinal piece; on the other hand, the constraint (3.3) fixed a longitudinal piece[9] (W^i) of K_{ij}, so we expect α to fix a scalar piece of K_{ij}. We may choose that scalar to be the trace $K\equiv\gamma_{ij}K^{ij}$. As with β^i, it is the specification of the *time derivative* of K that sets the gauge.[12] For any given choice of ∂_tK, symbolized by taking

$$\partial_tK=u(x^i,t)\,, \tag{3.20}$$

the trace of equation (3.6) yields an elliptic equation on α,

$$\Delta\alpha-[K_{ij}K^{ij}+\tfrac{1}{2}(\rho+S)]\alpha-\beta^i\partial_iK=-u(x^i,t)\,, \tag{3.21}$$

where we have used the constraint (3.4). Alternatively, we may write this as

$$\Delta\alpha-[R+K^2-\tfrac{3}{2}(\rho-\tfrac{1}{3}S)]\alpha-\beta^i\partial_iK=-u(x^i,t)\,. \tag{3.22}$$

Thus, again we are led to an elliptic equation for the gauge variable.

The class of radiation gauges (3.19) and (3.22) are as far as our formal analogy with the radiation gauge in electrodynamics will take us. We will now consider a few specialized subclasses. Let us start with α. From (3.20), it is clear that the spatial dependence of $K(x^i)$ must be specified as part of the initial data. The simplest choice is to take[4]

$$K(x^i)=K_0=\text{constant} \tag{3.23}$$

and maintain the spatial homogeneity by setting

$$\partial_t K=u(t). \tag{3.24}$$

The elliptic equation for α then becomes

$$\Delta\alpha-[K_{ij}K^{ij}+\tfrac{1}{2}(\rho+S)]\alpha=-u(t). \tag{3.25}$$

This subset of gauges has been very useful in a wide range of problems in general relativity. These include the maximal[8] ($K_0=0=u$) and hyperboloid[13,14] ($K_0\neq 0$, $u=0$) slicings of black holes[15] and star collapse,[16,17] as well as the slicings[4] ($u\neq 0$) of closed cosmologies. Furthermore, with the choice of (3.23), the elliptic equation for α (3.22) decouples from the elliptic equation for β^i (3.19). This is very useful for numerical work since (3.22) can be solved first and the resulting $\alpha(x^i)$ can then be considered as a given source term in solving (3.19) for β^i. Since the properties of the α gauges have been extensively reviewed recently,[18] we will not pursue the consequences of (3.25) in this paper.

Turning to the β^i equation (3.19), we present a very useful choice of N. Our motivation is that since we are choosing a radiation gauge, we should try to use the β^i freedom to separate[12,17] "coordinate waves" from "pure" gravitational waves. As discussed in earlier papers,[4,9] the conformal three-geometry, represented by

$$\tilde{\gamma}_{ij}=\gamma^{-1/3}\gamma_{ij}, \tag{3.26}$$

can be viewed as the carrier of the dynamical degrees of freedom of γ_{ij}. This suggests a choice of $N=\frac{1}{3}$ in (3.17). For this choice, together with the gauge restriction (3.23) and (3.24), we arrive at the elliptic equation,[12,17]

$$\Delta\beta+\tfrac{1}{3}D^i(D\circ\beta)+R^i_l\beta^l=2\alpha j^i+2(K^{il}-\tfrac{1}{3}K\gamma^{il})D_l\alpha. \tag{3.27}$$

One way to interpret the action of this β^i on γ_{ij} is to note that (3.27) can be derived from a variational principle[17] (see Appendix A):

$$\delta\mathfrak{F}\equiv\delta\int_{\mathcal{T}}(\gamma^{1/3}\partial_t\tilde{\gamma}_{ij})(\gamma^{1/3}\partial_t\tilde{\gamma}_{kl})\gamma^{ik}\gamma^{jl}\sqrt{\gamma}\;d^3x=0. \tag{3.28}$$

Since the metric γ_{ij} and determinant γ are specified on the spacelike hypersurface of integration $\mathcal{T}$, we see that the choice of β^i in (3.27) causes a global *minimization of the time rate of change of the conformal three-geometry*. That is, the radiation gauge for which β^i satisfies (3.27) puts into the gauge variables as much of the coordinate waves as possible for a given slicing (i.e., α gauge). This "moding" out of longitudinal components of $\tilde{\gamma}_{ij}$ is made precise in Appendix A. For reasons more clearly spelled out in another paper,[18] we refer to the gauge (3.27) as the "minimal-distortion" gauge.

There is one other choice of N that we believe is worth considering. Instead of using the conformal metric (corresponding to $N=\frac{1}{3}$), we may use the full three-metric by setting $N=0$. This condition is then $D_j\dot{\gamma}^{ij}=0$ or, equivalently (3.19) with $N=0$. This result may be derived by varying the shift in

$$\int_{\mathcal{T}}(\partial_t\gamma_{ij})(\partial_t\gamma_{kl})\gamma^{ik}\gamma^{jl}\sqrt{\gamma}\;d^3x. \tag{3.29}$$

Therefore, in this case there is a global minimization of the time rate of change of the full three-geometry. However, the principle (3.28) leads to results more closely related to radiation gauge conditions, as demonstrated in the next section.

IV. LINEARIZED GRAVITY

To justify the name radiation gauge, we now consider the linearized vacuum theory of gravity. Besides providing an example of our gauge with gravity waves present, the weak-field limit allows us to compare our gauge to the gauge conditions[2] of Dirac and of Arnowitt, Deser, and Misner (ADM). We separate the spacetime metric into a flat background and a perturbation:

$$g_{\mu\nu}=f_{\mu\nu}+h_{\mu\nu}. \tag{4.1}$$

We use the standard time variable of Minkowski spacetime,

$$f_{00}=-1,\;\; f_{0i}=\partial_t f_{ij}=0, \tag{4.2}$$

but we allow the use of any type of curvilinear spatial coordinates. We introduce $\bar{D}_i$, the covariant derivative with respect to f_{ij}, $\bar{\Delta}=\bar{D}_i\bar{D}^i$ for the covariant Laplacian, and traces are computed using f_{ij}. The three-metric perturbation will be decomposed into a trace and tracefree part:

$$h_{ij}=\tfrac{1}{3}\bar{h}f_{ij}+\psi_{ij},\;\;\psi^i_i=0,\;\;\bar{h}=f^{ij}h_{ij}. \tag{4.3}$$

In terms of these parts, the conformal three-geometry is represented by

$$\tilde{\gamma}_{ij}=f^{-1/3}(f_{ij}+\psi_{ij}),\;\; f\equiv\det(f_{ij}). \tag{4.4}$$

The linearized version of our radiation gauge reads

$$\partial_t\bar{K}=0,\;\;\bar{K}=f^{ij}K_{ij}, \tag{4.5}$$

$$\bar{D}^j\partial_t\tilde{\gamma}_{ij}=\partial_t\bar{D}^j\tilde{\gamma}_{ij}=f^{-1/3}\partial_t\bar{D}^j\psi_{ji}=0. \tag{4.6}$$

That is, we do not specify $\bar{K}$ or $\bar{D}^j\psi_{ij}$ at $t=0$, but rather we demand that their time derivatives vanish. Thus, we preserve whatever value these quantities had originally.

Let us now investigate the ADM gauge. They originally[2] used Cartesian coordinates in flat space. We will rewrite this in three-covariant form. In place of K_{ij}, ADM used the canonical momentum tensor density:

$$\pi^{ij}=\gamma^{1/2}(\gamma^{ij}K-K^{ij}). \tag{4.7}$$

In these terms the ADM isotropic gauge reads [Ref. 2, Eqs. (7-4.22c), (7-4.22d)]

$$\bar{\pi}=0, \tag{4.8}$$

$$\bar{\Delta}\bar{D}^j h_{ij}-\tfrac{1}{4}\bar{D}_i\bar{D}^j\bar{D}^k h_{jk}-\tfrac{1}{4}\bar{D}_i\bar{\Delta}(\bar{h})=0. \tag{4.9}$$

Their time coordinate is chosen by demanding that each time slice be maximal (4.8) to linear order. Thus, it is a special case of our condition (4.5), where $\bar{K}=0$ is taken as the initial value. We can simplify the ADM spatial coordinate condition (4.9) by inserting the decomposition (4.3). This reduces (4.9) to the form

$$(\delta_i^k\bar{\Delta}-\tfrac{1}{4}\bar{D}_i\bar{D}^k)(\bar{D}^j\psi_{jk})=0. \tag{4.10}$$

Since $(\delta_i^k\bar{\Delta}-\frac{1}{4}\bar{D}_iD^k)$ is a flat-space linear second-order elliptic operator, the required vanishing of the vector $(\bar{D}^j\psi_{jk})$ at infinity and (4.10) imply that the third-order condition (4.9) is equivalent to the simple first-order requirement:

$$\bar{D}^j\psi_{jk}=0. \tag{4.11}$$

Again, we see that this is a special case of our radiation gauge (4.6) where $\bar{D}^i\psi_{ij}=0$ is chosen as an initial restriction of the spatial coordinates and is maintained on each slice.

Given that the ADM gauge is a radiation gauge, we should be able to use our equivalent elliptic equations for (α,β^i) to solve for h^{00} and h^{0i}. From (3.25) we see that the equation for α reads (assuming no matter present)

$$\bar{\Delta}\alpha=0, \tag{4.12}$$

which, with the boundary condition $\alpha\to 1$ as $r\to\infty$, implies $\alpha=1$ or

$$h^{00}=0. \tag{4.13}$$

Using this result, (3.27) reads

$$\bar{\Delta}\beta^i+\tfrac{1}{3}\bar{D}^i(\bar{D}\circ\beta)=0, \tag{4.14}$$

which, when we demand the boundary condition $\beta^i\to 0$ as $r\to\infty$, implies $\beta^i=0$ or

$$h^{0i}=0. \tag{4.15}$$

Thus, we have specified the form of the metric perturbation by use of the ADM gauge:

$$h^{0\mu}=0, \tag{4.16}$$

$$h^{ij}=\tfrac{1}{3}f^{ij}\bar{h}+h_{ij}^{tt}, \tag{4.17}$$

where we have written ψ_{ij} as h_{ij}^{tt} to emphasize its spatially transverse-traceless character. For pure gravitational waves the trace term $\bar{h}$ can be set to zero by use of the constraint equation (3.4)[19]; this puts us in the full standard "TT gauge" as described, for instance, by Misner, Thorne, and Wheeler (MTW).[5] If matter is present then this trace is not zero because of ρ in (3.4), h^{00} is nonzero because of the ρ and S in (3.22), and h_{0i} is nonzero because of the j_i in (3.27). The analogs of (4.16) and (4.17) are then given on page 950 of MTW.[5]

Dirac's gauge conditions are quite similar to those above. He proposed the full maximal slicing condition (4.5), which differs from the ADM condition by terms of order $h\partial_t h$ and thus is equivalent to the latter in the linearized formalism.

Dirac's 3-coordinate gauge was stated in a form quite different from ADM's (4.9). As remarked in the last section, he chose (3.14) with $N=\frac{1}{3}$. This is a very awkward condition in practice, as it would, for example, forbid the use of spherical polar coordinates even if γ_{ij} were flat. However, by thinking of (3.14) in terms of Cartesian coordinates, we can easily transcribe Dirac's condition into a covariant statement relative to a background metric f_{ij}. In the linearized theory, this gives

$$\bar{D}_j\tilde{\gamma}^{ij}=f^{-1/3}\bar{D}_j\psi^{ij}=0, \tag{4.18}$$

which is identical to ADM (4.11). Again Dirac's gauge condition imposed on each slice is a special case of the maximal-slicing–minimal-distortion gauge.

Therefore, our radiation gauge in the linearized theory of gravity subsumes the well-known gauges of ADM and Dirac. We see explicitly that the conformal metric $\tilde{\gamma}_{ij}$ is directly related to the gravitational wave variables h_{ij}^{tt} (4.4) and (4.17) in this gauge. However, the gauge holds in strong fields as well, since R_{ij} appears in the elliptic equation for β^i (3.27). Therefore, one would expect that gravitational waves would manifest themselves in the time dependence of $\tilde{\gamma}_{ij}$, even in strong fields, *if* the maximal-slicing–minimal-distortion gauge is used.

For example, when the three-metric is conformally flat at $t=0$, then initially $\tilde{\gamma}_{ij}=f^{-1/3}f_{ij}\equiv\tilde{f}_{ij}$. Later, one has

$$\tilde{\Delta}_{ij}(t,\vec{x})\equiv\tilde{\gamma}_{ij}(t,\vec{x})-\tilde{f}_{ij}(\vec{x})\neq 0, \tag{4.19}$$

where $\vec{x}$ at $t=0$ and $\vec{x}$ at $t\neq 0$ are identified using our choice of β^i. This provides a well-motivated definition of a generalized wave amplitude at a position defined by $\vec{x}$ on $t=$constant. Far from the strong-field region one has

$$\Delta_{ij}(t,\vec{x})\equiv f^{1/3}\tilde{\Delta}_{ij}(t,\vec{x})=h_{ij}^{tt}+O(h^2), \tag{4.20}$$

in agreement with the definition of wave amplitudes in the linearized theory.

V. BOUNDARY CONDITIONS

In order to discuss boundary conditions[20] at spatial infinity in asymptotically flat spacetimes, we return to the perturbation form of γ_{ij} given in (4.1). We define an "asymptotic translational Killing vector" k^i as a solution of

$$\mathcal{L}_k f_{ij} = \bar{D}_i k_j + \bar{D}_j k_i = 0\,, \tag{5.1}$$

$$k_i = f_{ij}k^j\,, \quad k^i k_i = \text{constant}. \tag{5.2}$$

There are three linearly independent solutions $k^i_{(j)}$; we denote Lie derivatives along these directions by $\mathcal{L}_{(j)}$, $j = 1, 2, 3$.

As asymptotic conditions, we assume that the time slices are such that for $r \to \infty$ we have

$$h_{ij} = O(r^{-1})\,, \tag{5.3}$$

$$\mathcal{L}_{(i)} h_{jk} = O(r^{-2})\,, \tag{5.4}$$

$$K_{jk} = O(r^{-2})\,, \tag{5.5}$$

where r is a "radial distance" defined by f_{ij}; for example, $r = (x^2 + y^2 + z^2)^{1/2}$ if $f_{ij} = \delta_{ij}$ (Cartesian coordinates).

These coordinates are appropriate for our elliptic operators (see Appendix B) and guarantee that the total energy and linear momentum of the spacetime are well defined by the standard two-surface integrals.[21] Observe that (5.3)–(5.5) would not be appropriate for hyperboloid type slicings with $K = \text{constant} \neq 0$, where a different set of conditions, which we shall not describe here, must be used. Also notice that (5.3)–(5.5) exclude sources of gravitational radiation that have been active since the infinitely remote past. This would lead to infinite total energy and to some of the h^i_j's and K^i_j's being $O(r^{-1})$, with $O(r^{-1})$ derivatives, for arbitrarily large r.

The simplest reasonable asymptotic conditions on the lapse and shift that ensure the preservation of (5.3)–(5.5) in the time evolution are

$$\alpha - 1 = O(r^{-1})\,, \quad \mathcal{L}_{(i)}\alpha = O(r^{-2})\,, \tag{5.6}$$

$$\beta_{(i)} = \beta^j k_{(i)j} = O(r^{-1})\,, \quad \mathcal{L}_{(i)}\beta_{(j)} = O(r^{-2})\,. \tag{5.7}$$

The postulated fall off of the lapse and shift are achieved automatically by simply demanding[22] that $\alpha \to 1$ and $(\vec{\beta} \cdot \vec{k}_{(i)}) \to 0$ as $r \to \infty$ in elliptic equations such as those arising from maximal slicing and minimal distortion. This is described in Appendix B.

We have emphasized that appropriate gauge conditions are found not by postulating initial coordinate conditions but rather velocity conditions such as

$$D^j \partial_t \tilde{\gamma}_{ij} = \partial_t K = 0\,. \tag{5.8}$$

However, we may ask whether there exist initial restrictions on the Cauchy data that simplify their form and which are automatically preserved in the time development when gauge condition (5.8) is satisfied.

There are such conditions and they require that (5.3)–(5.5) hold together with the following four additional "asymptotic coordinate gauge conditions":

$$f^{1/3} k^i_{(l)} \bar{D}^j \tilde{\gamma}_{ij} = O(r^{-3})\,, \tag{5.9}$$

$$K = O(r^{-3})\,. \tag{5.10}$$

In Cartesian coordinates we have $f_{ij} = \delta_{ij}$, and (5.9) is simply $\partial_i \tilde{\gamma}_{ij} = O(r^{-3})$. The right-hand sides of (5.9) and (5.10) would be $O(r^{-2})$ without this additional restriction on the initial gauge, as one can see by noting that (5.9) would give

$$k^i_{(l)}[\bar{D}_j(h^j_i - \tfrac{1}{3}\delta^j_i \bar{h})] = O(r^{-2}) \tag{5.11}$$

if we only impose (5.3)–(5.5). From (5.8), it is not difficult to show that (5.9) and (5.10) are conserved in the evolution. One can say that Cauchy data satisfying (5.9) and (5.10) are asymptotically conformally simple.

The effect of (5.9) is to put the $O(r^{-1})$ part of the metric into "isotropic" form as nearly as possible, which facilitates interpretation of the metric functions by allowing a direct comparison with Schwarzschild Cauchy data in its familiar isotropic form.

As an example, let us consider data on a slice boosted relative to the standard Schwarzschild slices.[23] We find that (5.3)–(5.5) are satisfied and that one can choose a particular three-coordinate transformation on the boosted slice, so that (5.9) is satisfied to any desired order in V. The $O(r^{-1})$ parts of the metric on the boosted slice acquire the simple form $(8\pi G = c = 1)$

$$\begin{aligned}\gamma_{ij} &= f_{ik}[\delta^k_j(1 + E/4\pi r) + \psi^k_j] \\ &= f_{ij}(1 + E/4\pi r) + \psi_{ij}\,,\end{aligned} \tag{5.12}$$

where

$$\psi^k_j = h^k_j - \tfrac{1}{3}\delta^k_j h^l_l = O(r^{-1})\,, \tag{5.13}$$

$$\mathcal{L}_{(i)}\psi^k_j = O(r^{-2})\,, \tag{5.14}$$

and (5.9) holds:

$$k^i_{(l)}\bar{D}_j \psi^j_i = O(r^{-3})\,. \tag{5.15}$$

Here the total energy $E = (M^2 + P^2)^{1/2}$, M = Schwarzschild mass parameter = rest mass, and P is the magnitude of the linear momentum[21] vector P^i. Notice that E appears in the $O(r^{-1})$ part of the conformal factor just as it does in the standard isotropic rest-frame coordinates. Moreover, the

$O(r^{-1})$ part of ψ^i_j vanishes if and only if $P^i=0$. It seems likely that similar results may hold for any Cauchy data satisfying (5.3)–(5.5) under physically reasonable hypotheses, but this has not been rigorously demonstrated.

Condition (5.10) may be said to state that the slice is asymptotically maximal since it requires $K=O(r^{-3})$ instead of $O(r^{-2})$. In the example of boosted Schwarzschild data cited above, (5.10) is achieved to any order in V by "wiggling" the boosted slice in such a way as to preserve E and P^i. One finds that the $O(r^{-2})$ part of K^i_j comes out in the simple form

$$K^i_j=\frac{-3P^k}{16\pi r^2}(\delta^i_k e_j+f_{jk}e^i-\delta^i_j e_k+e^i e_j e_k), \tag{5.16}$$

where e^i is the unit normal of the standard 2-sphere defined by f_{ij}, for example, in Cartesian coordinates $e^i=x^i/r$.

Let us now consider the question of inner boundary conditions on α and β^i which arises if there is a throat or horizon[24] or a vacuum-matter interface. We shall illustrate the boundary conditions that it would be natural to impose by recalling the boundary integrals in the variational principles (see Appendix A) which gave the maximal-slicing and minimal-distortion equations. In the latter case we had (up to a constant multiplier), for the boundary term (A16),

$$\oint_B \Sigma_{ij}\delta\beta^i d^2S^j\equiv\tfrac{1}{2}\oint_B[(L\beta)_{ij}+2\alpha\sigma_{ij}]\delta\beta^i d^2S^j, \tag{5.17}$$

where B is the inner boundary. We can obtain boundary conditions by requiring the integrand of (5.17) to vanish on B. This happens if $\delta\beta^i=0$ on B, meaning that β^i is fixed on the boundary (Dirichlet condition). Alternatively, we can require that

$$\Sigma_{ij}e^j=0 \text{ on } B, \tag{5.18}$$

where e^j is the unit normal of B. Using (A10) then yields three (Neumann) boundary conditions on the "normal derivative" of β^i, i.e., on $\mathcal{L}_e\beta^i$.

It should be mentioned that in at least some cases it will be possible to solve for the minimal-distortion shift vector with either Dirichlet or Neumann conditions at B. In general, these solutions will differ and give different results for the actual minimum value of $\mathfrak{F}$. Therefore the minimizing of total time rate of change of the conformal three-geometry will only yield a unique result for fixed asymptotic boundary conditions if $B=0$.

For the lapse function, the situation is quite similar. We have for the boundary term

$$\oint_B (D_i\alpha)\delta\alpha d^2S^i. \tag{5.19}$$

Hence we obtain either a Dirichlet condition fixing α on B or a Neumann condition with the normal derivative $e^iD_i\alpha=0$ on B.

If spatial infinity is regarded as a boundary, as it necessarily is in numerical calculations, then situations that called for mixed conditions can occur. For example, in the maximal slicing of Schwarzschild spacetimes of Estabrook *et al.* and Reinhart,[15] one has $e^iD_i\alpha=0$ at the inner boundary (throat) and $\alpha=1$ at the outer boundary (spatial infinity). A similar situation arises for the Boyer-Lindquist β^i of the Kerr spacetimes,[18] which is readily shown to be a minimal-distortion shift vector. Here, $\beta^i\to 0$ at infinity and (5.18) holds on the horizon $r=r_+$.

VI. LORENTZ AND DE DONDER GAUGES

The above concludes our discussion of the radiation gauge. We now briefly review the other major gauge choice: the Lorentz gauge in electrodynamics,

$$\nabla^\mu A_\mu=0, \tag{6.1}$$

and the de Donder[25] or four-harmonic[26] gauge in general relativity,

$$g^{-1/2}\partial_\lambda(g^{1/2}g^{\lambda\mu})=g^{\alpha\beta}\Gamma^\mu_{\alpha\beta}\equiv\Gamma^\mu=0. \tag{6.2}$$

These are, formally, four-dimensional divergence conditions paralleling the three-dimensional radiation gauge conditions (2.8) and (3.14), respectively. However, their consequences are in general quite different because of the indefinite signature of the spacetime metric.

Let us rewrite (6.1) and (6.2) in terms of our gauge variables. The Lorentz condition becomes

$$\partial_t\varphi+D^iA_i=0, \tag{6.3}$$

while the de Donder gauge becomes four coupled equations:

$$\begin{aligned}\Gamma^0=0:&\quad(\partial_t-\beta^j\partial_j)\alpha=-\alpha^2K,\\ \Gamma^i=0:&\quad(\partial_t-\beta^j\partial_j)\beta^i=-\alpha^2[\gamma^{ij}\partial_j\ln\alpha+\gamma^{jk}\Gamma^i_{jk}(\gamma)],\end{aligned} \tag{6.4}$$

where $i,j,k,\ldots=1,2,3$ refers to a basis of spatial coordinates. To use these conditions in solving the evolution problem we evidently need to regard the gauge variables as possessing initial values; these are then updated by hyperbolic equations, in contrast to resolving the elliptic equations which result from the radiation gauge.

There are two sets of equations which can be used in solving the evolution problem in this gauge. The first set consists of simply adding the gauge evolution equations (6.3) or (6.4) to the spatial field evolution equations (2.4) and (2.5) or (3.5) and (3.6). The second equivalent set uses the gauge equations to simplify the form of the spacetime evolution equations. For instance, the flat spacetime electrodynamics equations take the form

$$\nabla^\mu F_{\mu\nu} = \nabla^\mu \nabla_\mu A_\nu - \nabla_\nu(\nabla^\mu A_\mu) = -4\pi J_\nu \tag{6.5}$$

before any gauge condition is imposed. By using the Lorentz gauge (6.1), these simplify to

$$\nabla^\mu \nabla_\mu A_\nu = -4\pi J_\nu \,. \tag{6.6}$$

This form of the equations is termed the *reduced form*.[27] Splitting (6.6) into space and time components yields

$$\Box\begin{pmatrix} -\varphi \\ A_i \end{pmatrix} = -4\pi \begin{pmatrix} -\rho \\ j_i \end{pmatrix} \tag{6.7}$$

and we see that *both* the gauge and dynamic variables satisfy wave equations. A similar result obtains for the Einstein equations.[27]

This decoupling of second-derivative terms is useful for the mathematical proofs of existence and uniqueness of solutions of Einstein's equations.[27] A remarkable fact about the second set of equations is that if the gauge conditions (6.2) are imposed on the initial-time slice, then the evolution by use of the reduced equations preserves these gauge conditions. Thus, for many applications, particularly in weak-field situations for gravity, the use of the simpler reduced equations is very helpful.

In order to compare the de Donder gauge to the radiation gauge, let us return to the linearized vacuum theory of gravity[28] with the notation of Sec. IV. It is more natural for the de Donder gauge to first introduce a spacetime perturbation of the full metric:

$$g_{\mu\nu} = f_{\mu\nu} + h_{\mu\nu} \,. \tag{6.8}$$

Again using covariant derivatives $\overline{\nabla}^\mu$ with respect to $f_{\mu\nu}$ the de Donder condition can be written in background form:

$$\overline{\nabla}^\mu (h_{\mu\nu} - \tfrac{1}{2} f_{\mu\nu} h^\sigma_\sigma) = 0 \,, \quad h^\sigma_\sigma = f^{\mu\nu} h_{\mu\nu} \,. \tag{6.9}$$

We first write (6.9) in 3+1 form as

$$\partial_t \alpha = -\overline{K} \,, \quad \overline{K} = f^{ij} K_{ij} \,, \tag{6.10}$$

$$\partial_t \beta^i = -\overline{D}^i \alpha + \overline{D}_j (h^{ij} - \tfrac{1}{2} \overline{h} f^{ij}) \,. \tag{6.11}$$

Now assume that the initial data (γ_{ij}, K_{ij}) satisfy

$$\overline{K} = 0 \,, \quad \overline{D}_j (h^{ij} - \tfrac{1}{2} f^{ij} \overline{h}) = 0 \,, \tag{6.12}$$

i.e., the data are maximal and *spatially* harmonic (in the background form) to linear order. Choosing the initial values $\alpha = 1$, $\beta^i = 0$, we see from (6.10) and (6.11) that these values remain fixed. Thus, we again force $h^{0\mu} = 0$. From (6.12) and the linearization of the vacuum Hamiltonian constraint (3.4) ($\rho = 0$), we have $\overline{\Delta}\overline{h} = 0$, so $\overline{h} = 0$. We have from (6.12) $h_{ij} = h^{tt}_{ij}$ (vanishing spatial trace and divergence). Therefore we again find the full *TT* gauge in the linearized vacuum theory.

We see that special cases of the radiation gauge and of the harmonic gauge can be made to agree in the linearized vacuum theory, which is not surprising because of the naturalness of the *TT* conditions in this case. However, the coincidence of these two approaches to gauge conditions manifestly does not occur in the linearized nonvacuum theory or the full nonlinear theory. Indeed, the harmonic conditions have at least two drawbacks: (1) They cannot be imposed in a covariant way (in either the spatial or spacetime sense) without the introduction "by hand" of a background metric. The radiation gauge, in contrast, requires no background. (2) They cannot be obtained as minima of natural non-negative functionals, again in contrast to the radiation gauge (Appendix A). Furthermore, we speculate that the harmonic conditions may not be well suited to strong-field situations, e.g., in the presence of a black hole. Reasons for this speculation are that the harmonic conditions are hyperbolic, rather than elliptic, and they do not explicitly account for the presence of sources. [Compare (6.4) and (3.22), (3.27).] On the other hand, there is no doubt about the technical utility of the harmonic conditions in the mathematical analysis of Einstein's equations.

We think it worth noting that the "geometrizing algorithm" of Sec. III can also be applied to the harmonic conditions. When they are imposed, one has $\Gamma^\mu = 0$, $\partial_t \Gamma^\mu = 0$. We write the latter as a "velocity condition,"

$$\partial_t \partial_\nu (\sqrt{-g}\, g^{\mu\nu}) = \partial_\nu \partial_t \sqrt{-g}\, g^{\mu\nu}) = 0 \,. \tag{6.13}$$

Replacing ∂_ν by ∇_ν and ∂_t by the Lie derivative along some timelike vector field t^μ, we have

$$\nabla_\nu \mathcal{L}_t (\sqrt{-g}\, g^{\mu\nu}) = 0 \,, \tag{6.14}$$

or

$$\Box t^\mu + R^\mu_\nu t^\nu = 0 \,, \tag{6.15}$$

where $\Box \equiv g^{\alpha\beta} \nabla_\alpha \nabla_\beta$. These equations are similar to the charge-free Maxwell equations in the Lorentz gauge

$$\Box A^\mu - R^\mu_\nu A^\nu = 0 \,. \tag{6.16}$$

Hence, the new equations are identical to the Maxwell equations in vacuum ($R^\mu_\nu = 0$), but would have, for example, different boundary conditions. Whether (6.15) is useful in practice remains to be seen.

VII. CONCLUSIONS

We have shown how our class of radiation gauges (3.18) and (3.21) stand in analogy to the Coulomb or transverse gauge of electrodynamics.

To demonstrate how they gauge away unwanted coordinate-wave freedom, we showed that they reduce to the ADM and Dirac radiation gauges for the linearized theory of gravity. However, being spatially covariant, our gauge is adapted also to strong time-dependent gravitational fields. The details of the strong-field behavior are left to a companion paper.[18]

The radiation gauge leads to a set of four spatially covariant elliptic equations for the metric gauge variables $g^{0\mu}$. These include the well-known maximal-time-slicing equation and our minimal-distortion shift vector equation. An extensive discussion of boundary conditions shows that our gauge is natural for asymptotically flat spacetimes. The elliptic equations can be derived from variational principles which shed further light on the physical interpretation of the radiation gauge.

Finally, we noted that the other preferred gauge choice, the de Donder gauge or harmonic gauge, can be restricted so that it agrees with the radiation gauge in linearized vacuum theory. However, the de Donder gauge leads to hyperbolic equations for the gauge variables in contrast to the elliptic equations resulting from the radiation gauge. This, together with the lack of spatial covariance for the de Donder gauge, suggests a limited usefulness of the de Donder gauge in strong-field problems.

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APPENDIX A

We consider here the rigorous sense in which the minimal-distortion gauge on β^i separates coordinate or longitudinal modes from dynamical wave modes. For reasons discussed in Sec. IV we focus on the time rate of change of conformal geometry represented by $\tilde{\gamma}_{ij}$. For simplicity we choose to work in terms of

$$2\Sigma_{ij} \equiv \gamma^{1/3}\partial_t\tilde{\gamma}_{ij} = \partial_t\gamma_{ij} - \tfrac{1}{3}\gamma_{ij}(\gamma^{mn}\partial_t\gamma_{mn}), \tag{A1}$$

$$\gamma^{ij}\Sigma_{ij} = 0, \tag{A2}$$

instead of $\partial_t\tilde{\gamma}_{ij}$. We decompose Σ_{ij} in a 3-covariant fashion[29]:

$$\Sigma_{ij} = \Sigma_{ij}^{tt} + (LX)_{ij}, \tag{A3}$$

where

$$\begin{aligned}(LX)_{ij} &\equiv D_iX_j + D_jX_i - \tfrac{2}{3}\gamma_{ij}D^mX_m \\ &= \gamma^{1/3}\mathcal{L}_X\tilde{\gamma}_{ij}\end{aligned} \tag{A4}$$

and

$$D^j\Sigma_{ji}^{tt} = 0. \tag{A5}$$

The following discussion assumes the slice is topologically R^3.

The vector X^i is then the solution of the elliptic equations

$$D^j(LX)_{ji} = D^j\Sigma_{ji}, \tag{A6}$$

where

$$D^j(LX)_{ji} \equiv (\Delta_L X)_i = \Delta X_i + \tfrac{1}{3}D_i(D^kX_k) + R_{ij}X^j. \tag{A7}$$

The two parts Σ_{ij}^{tt} and $(LX)_{ij}$ are globally orthogonal tensors. Thus we see that

$$2\gamma^{-1/3}\Sigma_{ij}^{tt} = (\partial_t\tilde{\gamma}_{ij})^{tt} \tag{A8}$$

represents precisely the part of the time development of the conformal metric that is unaffected by any change of the spatial coordinates from one slice to the next since this effect is given by the term (A4).

If we wish the coordinate effects represented by the longitudinal term $(LX)_{ij}$ to vanish, then, as a result of (A6)

$$D^j\Sigma_{ji} = 0. \tag{A9}$$

To see how to turn this into an equation for β^i, we use (3.5) to evaluate Σ_{ij}:

$$2\Sigma_{ij} = 2\alpha\sigma_{ij} + (L\beta)_{ij}, \tag{A10}$$

where σ_{ij} is the negative of the tracefree part of K_{ij}:

$$\sigma_{ij} = -(K_{ij} - \tfrac{1}{3}K\gamma_{ij}). \tag{A11}$$

The demand (A9) then becomes

$$D^j(L\beta)_{ij} \equiv (\Delta_L\beta)_i = -D^j(2\alpha\sigma_{ij}). \tag{A12}$$

To state this result in words, we note in (A12) that the term $(L\beta)_{ij}$ always represents the effect of a diffeomorphism of the slice to itself, whereas the first term $2\alpha\sigma_{ij}$ will in general generate a true change of the conformal geometry plus another three-dimensional diffeomorphism. The shift vector is therefore chosen in (A12) so as to nullify the latter. In this case

$$\partial_t\tilde{\gamma}_{ij} = (\partial_t\tilde{\gamma}_{ij})^{tt}. \tag{A13}$$

To obtain (A12) from a variational principle,[17] we note that Σ_{ij} is a functional of β^i. We can attempt to minimize in a global or "average" sense the coordinate effects measured by $\Sigma_{ij}(\beta^i)$ in a given slice. To do this, we define on a slice $\mathcal{T}$ the functional

$$\mathfrak{F}[\beta]=\int_{\mathcal{T}} \Sigma_{ij}(\beta)\Sigma^{ij}(\beta)\,dv\,, \tag{A14}$$

where $dv=\gamma^{1/2}d^3x$. Moreover, we will now drop the topological restriction of R^3 previously assumed. The value of $\mathfrak{F}$ is a non-negative measure of the total magnitude of the time-changing conformal geometry. We shall minimize this quantity with respect to the choice of β^i.

From (5.1) we have

$$\mathfrak{F}[\beta]=\tfrac{1}{4}\int_{\mathcal{T}} [4\alpha^2\sigma_{ij}\sigma^{ij}+4\alpha\sigma_{ij}(L\beta)^{ij} + (L\beta)_{ij}(L\beta)^{ij}]\,dv\,. \tag{A15}$$

Note that the first term in the integral is independent of β^i. Taking the first variation and assuming that the boundary terms vanish shows that the Euler-Lagrange equations of this variational problem are the shift equations (A12). Taking the second variation readily shows that (A12) minimizes $\mathfrak{F}$. Hence, the condition (A12) produces the minimal-distortion shift vector relative to given γ_{ij}, σ_{ij}, and α.

For slices that are compact, the variational principle is complete as there are no boundary terms. For asymptotically flat slices, it is necessary to examine the boundary terms in the first variation. Apart from an overall constant multiplying factor, the integral over the boundary $\partial\mathcal{T}$ of the slice $\mathcal{T}$ is given by

$$\tfrac{1}{2}\oint_{\partial\mathcal{T}} [(L\beta)_{ij}+2\alpha\sigma_{ij}]\delta\beta^i d^2S^j=\oint_{\partial\mathcal{T}} \Sigma_{ij}\delta\beta^i d^2S^j\,. \tag{A16}$$

We assume that $\partial\mathcal{T}$ is the union of a two-sphere at spatial infinity and an interior boundary or boundaries B enclosing compact regions. For example, in a black-hole metric, B could be the intersection of a $t=$const slice with the horizon. Both of these types of boundary conditions were discussed in Sec. V.

Finally we note that one can write down a variational principle for the α gauge condition (3.25). The functional $A[\alpha]$ reads

$$A[\alpha]=\int_{\mathcal{T}} [(D_i\alpha)(D^i\alpha)+\alpha^2(K_{ij}K^{ij}+\tfrac{1}{2}\rho+\tfrac{1}{2}S)]\,dv\,. \tag{A17}$$

The surface integral resulting from the integration by parts in the variation of (A17) is

$$\int_{\partial\mathcal{T}} (D_i\alpha)\delta\alpha\, d^2S^i\,,$$

as we used in Sec. V.

APPENDIX B

Elliptic operators on Riemannian spaces have proven to be very useful in analyzing the gravitational constraints and in other problems. In this paper, we have used primarily two such operators in connection with elliptic gauge conditions. This appendix briefly sketches some recent results that can be used to justify rigorously the properties of these operators that we assumed in the text.

The two operators in question are the standard covariant scalar Laplacian $\Delta\alpha=\gamma^{ij}D_iD_j\alpha$ and a vector Laplacian $(\Delta_L\beta)^i=\Delta\beta^i+\frac{1}{3}D^i(D_j\beta^j)+R^i_j\beta^j$. These operators are already well understood on compact slices[30] (without boundary). Here we shall deal with asymptotically flat Riemannian spaces of Euclidean topology. The principal tool is an isomorphism theorem proved by Cantor.[31] Here we shall specialize Cantor's results to second-order operators on three-dimensional spaces. For more details and other applications, see Cantor.[31]

Consider linear equations of the form $A\varphi=\rho$ where φ and ρ are a pair of vectors or a pair of scalars on R^3. A is a linear operator of the form

$$A=A^{ij}_{(2)}\partial_i\partial_j+A^i_{(1)}\partial_i+A_{(0)}\,. \tag{B1}$$

A is assumed to be elliptic in the sense that

$$\det(A^{ij}_{(2)}\xi_i\xi_j)\neq 0$$

for all $x\in R^3$ and for all $\xi_i\neq 0$ in R^3. In analogy to the standard flat-space Poisson equation $\Delta^{(0)}\varphi=\rho$, we need fall-off conditions on ρ that guarantee a reasonable potential $\varphi=O(r^{-1})$ at large distances from the origin. We also need to be sure that our operator A approaches a flat-space elliptic operator such as $\Delta^{(0)}$ at large distances, i.e., the curvature effects vanish towards infinity. For these purposes, following Nirenberg and Walker,[32] Cantor[33] introduced appropriate weighted norms with built-in fall-off conditions.

Introduce $\sigma(x)=(1+|x|^2)^{1/2}$, where $|x|^2=\delta_{ij}x^ix^j$ (Cartesian coordinates are used to simplify the analysis). Note that $\sigma\to|x|=r$ for large $|x|$. Let $|\;|_p$ be the standard L^p norm on scalars $R^3\to R$ or vectors $R^3\to R^3$. Then for $1\leq p\leq\infty$, $\delta\in R$, and s a non-negative integer, define

$$\|f\|_{p,s,\delta}=|f\sigma^\delta|_p+\sum_{i=1}^{3}|(\partial_i f)\sigma^{\delta+1}|_p + \sum_{i,j=1}^{3}|(\partial_i\partial_j f)\sigma^{\delta+2}|_p+\cdots\,, \tag{B2}$$

where the function f and all of its derivatives up to and including order s are included. The weighted Sobolev space $M^p_{s,\delta}$ of Cantor[33] is defined as the completion of the space of all C^∞ functions with compact support (from R^3 to R or to R^3) with respect to the norm $\| \|_{p,s,\delta}$.

As an example, suppose we have the flat-space Poisson equation $\Delta^{(0)}\varphi=\rho$. Then we expect that if ρ falls off faster than r^{-3}, there is a finite amount of mass and φ should go like r^{-1}. Moreover, in integrating for φ in the second-order equation, we expect the two integrations to smooth the falloff of φ so that not only $\varphi=O(r^{-1})$, but also $\partial_i\varphi=O(r^{-2})$ and $\partial_i\partial_j\varphi=O(r^{-3})$. The $M^p_{s,\delta}$ spaces are designed precisely to capture these properties. Actually they include slightly slower fall off than r^{-1} (e.g., $r^{-1}\ln r$), but this point will be ignored in the following. Thus, for the Poisson equation, we expect to get one and only one solution φ for a suitable ρ (i.e., $\Delta^{(0)}$ is an isomorphism) if $\rho\in M^p_{0,2}$ and $\varphi\in M^p_{2,0}$ for $p>3$, say $p=3+\epsilon$, $\epsilon>0$. That is,

$$\|\rho\|_{p,0,2}=|\rho\sigma^2|_{3+\epsilon}\,, \tag{B3}$$

$$\|\varphi\|_{p,2,0}=|\varphi|_{3+\epsilon}+\sum_{i=1}^{3}|\sigma\partial_i\varphi|_{3+\epsilon}+\sum_{i,j=1}^{3}|\sigma^2\partial_i\partial_j\varphi|_{3+\epsilon}\,. \tag{B4}$$

Having illustrated the reasonable properties of the $M^p_{s,\delta}$ spaces and norms, we can now state and briefly comment upon Cantor's isomorphism theorem[31] in the special case of interest here. We repeat that this version is neither the most general nor the strongest form of his theorem.

Let $\bar{A}=\bar{A}^{ij}_{(2)}\partial_i\partial_j$ be an elliptic homogeneous operator with constant coefficients $\bar{A}^{ij}_{(2)}$. Let A be an elliptic operator as in (B2). Assume that $A^{ij}_{(2)}(x)$, $A^i_{(1)}(x)$, and $A_{(0)}(x)$ are continuous and that

$$\lim_{\sigma\to\infty}\sup|(A^{ij}_{(2)}(x)-\bar{A}^{ij}_{(2)}|=0\,, \tag{B5}$$

$$\lim_{\sigma\to\infty}\sup|A^i_{(1)}(x)\sigma|<\epsilon\,,\quad \epsilon>0 \text{ and sufficiently small} \tag{B6}$$

$$\lim_{\sigma\to\infty}\sup|A_{(0)}(x)\sigma^2|<\epsilon\,, \tag{B7}$$

where sup stands for supremum. Moreover, suppose that for all $\lambda\in[0,1]$, $A_\lambda=\bar{A}+\lambda(A-\bar{A})$ is one-to-one. Then the operator A is an isomorphism from $M^p_{2,0}$ to $M^p_{0,2}$ for $p>3$.

In our applications involving Δ, the coefficients are determined by the metric γ_{ij} and its first derivatives. The usual assumptions that these are smooth and that $\gamma_{ij}=\delta_{ij}+O(r^{-1})$, $\Gamma^i_{jk}(\gamma)=O(r^{-2})$ satisfy the hypotheses of the theorem. In the case of the vector operator Δ_L, we need also to require that $[(\partial_l\Gamma^i_{jk})r^2]$ is sufficiently small as $r\to\infty$. This is because, as we have noted, Δ_L has curvature terms. This further requirement is assured by the usual hypothesis that $R_{ij}(\gamma)=O(r^{-3})$. We see that the asymptotic requirements on the lapse and shift stated in Sec. V are suitable. The fact that the relevant operator in the maximal equations is one-to-one was proved in a previous paper on maximal slices.[34] Similarly, in the shift equation, the one-to-one nature of the operator Δ_L follows from the fact that there are no conformal Killing vectors that are asymptotic to zero.[35] Hence, in view of Cantor's results, our elliptic gauge equations can be discussed correctly by relying on knowledge of flat-space equations of the Poisson type.

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[5] Our conventions are that the spacetime metric has a signature $(-+++)$; $i, j, k, \ldots=1,2,3$; $\mu, \nu, \ldots=0,1,2,3$. The units are such that $8\pi G=c=1$. The definitions of curvature, Ricci tensor, etc., agree with those of C. Misner, K. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973).

[6] Arnowitt, Deser, and Misner, Ref. 1. We use α for their N and β^i for their N^i.

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[10] Compare Arnowitt, Deser, and Misner, Ref. 1. They use $\pi^{ij}=(\det\gamma)^{1/2}(K\gamma^{ij}-K^{ij})$. Here K, S, and R are, respectively, the traces of K_{ij}, S_{ij}, and R_{ij} using the spatial metric. We have used (3.4) to write (3.6) in such a form that ρ does not appear explicitly; the only matter variables present are the S_{ij}.

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[13] See, e.g., A. J. Goddard, Ph.D. thesis (Oxford University, 1975) (unpublished). Also, see Ref. 15.

[14] D. Brill (Ref. 15); Y. Choquet-Bruhat, A. Fischer, and J. Marsden, in Proceedings of the 1976 Enrico Fermi

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[19]In linearized theory, we have the conformal factor for which one solves (3.4) is $\sim(1+\frac{1}{3}\bar{h})$ and $\bar{h}$ arises only from terms $\sim(hh)$ and ρ. Hence, in the linearized vacuum theory, $\bar{h}=0$.

[20]Appropriate asymptotic conditions specialized to Cartesian coordinates are described also, for example, in Arnowitt, Deser, and Misner (Ref. 1).

[21]$E=\frac{1}{2}\oint_\infty(\bar{D}^j h^i_j-\bar{D}^i\bar{h})\,d^2S_i$,
$P_j\, k^j_{(e)}=\oint_\infty(K^i_j-\delta^i_j K)\,k^j_{(e)}\,d^2S_i$.

[22]One may also allow "generalized translations" where $\alpha\to O(1)$, e.g., $\alpha\to f(x^i/r)$, and $\beta_{(i)}\to O(1)$; but this has the effect of changing the coordinates near infinity in a physically inessential way (asymptotic gauge transformation).

[23]There are a number of methods that can be used for this calculation. (The same results were obtained independently with different techniques by N. Ō Murchadha—private communication.) Here we outline our method. Treating the Schwarzschild isotropic coordinates $(\bar{t},\bar{x},\bar{y},\bar{z})$ as ordinary Minkowski spacetime coordinates, transform to a boosted $t=$constant slice from the resting slice $\bar{t}=$constant by using a Lorentz boost in the x direction. Here we consider the situation when only terms through $O(V^2)$ are kept. From the boosted spacetime metric in boosted (t,x,y,z) coordinates, compute the Cauchy data γ_{ij} and K_{ij} for the new slice $t=$constant. Keeping terms through $O(r^{-2})$ one verifies (5.3)–(5.5). Using Ref. 21, we find $E=M+\frac{1}{2}MV^2+O(V^4)$, $P^x=MV+O(V^3)$, $P^y=P^z=0$. We then satisfy conditions (5.9) and (5.10) by making asymptotic gauge transformations that preserve this E, $\vec{P}$, and (5.3)–(5.7). This is equivalent to adding a term of $O(V)$ and $O(r)$ to the transformation between $\bar{t}$ and t and to adding terms of $O(V^2)$ and $O(r)$ to each of the transformations from $(\bar{x},\bar{y},\bar{z})$ to (x,y,z). The final results for the boosted data have the forms quoted in (5.10), (5.12)–(5.15), and (5.16).

[24]For a discussion of these problems using (3.16) and maximal slicing, see A. Čadež, Ann. Phys. (N.Y.) 91, 58 (1975). The maximal-slicing–minimal-distortion conditions with $\beta^i=0$ on B in the Schwarzschild metric are discussed in the 1975 Princeton doctoral thesis of K. Eppley (unpublished).

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[28]Extensive discussions of related topics are found in a series of articles of K. S. Thorne and collaborators beginning with Astrophys. J. 200, 245 (1975).

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[35]Proofs of this fact for Killing vectors have been given by M. Cantor (Ref. 31) and Y. Choquet-Bruhat (private communication).