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Relativistic model of a spherical star emitting neutrinos

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We present a simple but complete relativistic model of a spherical star emitting neutrinos, with its basis in the coupled Einstein-Dirac equations. The interior of the star is assumed to be a perfect fluid—described by its energy-matter density, pressure p, and baryon number density n—bounded in space. Matter is considered transparent for neutrinos and the exterior region contains only neutrinos and the gravitational field. The question of compatibility of neutrinos with spherically symmetric gravitational fields is discussed and a redefinition proposed for the physical energy-momentum tensor of neutrinos, which enters the right-hand side of Einstein equations. The analytical solutions are shown to correspond to a description of emission of neutrinos with cooling and contraction of the configuration. The local conservation laws and the junction and boundary conditions of the exterior and interior solutions in the surface of the fluid are studied and allowed to characterize two classes of solutions. In one case the solution describes the stage of neutrino emission with consequent contraction of the configuration of the star immediately before the fluid is totally contained inside its Schwarzschild radius, when the emission of neutrinos ceases. The other possibility can correspond to a quasistatic configuration emitting neutrinos; the relativistic equation of radiative equilibrium for neutrinos is derived and permits us to define the equivalent of a "radiation pressure" for neutrinos, which has an additive contribution to the gravitational pressure and is not a purely relativistic effect.

I. INTRODUCTION

In the study of the interaction of neutrinos and gravitational fields, we can consider the paper by Brill and Wheeler as a basic reference.[1] To the theoretical motivations these two authors have given for the importance of considering the physics of neutrinos in a curved space-time, many substantial arguments have been added in the last twenty years. In cosmology, neutrinos are believed to play an important role in the question of the energy-density of the universe[2,3]; also astrophysical processes connected to the emission and absorption of neutrinos have been extensively discussed where, in certain cases (advanced stages of stellar evolution, etc.),[4] the general theory of relativity becomes important.

In this vein, Misner[5] examined the gravitational collapse of a spherically symmetric perfect fluid with neutrino production, neutrinos being treated phenomenologically as a null fluid[6] and matter transparent to neutrinos. His paper is limited to the formulation of the basic equations, describing "a simple heat transfer process in which internal energy is converted into an outward flux of neutrinos." Also Vaidya,[7] with an analogous model, obtained some nonstatic solutions of Einstein equations for fluid spheres radiating electromagnetic energy. Although both authors use a null-fluid description for radiation (neutrinos or photons), Vaidya was led to consider the partial absorption of the radiation on traversing the medium, which is an effect of nongravitational origin and demands further assumptions.

We present here a class of analytical solutions corresponding to a model which has many similarities with the above two models. We consider a spherically symmetric bounded distribution of a perfect fluid (for instance, a sphere of a degenerate neutral baryon gas under self-gravity) with the following assumptions: (i) Neutrinos in interaction with gravitation are described by spinorial fields in the curved space-time; (ii) matter is transparent to neutrinos; (iii) the model is to be a solution of the coupled Einstein-Dirac equations. The question of compatibility of neutrinos (as source) with a spherically symmetric gravitational field is discussed in Appendix A and a redefinition proposed for the physical energy-momentum tensor of neutrinos. Einstein equations and the junction conditions of the exterior and interior solutions in the surface of the star determine many substantial properties of the model.

For a general review of spinors on a Riemannian space-time, see Ref. 1. Here we use four-component spinors from the point of view of the tetrad formalism, together with the Cartan calculus of differential forms[8] which we use in the calculations.

We choose a tetrad field $e_\alpha^{(A)}(x)$ such that locally the line element is reduced to[9]

$$ds^2 = \eta_{AB}\,\theta^A\theta^B\,, \tag{1.1}$$

where $\theta^A = e_\alpha^{(A)}dx^\alpha$. The Lagrangian for neutrinos is

$$i\sqrt{-g}\,(\bar\psi\gamma^A\nabla_A\psi - \nabla_A\bar\psi\gamma^A\psi)\,, \tag{1.2}$$

with the associated energy-momentum tensor

$$T_{AB}(\psi) = i(\bar\psi\gamma_{(A}\nabla_{B)}\psi - \nabla_{(A}\bar\psi\gamma_{B)}\psi)\,. \tag{1.3}$$

In the above formalism, γ^A are the constant Dirac matrices[10] and $\bar{\psi}=\psi^\dagger\gamma^0$, where γ^0 is the constant matrix. The spinor covariant differentiations are given by

$$\nabla_A\psi = e^\alpha_{(A)}\partial_\alpha\psi - \Gamma_A\psi\,, \qquad \nabla_A\bar{\psi} = e^\alpha_{(A)}\partial_\alpha\bar{\psi} + \bar{\psi}\,\Gamma_A\,, \tag{1.4}$$

where the Fock-Ivanenko coefficients Γ_A have the form

$$\Gamma_A = -\tfrac{1}{4}\gamma_{B\,CA}\gamma^B\gamma^C\,. \tag{1.5}$$

The Ricci rotation coefficients $\gamma_{AB\,C}$ are defined by

$$\gamma_{AB\,C} = -e^\alpha_{(A)\parallel\beta}\,e_{\alpha(B)}e^\beta_{(C)}\,. \tag{1.6}$$

The coupled Einstein-Dirac equations for neutrinos are expressed[11] as

$$R_{AB} - \tfrac{1}{2}\eta_{AB}R = -\kappa[T_{AB}(\psi) + T_{AB}(\text{matter})]\,, \tag{1.7a}$$

$$\gamma^A\nabla_A\psi = 0 \tag{1.7b}$$

and constitute the basis for the study of the interaction of neutrinos and gravitational fields, a solution of which — corresponding to a physical situation where this interaction should be dominant — is the object of the present paper.

II. THE INTERIOR PROBLEM: A CLASS OF SOLUTIONS

The interior region is constituted of a distribution of matter and neutrinos flowing outwards. The matter distribution is a perfect fluid characterized by a total density ρ, pressure p, baryon number density n, and radius r_s, and which emits neutrinos. Neutrinos are assumed to move radially when emitted, i.e., only radial neutrinos contribute to the energy-momentum tensor. Emitted neutrinos interact with gravitation only; they are not scattered or absorbed by adjacent matter. The above model is to be the solution of the Einstein equations, joined to the exterior solution of Sec. III on the surface of the fluid sphere.

In the coordinate system of Appendix A, the line element for the interior region is taken to be

$$ds^2 = \alpha^2 du^2 + 2\,du\,dr - \beta^2(d\theta^2 + \sin^2\theta\, d\varphi^2)\,, \tag{2.1}$$

where α and β are functions of u and r. All necessary calculations are given in Appendixes A and B, and all quantities are expressed in the local inertial frame determined by (A2).

The energy-momentum tensor used for (radial) neutrinos is

$$\tilde{T}_{AB} = 2\pi i \frac{1}{\alpha^2\beta^2}(\Lambda^\dagger\dot{\Lambda} - \dot{\Lambda}^\dagger\Lambda)k_A k_B \tag{2.2}$$

as derived in Appendix A, where $k_A = (1, -1, 0, 0)$. It is conserved locally and has the form of a null-fluid energy-momentum tensor, which is the usual phenomenological description of neutrinos in general relativity.[5,7] Here $\Lambda(u)$ is the arbitrary two-component spinor appearing in solution (A7) of Dirac's equation. The factor $2\pi i(\Lambda^\dagger\dot{\Lambda} - \dot{\Lambda}^\dagger\Lambda)/\alpha^2\beta^2$ can be interpreted as the energy density of neutrinos, as measured locally by the observer with four-velocity $\alpha^{-1}\delta^\mu_0$.

For the energy-momentum tensor of the perfect fluid we assume that an observer comoving with matter has four-velocity

$$V^A = \delta^A_0\,. \tag{2.3}$$

and ρ and p denote respectively the density of mass energy and the pressure of the fluid, as measured locally by the observer (2.3). The total energy-momentum tensor for the interior problem is then expressed as

$$T_{AB}(\text{tot}) = (\rho + p)\delta^0_A\delta^0_B - p\eta_{AB} + \frac{2\mathfrak{L}(u)}{\alpha^2\beta^2}k_A k_B\,, \tag{2.4}$$

where $\mathfrak{L}(u) = \pi i(\Lambda^\dagger\dot{\Lambda} - \dot{\Lambda}^\dagger\Lambda)$. The Einstein field equations $R_{AB} - \frac{1}{2}\eta_{AB}R = -\kappa T^{(\text{tot})}_{AB}$ can be reduced to the set of independent equations

$$R_{00} = -2\kappa\frac{\mathfrak{L}(u)}{\alpha^2\beta^2} - \kappa\rho + \tfrac{1}{2}R\,, \tag{2.5a}$$

$$R_{01} = 2\kappa\frac{\mathfrak{L}(u)}{\alpha^2\beta^2}\,, \tag{2.5b}$$

$$R_{22} = -\kappa p - \tfrac{1}{2}R\,, \tag{2.5c}$$

$$R_{11} + R_{01} - R_{22} = 0\,. \tag{2.5d}$$

From (2.5) we can see that $R_{00} + R_{11} + 2R_{01} = -\kappa(\rho + p)$, which yields the important relation

$$4\frac{\beta''}{\beta}\alpha^2 = -\kappa(\rho + p)\,. \tag{2.6}$$

For physically reasonable equations of state we must then have $\beta''/\beta < 0$ in all points of the interior region. Also the existence of matter in the interior region described by (2.1) depends essentially on β''/β being nonzero.

An inspection of (2.5) and of the expressions in Appendix B can convince us of the difficulty of finding an explicit solution of the field equations. For the present case of matter transparent to neutrinos, we try a solution by separation of variables. We take

$$\alpha = R_1(r)T_1(u)\,, \qquad \beta = R_2(r)T_2(u)\,. \tag{2.7}$$

Using (2.7) in field equation (2.5d) we have

$$\left[R_1R_1'' + R_1'^2 + \frac{R_2''}{R_2}R_1^2 - \left(\frac{R_2'R_1}{R_2}\right)^2\right]T_1^2 + \frac{1}{R_2^2}T_2^{-2} + 2\frac{R_2'}{R_2}\frac{\dot{T}_2}{T_2} = 0\,. \tag{2.8}$$

We make the choice $\dot{T}_2/T_2 = \xi T_2^{-2}$, that implies from (2.8)

$$\dot{T}_2 = \xi T_2^{-1}, \tag{2.9}$$

$$T_1^2 T_2^2 = 1/\eta^2, \tag{2.10}$$

$$R_1 R_1'' + R_1'^2 + \frac{R_2''}{R_2} R_1^2 - \left(\frac{R_2' R_1}{R_2}\right)^2 = -\eta^2\left(\frac{1}{R_2^2} + 2\xi \frac{R_2'}{R_2}\right), \tag{2.11}$$

where ξ and η are arbitrary separation constants. We examine now equation (2.5b). Using (2.7), (2.9), and (2.10), Eq. (2.5b) results in

$$\mathcal{L} = -\frac{\chi}{\kappa} T_2^{-2}, \tag{2.12}$$

$$R_1' = \frac{\eta^2}{\xi}(\chi + 2\xi^2 R_2^2)\frac{1}{2R_1 R_2^2}, \tag{2.13}$$

where χ is a separation constant. From (2.12) we see that the functional $\mathcal{L}(u)$ of the arbitrary spinorial field $\Lambda(u)$ of the neutrino can be described by the metric function T_2 or vice versa. Also relation (2.12) will be useful in relating the sign of χ and of the total mass parameter derivative $\mathring{m}(u)$, for a class of junction conditions.

Substituting (2.13) in (2.11) we obtain

$$\frac{R_2''}{R_2} = \frac{R_2'}{R_2}\left(\frac{R_2'}{R_2} + 2\frac{R_1'}{R_1}\right) - \frac{1}{R_1^2}\left(\frac{\eta^2}{R_2^2} + 4\xi\eta^2\frac{R_2'}{R_2}\right). \tag{2.14}$$

(2.13) and (2.14) constitute a pair of coupled differential equations for the two metric functions R_1 and R_2. Once we have a solution (R_1, R_2), $\rho + p$ is determined by (2.6), and assuming an equation of state $p = \lambda\rho$ for the fluid, the total density is determined as

$$\rho = -\frac{4}{\kappa(1+\lambda)\eta^2}\left(R_1^2 \frac{R_2''}{R_2}\right) T_2^{-2}. \tag{2.15}$$

However we must verify that the solutions given by (2.9), (2.10), (2.13), and (2.14) are compatible with the remaining field equations (2.5a) and (2.5c). To this end we initially remark that equation (2.5c) can be obtained by a convenient linear combination of (2.5a), (2.5b), (2.5d), and (2.7). Also Eq. (2.7) does not impose further restrictions on the solutions but only defines the additional variable $\rho + p$. Thus the only remaining condition to be satisfied by the solutions is Eq. (2.5a), or equivalently (by using anterior expressions)

$$R_{22} = -\kappa(\rho - p). \tag{2.16}$$

Now (2.16) together with (2.6) determine uniquely ρ, p, and the equation of state $p = p(\rho)$. In fact, from the expression for R_{22} we have

$$\kappa\rho = 3\left[8\xi\eta^2\frac{R_2'}{R_2} - 4\frac{R_2'}{R_2}R_1'R_1 - 2\left(\frac{R_2'R_1}{R_2}\right)^2 + 2\frac{\eta^2}{R_2^2}\right]T_1^2, \tag{2.17}$$

$$p = -\tfrac{1}{3}\rho. \tag{2.18}$$

The equation of state (2.18), though satisfying energy conditions,[12] implies the existence of negative scalar pressures. In this case, the neutrinos interact with matter only through gravitation; they are otherwise completely decoupled, because for (2.18) the energy-momentum tensors of neutrinos (2.2) and of matter have independently null covariant divergence.

A possible configuration of the star, described by (2.18), can occur for critical values of densities of matter (above 10^{16} g/cm^3) where general relativistic effects play an important role in the equation of state. As discussed by Sakharov,[13] for very high densities the exchange and correlation gravitational interactions of baryons become comparable in order of magnitude with the Fermi energy, having a decisive contribution in the equation of state and allowing for an effective behavior in which negative scalar pressures appear. For less critical values of density we adhere to the view that the existence of negative scalar pressures is not physically satisfactory. To circumvent this we later introduce in the total energy-momentum tensor (or equivalently, in field equations) a term which describes the cooling of the fluid by emission of neutrinos. For both cases, we shall have physically distinct junction and boundary conditions.

From the local conservation law we obtain the useful expression

$$\left(\frac{R_2''}{R_2}\right)' + 4\xi\eta^2 R_1^{-2}\frac{R_2''}{R_2} = 0. \tag{2.19}$$

We discuss now the behavior of ρ with r. Writing $\rho(r,u) = \rho(r)T_1^2$, where

$$\rho(r) = \frac{6}{\kappa}\frac{R_2''}{R_2}R_1^2,$$

we have from (2.13) and (2.19)

$$\frac{d\rho(r)}{dr} = \rho(r)\frac{\eta^2}{R_1^2}\left(\frac{\chi}{\xi}\frac{1}{R_2^2} - 2\xi\right),$$

and since $\rho(r) > 0$, the sign of $d\rho/dr$ is given by the sign of

$$\frac{\chi}{\xi} - 2\xi R_2^2.$$

The density ρ decreases or increases with r, respectively, for the inequalities

$$\frac{\chi}{\xi} - 2\xi R_2^2 \lessgtr 0 \tag{2.20}$$

which are verified for the following cases: (i) $\chi>0, \xi>0, |R_2(r)| \geq (\chi/2\xi^2)^{1/2}$; (ii) $\chi>0, \xi<0, |R_2(r)| \leq (\chi/2\xi^2)^{1/2}$; (iii) $\chi>0, \xi \gtrless 0$, for any value of $R_2(r)$. If at some value $r=r_m$ corresponding to the interior of the star we have $R_2(r_m)=(\chi/2\xi^2)^{1/2}$, the density ρ has an extremum on the two spheres with radius $R_2(r_m)$. For the two regions $0<r<r_m$ and $r_m<r<r_s$, where r_s is the value of r corresponding to the radius of the star, compatible choices of (i)–(iii) can be made. Since the junction of both choices must hold for any u, they must have ξ with the same sign, and hence for $\chi<0$ the density ρ must be a monotonic function of r.

We now interpret the parameter ξ. The congruence of observers comoving with the fluid is defined by the velocity field [cf. (2.3)]

$$V^\mu = e^\mu_{(0)} = \alpha^{-1}\delta^\mu_0 . \tag{2.21}$$

For our choice of observers in the interior of the future light cone we have the condition $\alpha>0$. For the congruence determined by (2.21) the expansion parameter[14] $\theta = V^\mu{}_{\|\mu}$ is calculated as

$$\theta = \xi \frac{3}{R_1 T_1 T_2{}^2}. \tag{2.22}$$

Since $\alpha>0$, the sign of θ is determined by the sign of ξ, i.e., the fluid is contracting for $\xi<0$ and expanding for $\xi>0$.

Reinterpretation of the equation of state

For the complete analytic solution discussed above, the field equations imply the equation of state $p=-\frac{1}{3}\rho$, involving thus negative scalar pressures. Although the fluid can present this behavior for some critical configurations, this solution can be considered to describe more regular configurations in which the fluid is cooling by emission of neutrinos. To this end we introduce in the energy-momentum tensor a Λ term, namely

$$R_{AB} - \tfrac{1}{2}\eta_{AB}R = -\kappa T_{AB}(\text{tot}) + \Lambda(x)\eta_{AB} . \tag{2.23}$$

Using the form (2.4) for T_{AB}(tot), the right-hand side of (2.23) can be rewritten

$$-\kappa \left[\left(\tilde\rho - \frac{\Lambda}{\kappa} + \tilde p + \frac{\Lambda}{\kappa}\right)\delta^0_A\delta^0_B - \left(\tilde p + \frac{\Lambda}{\kappa}\right)\eta_{AB}\right] - \kappa T_{AB}(\text{neutrino}) , \tag{2.24}$$

where $\tilde\rho$ and $\tilde p$ are the actual energy density and pressure of the fluid, as measured locally by the comoving observer (2.23). (2.29) has the form (2.4) for density ρ and pressure p defined by

$$\rho = \tilde\rho - \Lambda/\kappa , \tag{2.25a}$$

$$p = \tilde p + \Lambda/\kappa . \tag{2.25b}$$

If we take (ρ, p) as given by (2.17) and (2.18), we find that $\tilde\rho$ and $\tilde p$ correspond to the same metric solution discussed above, for Einstein equations (2.23), with the Λ term satisfying

$$\tilde p + \tfrac{1}{3}\tilde\rho = -\tfrac{2}{3}\Lambda/\kappa . \tag{2.26}$$

From (2.23) the Bianchi identities imply the conservation law

$$\kappa T_\mu{}^\nu{}_{\|\nu}(\text{tot}) = \Lambda_{|\mu} . \tag{2.27}$$

The right-hand side of (2.27) permits us to describe the heat output (input) rate of the system, and which is interpreted as the rate of cooling (heating) of the fluid due to emission (absorption) of neutrinos. To see this let us write (2.27) in the local basis (A2),

$$\kappa[(\tilde\rho+\tilde p)^{\bullet}\alpha^{-1}\eta_{A0} - (\tilde\rho+\tilde p)\gamma_{0A0} - (\tilde\rho+\tilde p)\eta_{A0}\gamma_0{}^B{}_B - \dot{\tilde p} e^0_{(A)} - \tilde p' e^1_{(A)}] = e^\mu_{(A)}\Lambda_{|\mu} . \tag{2.28}$$

For $A=0$ we have

$$\kappa\left[\dot{\tilde\rho} - (\tilde\rho+\tilde p)\left(\frac{\dot\alpha}{\alpha} - 2\frac{\dot\beta}{\beta}\right)\right] = \dot\Lambda . \tag{2.29}$$

Denoting by n the baryon number density as measured locally by the comoving observer (2.23), matter conservation is expressed as

$$\dot n - n\left(\frac{\dot\alpha}{\alpha} - 2\frac{\dot\beta}{\beta}\right) = 0 . \tag{2.30}$$

Defining a specific internal energy ϵ by $\tilde\rho = n(\mu_0 + \epsilon)$, where μ_0 is the rest mass of the baryon, we obtain, from (2.29),

$$\dot\epsilon + \tilde p\left(\frac{1}{n}\right)^{\bullet} = \frac{1}{\kappa}\frac{\dot\Lambda}{n} . \tag{2.31}$$

Equation (2.31) is the expression, in the rest frame of the fluid, of the first law of thermodynamics, where $\dot\Lambda/n$ is proportional to the heat output (input) rate per baryon of the fluid.

For $A=1$, (2.28) yields

$$\tilde p' + (\tilde\rho+\tilde p)\left(\alpha' + \frac{\dot\alpha}{\alpha^2}\right)\frac{1}{\alpha^2} = \frac{1}{\alpha^2}(\dot{\tilde\rho}+\dot{\tilde p}) + \frac{1}{\alpha}(\tilde\rho+\tilde p)\left(\frac{2\dot\beta}{\beta\alpha} - \frac{\dot\alpha}{\alpha^2}\right) - \Lambda'/\kappa , \tag{2.32}$$

which substitutes the usual equation for static distributions $\tilde p' + (\tilde\rho+\tilde p)\alpha'/\alpha = 0$. Equation (2.32) is the equation of hydrodynamic equilibrium for the star configuration. We now consider quasistatic distributions. Since by (2.22) the expansion or contraction is determined by ξ, we define quasistatic configurations for values of ξ such that

$$\xi^2 \ll |\xi| . \tag{2.33}$$

An immediate integration of (2.9) and (2.10) gives

$$T_2{}^2 = 2\xi u + \xi_0 , \tag{2.34}$$

$$T_1{}^2 = \frac{1}{\eta^2(2\xi u + \xi_0)}, \qquad (2.35)$$

where ξ_0 is a constant of integration. For (2.33) and finite values of u, the functions (2.34) and (2.35) have approximately the constant values $T_2{}^2 \simeq \xi_0$, $T_1{}^2 \simeq 1/\eta^2\xi_0$. We remark also that all u derivatives in (2.32) depend linearly on ξ and can thus be neglected for (2.33) and finite values of u. Equation (2.32) reduces then to

$$\tilde{p}' - \left(-\frac{\Lambda}{\kappa}\right)' + (\tilde{\rho} + \tilde{p})\frac{\alpha'}{\alpha} = 0\,, \qquad (2.36)$$

which is the equation of neutrino radiative equilibrium for a spherically quasistatic configuration. The quantity $-\Lambda/\kappa$ can be interpreted as a pressure associated with the neutrino radiation, which we denote[15] as

$$p_r = -\Lambda/\kappa\,, \quad \xi^2 \ll |\xi|\,. \qquad (2.37)$$

In fact we can think of the equation $\tilde{p}' - p_r' + (\tilde{\rho} + \tilde{p})\alpha'/\alpha = 0$ as the relativistic analog *for neutrinos* of the Chandrasekhar equation[16] of radiative equilibrium for photons, of a star in Newtonian approximation. The terminology "neutrino radiation pressure" is by formal analogy with the photon radiation pressure in Chandrasekhar's equation. The basic difference is that the gradient of the neutrino radiation pressure (2.37) contributes negatively in the equation of radiative equilibrium (2.36). This was expected since neutrinos have no interaction with the fluid and their effect is to cool the configuration, this cooling corresponding to a pressure gradient in the inverse direction, additive to the gravitational compression.

III. THE EXTERIOR SOLUTION

The exterior region is supposed to contain only neutrinos in interaction with the gravitational field. Since the problem is nonstationary, spherically symmetric, and the trace of $T_{\alpha\beta}(\psi)$ is null, we take for the exterior region the Schwarzschild radiating metric[17,18] which, in the coordinates of Appendix A, assumes the form

$$ds^2 = \alpha^2 du^2 + 2du\,dr - r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)\,, \qquad (3.1)$$

where $\alpha^2 = 1 - 2m(u)r^{-1}$. Light signals propagate along null lines of constant u, such that du is the proper time (or Newtonian time[17]) of an observer at rest at infinity. For neutrinos propagating radially along the light cones of (3.1) we use the energy-momentum tensor

$$\tilde{T}_{AB} = 2\pi i \frac{1}{\alpha^2 r^2}(\dot{\lambda}^\dagger \lambda - \dot{\lambda}^\dagger \lambda) k_A k_B\,, \qquad (3.2)$$

as derived in Appendix A, where $k_A = (1, -1, 0, 0)$. It has the form of the energy-momentum tensor of a null fluid, which is the usual phenomenological description for neutrinos in general relativity.[5,7] Here, $\lambda(u)$ is the arbitrary two-component spinor of the solution (A7) of Dirac's equation.

The Ricci tensor for (3.1) has the form

$$R_{AB} = \frac{4\dot{m}}{\alpha^2 r^2} k_A k_B \qquad (3.3)$$

and the Einstein equations imply

$$\dot{m} = \frac{k\pi i}{2}(\dot{\lambda}^\dagger \lambda - \lambda^\dagger \dot{\lambda})\,. \qquad (3.4)$$

The geometric properties of the Schwarzschild radiating space-time have been extensively studied by Lindquist, Schwarz, and Misner,[19] but we have some comments here. If we examine the curvature tensor R_{ABCD} of the metric (3.1) for r sufficiently large, we see that the leading term (corresponding to the lowest power of r^{-1}) has the expression $R_{ABCD} \sim \dot{m}/r^2$. The radiated part of the space-time thus comes from $\dot{m} \neq 0$ which by (3.4) is due only to the radiated neutrinos, and no gravitational radiation emission is present, a fact that was expected because of the spherical symmetry of the space-time. The total radiated power (which is associated to neutrinos only, denoted "neutrino luminosity") can then be calculated from the energy-momentum tensor (3.2). Writing $v^\mu = \alpha^{-1}\delta_0^\mu$ (corresponding to a local inertial observer), the neutrino luminosity as measured by an asymptotic observer at rest is then given by

$$L_{\text{tot}} = \lim_{r\to\infty} 4\pi r^2 \tilde{T}_{\mu\nu}(\text{neutrino}) v^\mu v^\nu$$
$$= 8\pi^2 i(\lambda^\dagger \dot{\lambda} - \dot{\lambda}^\dagger \lambda)\,,$$

which can be expressed through (3.4) as $L_{\text{tot}} = -(16\pi/\kappa)\dot{m}$, a result obtained in Ref. 19 by a different approach.

We here neglect the region $r \leq 2m(u)$ ($\dot{m} < 0$) by assuming that the fluid has a boundary $r_s(u) > 2m(u)$, such that the emission of neutrinos takes place before the fluid is inside its Schwarzschild [the static limit corresponds to the Schwarzschild configuration for $r > 2m = \text{const}$ (cf. Sec. IV)]. We exclude the case of the whole mass of the object being emitted as neutrinos before the fluid reaches its Schwarzschild radius (i.e., $\dot{r}_s < \dot{m}$); the conditions which eliminate this possibility for our solutions are examined later [cf. Eq. (4.19)].

IV. JUNCTION (AND BOUNDARY) CONDITIONS FOR THE INTERIOR AND EXTERIOR SOLUTIONS

Here we denote the coordinate systems of the interior solution (2.1) and the exterior solution (3.1), respectively, by $x_{\text{I}}^\alpha = (U, R, \theta, \varphi)$ and $x_{\text{II}}^\alpha = (u, r, \theta, \varphi)$. For the exterior solution the coordinates x^α_{II} are admissible in the region of the space-time restricted by $r > 2m(u)$. For the interior solution we see by (2.13), (2.14), and (2.19) that the coordinate R is admissible in all its domain, excluding the points such that $R_2(r) = 0$ (we remark that R_2^{-2} is proportional to the curvature of $U, R = \text{const}$ spheres, and always assumed to be finite). The coordinate U must be restricted [cf. (2.34)],

$$U < -\frac{\xi_0}{2\xi}, \quad \xi < 0,$$

which is a consequence of $T_2 > 0$ and T_1 finite in the admissible domain of U. In the following analysis we are considering only the region of the space-time for the solutions covered by the above coordinates.

Let us consider the three-dimensional hypersurface Σ of the junction of the two solutions and D a finite neighborhood of Σ. D and Σ are chosen such that x^α_{I} and x^α_{II} are simultaneously admissible in D. Following Lichnerowicz[20] we assume reasonable conditions on the continuity of the metric and its partial derivatives on D, such that we have the continuity of the metric through Σ and the junction conditions[20,21]

$$G_\mu{}^\nu \phi_{|\nu} = \text{continuous through } \Sigma\,, \tag{4.1}$$

where the equation of Σ is given by $\phi = 0$. For the present admissible coordinate systems, we also assume that on D the transformations functions $x^\alpha_{I} \rightleftarrows x^\alpha_{II}$ have continuous first-order partial derivatives and piecewise continuous second-order derivatives (that is, the second-order partial derivatives may have different limits on each side of Σ), such that the junction conditions (4.1) are preserved under the transformations. From the Einstein equations and (4.1) we have the Israel-O'Brien-Synge junction conditions[21,22]

$$T_\mu{}^\nu \phi_{|\nu} = \text{continuous through } \Sigma\,, \tag{4.2}$$

which express the continuity of the flux of four-momentum through Σ.

We take as junction hypersurface a sphere with u-dependent radius, described in exterior coordinates by

$$\Sigma:\ r = r_s(u)\,. \tag{4.3}$$

In interior coordinates x^α_{I}, we note that R is a comoving coordinate ($e^\mu_{I(0)}\partial R/\partial x^\mu_{I} = 0$) and Σ is described by

$$R = R_s = \text{const.} \tag{4.4}$$

Parametrizing the extrinsic coordinates of points on Σ as $x^\alpha_{II} = (u, r_s(u), \theta, \varphi)$ and $x^\alpha_{I} = (U, R_s = \text{const}, \theta, \varphi)$, the metrics induced on Σ by the exterior and interior metrics can be respectively calculated,

$$(ds^2)_\Sigma = \left(1 - \frac{2m(u)}{r_s(u)} + 2\dot{r}_s\right)du^2 - r_s{}^2 d\Omega^2 \tag{4.5}$$

and

$$(ds^2)_\Sigma = \alpha^2(R_s, U)dU^2 - \beta^2(R_s, U)d\Omega^2\,. \tag{4.6}$$

The first junction condition can be expressed by the equality of (4.5) and (4.6), which results in

$$\beta^2(R_s, U) = r_s{}^2(u)\,, \tag{4.7}$$

$$\alpha^2(R_s, U)dU^2 = \left(1 - \frac{2m(u)}{r_s(u)} + 2\dot{r}_s\right)du^2\,. \tag{4.8}$$

This equality of the first fundamental forms of Σ guarantees the continuity of the metric through Σ. Equation (4.8) relates the proper time interval du of an observer at rest at infinity and the interval dU of an observer on the surface Σ. We remark that if $\dot{r}_s = 0$ from Eq. (4.8) we could take $U \equiv u$ and extend the exterior coordinates naturally to the interior region $R < r_s$, corresponding to the complete solution of the Schwarzschild problem for a static fluid sphere. Defining coordinate transformations $u = F(U, R)$, $r = G(U, R)$, differentiating and calculating in $R = R_s$, we have from (4.8) that $F'(R_s, U) = 0$. The continuity of the metric through Σ can then be expressed as

$$\begin{aligned} &G^2(R_s, U) = R_2{}^2(R_s)T_2(U)\,,\\ &(\dot{F}G')_{R=R_s} = 1\,, \\ &\{[1 - 2m(F)G^{-1}]\dot{F}^2 + 2\dot{F}\dot{G}\}_{R=R_s} = R_1{}^2(R_s)T_1{}^2(U)\,. \end{aligned} \tag{4.9}$$

Equations (4.12) define a class of coordinate transformations in a finite neighborhood of Σ which are compatible with the junction conditions discussed above.

By using (4.7), (4.8), (2.9), and (2.10) we can calculate

$$\frac{dr_s(u)}{du} = \xi\eta\frac{R_2(R_s)}{R_1(R_s)}\left(1 - \frac{2m(u)}{r_s(u)} + 2\dot{r}_s(u)\right)^{1/2}, \tag{4.10}$$

where the variable u is related to U by $u = F(R_s, U)$. Equation (4.10) determines the evolution of the surface Σ. We remark that whenever the square root is taken in obtaining the above expressions, it is geometrically reasonable to consider the positive root only. Since from (4.8) $(1 - 2m/r_s + \dot{r}_s)^{1/2}$ is always positive, we see that the sign of $\dot{r}_s$ is given by ξ. Hence $\dot{r}_s$ is greater than or less than zero if the fluid is expanding or contracting, corresponding respectively to an increasing or decreasing of the area of Σ.

We examine now the junction conditions (4.2) in the junction surface (4.3), described in exterior coordinates. Since the metric is continuous through Σ the junction conditions (4.2) can be expressed in the tetrad basis as

$$e^{\mu}_{\mathrm{II}(A)} e^{(B)}_{\mathrm{II}\nu} e^{\rho}_{\mathrm{II}(B)} T_{\mu}{}^{\nu} \Sigma_{|\rho} = \text{continuous through } \Sigma$$

or (4.11)

$$T_A{}^B\, \Sigma_{|B} = \text{continuous through } \Sigma\,,$$

where

$$\Sigma_{|B} = e^{\mu}_{\mathrm{II}(B)} \Sigma_{|\mu} = (-\alpha_{\mathrm{II}}{}^{-1} \dot{r}_s, \alpha_{\mathrm{II}}{}^{-1}(\dot{r}_s + \alpha_{\mathrm{II}}{}^2), 0, 0) \tag{4.12a}$$

are the tetrad components of the normal to the surface Σ. Here $\alpha_{\mathrm{II}}{}^2 = 1 - 2m(u) r^{-1}$, since the normal is expressed in terms of the exterior solution. From (4.11) we have

$$(\Delta T_A{}^B) \Sigma_{|B} = 0\,, \tag{4.12b}$$

where $\Delta T_A{}^B$ denotes the discontinuity in $T_A{}^B$ on crossing Σ, explicitly $\Delta T_A{}^B = T_{\mathrm{I}A}{}^B(\Sigma) - T_{\mathrm{II}A}{}^B(\Sigma)$. In (4.12) $\alpha_{\mathrm{II}}{}^2$ is calculated on Σ, $\alpha_{\mathrm{II}}{}^2(\Sigma) = 1 - 2m(u) r_s{}^{-1}$, where $u = F(R_s, U)$. We follow an analogous notation in all cases.

Noting that the quantities in (4.12) are expressed in a tetrad basis, and are then invariant under coordinate transformations, we can use directly the components (2.4) and (3.2) of the interior and exterior energy-momentum tensors in the calculation of $\Delta T_A{}^B$. With the notation $l(u) = 2\pi i(\lambda^\dagger \dot{\lambda} - \dot{\lambda}^\dagger \lambda)$ and using (4.7) and (4.12), the junction conditions (4.12) yield

$$\Delta\rho \dot{r}_s = \frac{2}{r_s{}^2}\left(\frac{\mathcal{L}}{\alpha_{\mathrm{I}}{}^2(\Sigma)} - \frac{l}{\alpha_{\mathrm{II}}{}^2(\Sigma)}\right) \alpha_{\mathrm{II}}{}^2(\Sigma)\,, \tag{4.13}$$

$$-\Delta p[\dot{r}_s + \alpha_{\mathrm{II}}{}^2(\Sigma)] = \frac{2}{r_s{}^2}\left(\frac{\mathcal{L}}{\alpha_{\mathrm{I}}{}^2(\Sigma)} - \frac{l}{\alpha_{\mathrm{II}}{}^2(\Sigma)}\right). \tag{4.14}$$

Since ρ and p must satisfy $p = -\frac{1}{3}\rho$, this implies their discontinuity through Σ must satisfy

$$\Delta p = -\tfrac{1}{3}\Delta\rho \tag{4.15}$$

which determines that the actual solution $(\tilde{\rho}, \tilde{p}, \Delta)$ must be discontinuous through Σ. Also, since it is physically reasonable to have $\Delta\tilde{p} = 0$, we have from (2.25)

$$\Delta\rho = \Delta\tilde{\rho} - \Delta\Lambda/\kappa\,,$$
$$\Delta p = \Delta\Lambda/\kappa\,. \tag{4.16}$$

On examining the junction conditions (4.13) and (4.14) we can distinguish two relevant situations:

(a) $\Delta\rho, \Delta p \neq 0$.

Substituting (4.13) and (4.14) in (4.15), we obtain

$$2\frac{\alpha_{\mathrm{II}}{}^2(\Sigma)}{r_s{}^2}\left(\frac{\mathcal{L}}{\alpha_{\mathrm{I}}{}^2(\Sigma)} - \frac{l}{\alpha_{\mathrm{II}}{}^2(\Sigma)}\right)\left(\frac{1}{3}\,\frac{\dot{r}_s + \alpha_{\mathrm{II}}{}^2(\Sigma)}{\dot{r}_s} - 1\right) = 0\,,$$

which implies

$$\frac{\mathcal{L}}{\alpha_{\mathrm{I}}{}^2(\Sigma)} - \frac{l}{\alpha_{\mathrm{II}}{}^2(\Sigma)} = 0 \tag{4.17}$$

or

$$\alpha_{\mathrm{II}}{}^2(\Sigma) = 2\dot{r}_s\,. \tag{4.18}$$

The case (4.17) will be examined later. From Eq. (4.18), together with (4.10), we have the expression for $m(u)$:

$$m(u) = \left(1 - 8\xi^2\eta^2 \frac{R_2{}^2(R_s)}{R_1{}^2(R_s)}\right)\frac{r_s(u)}{2}\,. \tag{4.19}$$

Because the present coordinate system is admissible only for $r > 2m(u)$, the parameters shall be restricted by

$$8\xi^2\eta^2 \frac{R_2{}^2(R_s)}{R_1{}^2(R_s)} < 1\,.$$

Equation (4.19), which basically results from the choice (4.18), contains the important information that, in the static limit $\xi \to 0$, the surface Σ coincides with the Schwarzschild surface of the star, that is, for $\xi \to 0$ — when the emission of neutrinos ceases — the fluid is entirely contained inside its Schwarzschild radius. Using (2.34) and (4.7) in (4.19), the Schwarzschild mass for the static limit of a star emitting neutrinos, immediately before passing its Schwarzschild radius, can be evaluated:

$$m = \xi_0 \frac{R_2(R_s)}{2}\,. \tag{4.20}$$

From the above discussions, the choice (4.18) is not satisfactory for quasistatic distributions in radiative equilibrium [cf. (2.36)]. For this we consider the following:

(b) Distributions $(\tilde{\rho}, \tilde{p}, \Lambda)$ vanishing smoothly on Σ, without discontinuity: $\Delta\tilde{\rho}, \Delta\tilde{p}, \Delta\Lambda = 0$. From (4.13) and (4.14) we have

$$\frac{\mathcal{L}}{\alpha_{\mathrm{I}}{}^2(\Sigma)} - \frac{l}{\alpha_{\mathrm{II}}{}^2(\Sigma)} = 0\,, \tag{4.17}$$

which is the first case of (a). Using (2.10), (2.12), (3.4), and $\alpha_{\mathrm{I}}{}^2(\Sigma) = R_1{}^2(R_s) T_1{}^2(U)$, we obtain from (4.17) the differential equation for $m(u)$:

$$4\dot{m} = \frac{\eta^2\chi}{R_1{}^2(R_s)}\left(1 - \frac{2\dot{m}(u)}{r_s(u)}\right). \tag{4.21}$$

We now know that for neutrino emission $\dot{m} < 0$ and for neutrino absorption $\dot{m} > 0$. Since $1 - 2m/r_s$ is always positive, we have

(b.1) $\chi < 0$ emission,

(b.2) $\chi > 0$ absorption

independently of the sign of ξ.[23] So in this case we can eventually have emission with expansion, though these situations seem physically improbable.

Finally we determine the function $\Lambda(R,U)$ and the actual density $\tilde{\rho}$, for the case of the junction condition (4.18). Since our solution has a natural static limit defined by $\xi\to 0$, we define the total mass-energy of the fluid, in a way analogous to the static case[24,25]

$$m(R_s,U)=4\pi\int_0^{R_s}\tilde{\rho}(R,U)\beta^2(R,U)\beta'(R,U)\,dR\,, \tag{4.22}$$

correctly including not only the rest, the internal, and the baryon interaction energies but also the energy of gravitational interaction. Because no gravitational radiation is emitted simultaneously with neutrinos, the problem of localization of the energy of gravitational waves is not present here, and the total energy (4.22) is localized since other fluxes of energy (neutrinos, for instance) are always locally measurable [cf. (2.31) and (2.32)].

Using that $\tilde{\rho}(R,U)=\rho(R)T_2^{-2}(U)+(\Delta/k)(R,U)$ and $\beta=R_2(R)T_2(U)$, the equality of (4.22) to (4.19) results in

$$\gamma^2T_2\left(R_2'(R)+\frac{R_2(0)}{R_s}\right)=4\pi\rho(R)R_2^2(R)R_2'(R)T_2+\frac{4\pi}{\kappa}\Lambda(R,U)R_2^2(R)R_2'(R)T_2^3$$

up to a function of (R,U) whose integral in R between 0 and R_s is identically zero; here $\gamma^2=\frac{1}{2}[1-8\xi^2\eta^2R_2^2(R_s)/R_1^2(R_s)]$. We then have for this case the expressions

$$\tilde{\rho}(R,U)=\frac{\gamma^2}{4\pi R_2^2(R)}T_2^{-1}(U)\left(1+\frac{R_2(0)}{R_sR_2'(R)}\right), \tag{4.23}$$

$$\frac{\Lambda}{\kappa}(R,U)=-\rho(R,U)+\frac{\gamma^2}{4\pi R_2^2(R)}T_2^{-2}(U)\left(1+\frac{R_2(0)}{R_sR_2'(R)}\right), \tag{4.24}$$

where $\rho(R,U)$ is given by (2.17). In the present case where $\Delta T_{AB}(\Sigma)\neq 0$, the value R_s corresponding to the radius of the star can be determined by the condition $\Delta\tilde{p}=\tilde{p}(R_s)=0$ subject to the condition $R_2''/R_2<0$ for $R<R_s$ [cf. (2.6)].

V. CONCLUSIONS

We have presented a complete relativistic model of a spherically symmetric star, with its basis in the coupled Einstein-Dirac equations. The interior of the star is assumed to be a perfect fluid — described by its total density ρ, pressure p, and baryon number density n — and bounded by a spherical surface of radius r_s. The matter of the star is assumed transparent to neutrinos. The energy-momentum tensor used for neutrinos has the form of the energy-momentum tensor of a null fluid, which is the usual phenomenological description of neutrinos in general relativity. It has been derived in Appendix A, starting from the classical spinorial field ψ, the solution of Dirac's equation for neutrinos in the metric of the spherically symmetric space-time, corresponding to a radial current $j^\alpha=\bar{\psi}\gamma^\alpha(x)\psi$ along the local light cones. In the coordinate system used, the Dirac equation is completely integrable for the radial neutrinos considered, and all observable quantities constructed with these solutions correspond to an isotropic emission. This makes the average proposed for the energy-momentum tensor, leading to the null-fluid description, legitimate. The interior solution is obtained by separation of variables, the field equations implying the equation of state $p=-\frac{1}{3}\rho$. Although negative scalar pressures can occur for very high values of the density,[13] for less critical configurations negative scalar pressures are not physically satisfactory. To circumvent this we introduce a Λ term in the total energy-momentum tensor (equivalently, in the field equations). To the above solution corresponds a solution for actual energy density, and pressure $\tilde{\rho}=\rho+\Lambda/\kappa$, $\tilde{p}=p-\Lambda/\kappa$, respectively, with $\tilde{p}+\tilde{\rho}/3=\frac{2}{3}\Lambda/\kappa$. From the conservation equations which follow from the Bianchi identities, we have that $\dot{\Lambda}$ is proportional to the rate of cooling of the fluid. Also for quasistatic distributions (this limit being well defined for our solution) these equations provide us with the relativistic analog *for neutrinos* of the radiative equilibrium equation of Chandrasekhar, where $(-\Lambda/\kappa)$ appears as the equivalent of a radiation pressure for neutrinos (perhaps a better designation would be "gravitational pressure due to neutrinos"). Contrary to the photon radiation pressure in Chandrasekhar's equation, the gradient of $(-\Lambda/\kappa)$ has a negative sign — in fact the effect of neutrinos is to cool the configuration (they do not interact with the matter of the star), this cooling being equivalent to a pressure in the inverse direction of photon pressure, additive to the gravitational compression. The exterior metric is the Schwarzschild radiating metric and we have used Israel-O'Brien-Synge junction conditions of the exterior and interior solutions. Two physically distinct situations arise. In one case the solution describes a stage of emission of neutrinos, with consequent contraction of the configuration immediately before the fluid is entirely inside its Schwarzschild radius, when the emission of neutrinos ceases ($\xi\to 0$). The Λ function is explicitly determined for this case. The other possibility can, for instance, correspond to a quasistatic configuration, where $(-\Lambda/\kappa)$ has the interpretation of a radiation pressure for neutrinos.

Our detailed geometrical treatment of the problem has nevertheless been unilateral because of our description of neutrinos as classical spinorial fields. An improvement of it would be the quantization of the neutrino field on the classical curved space-time. However, the only Killing vectors of the present model are the generators of the spatial rotation group. These are not sufficient to construct a basis for an invariant decomposition into excitation modes of the field, and an invariant definition of particle and antiparticle states and vacuum states.[26] Alternatively, since the exterior solution is asymptotically flat we could assay an asymptotic quantization of the neutrino field, choosing the asymptotic basis functions as the usual basis functions in flat space[27]; for the junction condition (4.17) the form (2.12) would be an additional guide in the choice of the basis to express $l(u)$. From the classical form of $l(u)$ and following the usual scheme of quantization it would in principle be possible to characterize the parameter χ as function of the generalized momenta corresponding to the basis chosen. At present we do not know if a complete quantization of the radiated neutrino field in the background metric would provide a drastic change in the problem of these particles contributing to the curvature of the background and if our solution as a first approximation would be significant at all.

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APPENDIX A

For both the exterior and interior problem, our choice of coordinates is $x^\alpha = (u, r, \theta, \varphi), -\infty \leq u \leq \infty$, $0 < r < \infty$, with the following properties: (i) the hypersurfaces $u = \text{const}$ are null hypersurfaces tangent in each point to the local light cone; (ii) r is an affine parameter along the congruence of null geodesics with tangent $k^\alpha = g^{\alpha\beta} u_{|\beta}$, and can be interpreted as a luminosity distance in the usual sense.[28] In this coordinate system, a spherically symmetric line element can always be expressed as

$$ds^2 = \alpha^2 du^2 + 2\, du\, dr - \beta^2 (d\theta^2 + \sin^2\theta\, d\varphi^2), \qquad \text{(A1)}$$

where α and β are functions of u and r. We choose a tetrad basis $(e_\alpha^{(A)})$ with non-null components

$$e_0^{(0)} = \alpha, \quad e_1^{(0)} = \alpha^{-1}; \\ e_1^{(1)} = \alpha^{-1}, \quad e_2^{(2)} = \beta, \quad e_3^{(3)} = \beta \sin\theta, \qquad \text{(A2)}$$

such that (A1) assumes the form (1.1). Ricci coefficients γ_{ABC} are calculated from (A2) and have the non-null components

$$\begin{aligned} &\gamma_{010} = \alpha' + \dot\alpha/\alpha^2, \quad \gamma_{122} = -\dot\beta/\beta\alpha + \beta'\alpha/\beta, \\ &\gamma_{011} = -\dot\alpha/\alpha^2, \quad \gamma_{133} = -\dot\beta/\beta\alpha + \beta'\alpha/\beta, \\ &\gamma_{022} = \dot\beta/\beta\alpha, \quad \gamma_{233} = \cot\theta/\beta, \\ &\gamma_{033} = \dot\beta/\beta\,\alpha, \end{aligned} \qquad \text{(A3)}$$

where a dot and a prime denote, respectively, derivatives with respect to u and r.

The Fock-Ivanenko coefficients (1.5) are then calculated[10]:

$$\begin{aligned} \Gamma_0 &= -\tfrac{1}{2}(\alpha' + \dot\alpha/\alpha^2)\gamma^0\gamma^1, \\ \Gamma_1 &= \tfrac{1}{2}\, \frac{\dot\alpha}{\alpha^2}\gamma^0\gamma^1, \\ \Gamma_2 &= -\frac{1}{2}\frac{\dot\beta}{\beta\alpha}\gamma^0\gamma^2 - \frac{1}{2}\left(\frac{\beta'\alpha}{\beta} - \frac{\dot\beta}{\beta\alpha}\right)\gamma^1\gamma^2, \\ \Gamma_3 &= -\frac{1}{2}\frac{\dot\beta}{\beta\alpha}\gamma^0\gamma^3 - \frac{1}{2}\left(\frac{\beta'\alpha}{\beta} - \frac{\dot\beta}{\beta\alpha}\right)\gamma^1\gamma^3 \\ &\quad - \frac{\cot\theta}{2\beta}\gamma^2\gamma^3. \end{aligned} \qquad \text{(A4)}$$

Since only radial neutrinos are considered, we are restricted to spinorial fields of the form

$$\psi = \begin{pmatrix} \varphi \\ \sigma^1 \varphi \end{pmatrix}, \qquad \text{(A5)}$$

where φ is a two-spinor and σ^1 is the constant Pauli matrix, corresponding to a four-current

$$j^\alpha = e^\alpha_{(A)} \bar\psi \gamma^A \psi = 2\alpha\varphi^\dagger\varphi\,\delta_1^\alpha \qquad \text{(A6)}$$

along the radial light cones of (A1). Noting that Γ_3 depends explicity on θ, we are then led to take $\varphi = \varphi(u, r, \theta)$. In the present representation of the γ^A, the Dirac equation (1.7b) reduces to

$$\begin{aligned} &\alpha\varphi' + \left(\frac{\alpha'}{2} + \frac{\alpha}{r}\right)\varphi = 0, \\ &\frac{\partial\varphi}{\partial\theta} + \tfrac{1}{2}\cot\theta\,\varphi = 0, \end{aligned}$$

which can be immediately integrated to give

$$\varphi(u, r, \theta) = \frac{1}{(\alpha \sin\theta)^{1/2}\beta}\, \Lambda(u), \qquad \text{(A7)}$$

where $\Lambda(u)$ is an arbitrary two-spinor.[29] (A7) corresponds to the most general solution (A5) of the Dirac equation in the metric (A1). In the four-current (A6) the solution (A7) yields

$$j^\alpha = \frac{2}{\sin\theta\,\beta^2}\Lambda^\dagger(u)\Lambda(u)\delta_1^\alpha. \qquad \text{(A8)}$$

The non-null components of the energy-momentum tensor (1.3) of the neutrino field (A5), (A7) are

$$T_{00}=T_{11}=-T_{01}=\frac{4i}{\sin\theta}\,\frac{1}{\alpha^2\beta^2}(\Lambda^\dagger\dot{\Lambda}-\dot{\Lambda}^\dagger\Lambda), \qquad (A9a)$$

$$T_{03}=-T_{13}=\frac{2}{\sin\theta}\,\frac{\cot\theta}{\beta^3\alpha}\Lambda^\dagger\sigma^1\Lambda. \qquad (A9b)$$

Expressions (A8) and (A9) show clearly why it has been widely stated in the literature[30] that neutrinos cannot generate a curvature compatible with spherical symmetry. The angle dependence of (A9b) is not so drastic because we could consider neutrino fields satisfying

$$\Lambda^\dagger\sigma^1\Lambda(u)=0 \qquad (A10)$$

which are not eigenstates of γ^5. Actually these components (A9b) should vanish by the Einstein equations for (A1).

Since, in principle, there is no reason why a spherically symmetric star (in some stage of its evolution) cannot emit neutrinos in interaction with and contributing to the gravitational field, the contrary possibility being physically intuitive, we must modify or reinterpret result (A9). Fortunately the θ dependence of (A8) and (A9a) is suggestive because it is exactly the factor $1/\sin\theta$ which corrects areas in measurements in a spherical coordinate system.

Indeed if we consider the measurement of a radial flux of neutrinos, we can easily see that the number of particles, by unit of time and area, measured in the direction θ, for r fixed, is proportional to

$$\sqrt{-g}\,j^r(\theta),$$

where $j^r=2\Lambda^\dagger\Lambda(u)/\beta^2\sin\theta$ and $\sqrt{-g}=\beta^2\sin\theta$ [cf. (A8)]. Hence the *observed* flux is independent of the direction of measurement, whether observed locally or globally. Analogously we can interpret T_0^μ as a current density of energy-momentum, which depends on θ as $1/\sin\theta$ and so corresponds to an isotropic (or spherically symmetric) flux of energy. Since all observable quantities constructed with (A5), (A7) are independent of the direction of measurement, it is legitimate to redefine

$$\tilde{T}_{\mu\nu}=\frac{1}{4\pi}\int_0^{2\pi}d\varphi\int_0^{\pi}d\theta\,\sin\theta\,T_{\mu\nu} \qquad (A11)$$

as the physical energy-momentum tensor for neutrinos, which shall enter the right-hand side of the Einstein equations, in a spherically symmetric space-time. Condition (A10) can be discarded as artificial because in (A11) we can take the θ integral as the principal value, which implies $\tilde{T}_{03}=\tilde{T}_{13}=0$, and we can eventually have emission of neutrinos of only one type, $\psi_{(\pm)}=\pm\gamma^5\psi_{(\pm)}$.

Redefinition (A11) has two important properties: (i) $\tilde{T}_{\alpha\beta}$ is still conserved locally in the metric (A1), $\tilde{T}_\alpha{}^\beta{}_{\|\beta}=0$; (ii) $\tilde{T}_{\alpha\beta}$ has the form of the energy-momentum tensor of a null fluid[6] which is the usual phenomenological description of neutrinos in general relativity.[5,7].

Substituting (A9) in (A11) we obtain the non-null components

$$\tilde{T}_{00}=\tilde{T}_{11}=-\tilde{T}_{01}=2\pi i\,\frac{1}{\alpha^2\beta^2}(\Lambda^\dagger\dot{\Lambda}-\dot{\Lambda}^\dagger\Lambda). \qquad (A12)$$

In the coordinate basis, $\tilde{T}_{\alpha\beta}=e_\alpha^{(A)}e_\beta^{(B)}\tilde{T}_{AB}$ can be expressed as

$$\tilde{T}_{\alpha\beta}=\frac{2\pi i}{\beta^2}(\Lambda^\dagger\dot{\Lambda}-\dot{\Lambda}^\dagger\Lambda)u_{|\alpha}u_{|\beta}$$

which shows property (ii).

We finally remark that, in the same context, Griffiths[31] proposed an anlogous average for the energy-momentum tensor of neutrinos, based on the following considerations. The results of the theorems (30) are physically irrelevant since they are in the realm of one-particle theory, and for describing neutrinos radiated from a star we must do a "statistical" approach. Starting from the one-particle theory with a radially propagating neutrino field, a first approximation to the many-neutrino energy-momentum tensor is constructed by summing all the individual one-neutrino tensors over neutrinos propagating in random directions.

APPENDIX B

The non-null components of the Ricci tensor R_{AB} for the metric (A1) in the tetrad basis (A2) are

$$R_{00}=-2\alpha\alpha''-2\alpha'^2+4\ddot{\beta}/\beta\alpha^2+4\dot{\beta}\alpha'/\beta\alpha$$
$$-4\alpha\alpha'\beta'/\beta-4\dot{\alpha}\beta'/\alpha\beta,$$

$$R_{01}=-4\ddot{\beta}/\beta\alpha^2+4\dot{\beta}'/\beta-4\dot{\beta}\alpha'/\beta\alpha$$
$$+4\beta'\dot{\alpha}/\beta\alpha,$$

$$R_{11}=2\alpha\alpha''+2\alpha'^2+4\ddot{\beta}/\beta\alpha^2-8\dot{\beta}'/\beta$$
$$-4\beta'\dot{\alpha}/\beta\alpha+4\dot{\beta}\alpha'/\beta\alpha$$
$$+4\beta''\alpha^2/\beta+4\alpha\alpha'\beta'/\beta,$$

$$R_{22}=R_{33}=-4\dot{\beta}'/\beta+2\beta''\alpha^2/\beta+4\alpha\alpha'\beta'/\beta$$
$$-2/\beta^2-4\dot{\beta}\beta'/\beta^2$$
$$+2\beta'^2\alpha^2/\beta^2.$$

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[10]We use a representation such that $(\gamma^A)^\dagger = \gamma^0 \gamma^A \gamma^0$, with $(\gamma^0)^2 = -(\gamma^k)^2 = 1$, $k = 1, 2, 3$.

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