PHYSICAL REVIEW D VOLUME 17, NUMBER 6 15 MARCH 1978

Axiomatic lower bound on the slope parameter

Seichi Naito
Department of Physics, Osaka City University, Sumiyoshiku, Osaka, Japan
(Received 24 August 1976; revised manuscript received 21 December 1977)

We derive the axiomatic lower bound on the s-channel slope parameter $B(s)$. The result is that $B(s)$ has the lower bound $B(s) \geqslant \text{const} \times s^{-5}(\ln s)^{-12}$ (>0) for some sequence of $s \to \infty$. In the special case when the s-channel scattering amplitude $F(s,t)$ satisfies the scaling limit for $s \to \infty$ $\lim[\partial_t^m \operatorname{Im}F(s,0)/\operatorname{Im}F(s,0)]/B(s)^m = \text{const}$ (>0) for some fixed m (≥ 3), $B(s)$ has the stronger lower bound $B(s) \geq \text{const} \times s^{-6/m}(\ln s)^{-2-4/m}$ (>0) for some sequence of $s \to \infty$.

It will be interesting to know what kinds of restrictions on the s- and u-channel slope parameters [denoted respectively by $B_1(s)$ and $B_2(u)$] can be derived from the general principles on the scattering amplitudes. In our previous paper[1] we investigated this problem and obtained the result by using as general principles the unitarity of the S matrix, analyticity in s and t, s-u crossing symmetry, and the polynomial upper boundedness of the scattering amplitude. The improved result[2] is that *at least one*[3] of $B_1(s)$ and $B_2(u)$ has the lower bound

$$B_1(s) \geqslant \text{const} \times s^{-5}(\ln s)^{-4} \quad (>0)$$

for some sequence of $s \to \infty$, or (1)

$$B_2(u) \geqslant \text{const} \times u^{-5}(\ln u)^{-4} \quad (>0)$$

for some sequence of $u \to \infty$. In this paper, we investigate the axiomatic lower bound on the *s-channel* slope parameter. The result is that $B_1(s)$ has the lower bound

$$B_1(s) \geqslant \text{const} \times s^{-5}(\ln s)^{-12} \,(>0) \tag{2}$$

for some sequence of $s \to \infty$. In the special case when the s-channel scattering amplitude $F_I(s,t)$ satisfies the scaling limit

$$\lim_{s\to\infty}[\partial_t^m \operatorname{Im}F_{\text{I}}(s,0)/\operatorname{Im}F_{\text{I}}(s,0)]/B_1(s)^m = \text{const} \quad (>0) \tag{3}$$

for some fixed m $(\geqslant 3)$, we obtain the stronger lower bound

$$B_1(s) \geqslant \text{const} \times s^{-6/m}(\ln s)^{-2-4/m} \quad (>0) \tag{4}$$

for some sequence of $s \to \infty$. In the following, we shall sketch the derivation of the new lower bound by referring to various equations in our previous paper[1] with a prefix N, i.e., (1.1) will be referred to as (N1.1). For simplicity we consider the spinless elastic scattering $A + B \to A + B$ (s channel) coupled by crossing to $A + \bar{B} \to A + \bar{B}$ (u channel). In order to avoid kinematical complications, we set the masses of A and B equal to unity. The scattering amplitude $F(s,t)$ satisfies the twice-subtracted dispersion relation[4]

$$F(s,t) = A(t) + B(t)s + \frac{s^2}{\pi}\int_4^\infty ds' \frac{\operatorname{Im}F_{\text{I}}(s',t)}{s'^2(s'-s)} + \frac{u^2}{\pi}\int_4^\infty du' \frac{\operatorname{Im}F_{\text{II}}(u',t)}{u'^2(u'-u)}, \tag{5}$$

with

$$s + t + u = 4,$$

$$F_{\text{I}}(s,t) \equiv F(s,t) \tag{6}$$

and

$$F_{\text{II}}(u,t) \equiv F(4-u-t,t).$$

The unitarity of the S matrix gives the constraints

$$0 \leqslant |f_l^{\text{II}}(u')|^2 \leqslant \operatorname{Im}f_l^{\text{II}}(u') \leqslant 1, \tag{7}$$

so that

$$\partial_t^m \operatorname{Im}F_{\text{II}}(u',0) = \frac{16\pi}{m!}\left(\frac{u'}{u'-4}\right)^{1/2}\frac{1}{(u'-4)^m} \times \sum_{l=m}^\infty (2l+1)\frac{(l+m)!}{(l-m)!}\operatorname{Im}f_l^{\text{II}}(u') \quad (>0),$$

with the simplified notation

$$\partial_t^m F_{\text{II}}(u,0) \equiv \left(\frac{\partial}{\partial t}\right)^m F(4-u-t,t)\bigg|_{t=0}. \tag{8}$$

(Throughout this paper we explicitly give relations involving one of F_{I} and F_{II}, since the other relations can be similarly obtained.) Similarly to (N2.11), the polynomial upper boundedness

$$|\partial_t^m F_{\text{I}}(s,0)| \leqslant |s|^N \text{ as } |s| \to \infty \tag{9}$$

can be derived from the polynomial upper boundedness of the scattering amplitude.

With the help of the general principles (5)–(7) and (9), we shall investigate the lower bound on $\partial_t^m \operatorname{Im}F_{\text{I}}(s,0)$ *for any fixed* m ($m = 3, 4, \ldots$). The result is that $\partial_t^m \operatorname{Im}F_{\text{I}}(s,0)$ has the lower bound

$$\lim_{s\to\infty}\sup s^5(\ln s)^{2m+2}\partial_t^m \,\mathrm{Im}F_{\mathrm{I}}(s,0)>0\,. \qquad (10)$$

In the case when at least one of the two conditions

$$\lim_{s\to\infty} s^2\partial_t^m \,\mathrm{Re}F_{\mathrm{I}}(s,0)=0 \qquad (11)$$

and

$$\lim_{s\to\infty} s^4\partial_t^m \,\mathrm{Im}F_{\mathrm{I}}(s,0)=0 \qquad (12)$$

does not hold, we find (10) in a similar fashion to (N3.11). In what follows, we investigate the case when both (11) and (12) hold simultaneously. From (5), we find

$$\partial_t^m F_{\mathrm{I}}(s,0)=c_1 s+c_2+\frac{1}{\pi}\int_4^\infty ds'\frac{\partial_t^m \mathrm{Im}F_{\mathrm{I}}(s',0)}{s'-s}+\frac{1}{\pi}\int_4^\infty du'\left[\frac{s}{u'^2}-\frac{u'+4}{u'^2}+\frac{1}{u'-4+s}\right]\partial_t^m \mathrm{Im}F_{\mathrm{II}}(u',0)$$
$$+\frac{m}{\pi}\int_4^\infty du'\left[\frac{1}{u'^2}-\frac{1}{(u'-4+s)^2}\right]\partial_t^{m-1}\mathrm{Im}F_{\mathrm{II}}(u',0)+\sum_{i=2}^m I_i(s)\,, \qquad (13)$$

where

$$c_1\equiv\partial_t^m B(t)|_{t=0}-\frac{1}{\pi}\int_4^\infty ds'\frac{\partial_t^m \mathrm{Im}F_{\mathrm{I}}(s',0)}{s'^2}\,,$$

$$c_2\equiv\partial_t^m A(t)|_{t=0}-\frac{1}{\pi}\int_4^\infty ds'\frac{\partial_t^m \mathrm{Im}F_{\mathrm{I}}(s',0)}{s'}\,, \qquad (14)$$

and

$$I_i(s)\equiv(-1)^i\frac{m!}{(m-i)!}\frac{1}{\pi}\int_4^\infty du'\frac{\partial_t^{m-i}\mathrm{Im}F_{\mathrm{II}}(u',0)}{(u'-4+s)^{1+i}}\,.$$

In the following discussion it will be useful to notice

$$\frac{s}{u'^2}-\frac{u'+4}{u'^2}+\frac{1}{u'-4+s}=\frac{(s-4)^2}{u'^2(u'-4+s)}\quad(>0) \qquad (15)$$

and

$$\frac{1}{u'^2}-\frac{1}{(u'-4+s)^2}=\frac{(s-4)(s-4+2u')}{u'^2(u'-4+s)^2}\quad(>0)\,. \qquad (16)$$

Considering the case where (12) holds, we have an inequality

$$0\leq\partial_t^m \mathrm{Im}F_{\mathrm{I}}(s,0)<s^{-4}\quad\text{as}\quad s\to\infty\,. \qquad (17)$$

Furthermore, we have

$$0\leq\partial_t^j \mathrm{Im}F_{\mathrm{II}}(u,0)<\mathrm{const}\times u(\ln u)^{2j+2}\quad\text{as}\quad u\to\infty\,, \qquad (18)$$

in much the same way as we obtain the Froissart bound.[5] With the help of (17) and (18), we can evaluate various integrals in (13) and (14). First, c_1 and c_2 in (14) are found to be finite. Second, we find from (14)

$$\lim_{s\to\infty} s^{i-2}I_i(s)=0\,. \qquad (19)$$

Finally, we have

$$\int_4^\infty ds'\frac{1}{s'-s}\partial_t^m \mathrm{Im}F_{\mathrm{I}}(s',0)+\frac{1}{s}\int_4^\infty ds'\partial_t^m \mathrm{Im}F_{\mathrm{I}}(s',0)+\frac{1}{s^2}\int_4^\infty ds's'\partial_t^m \mathrm{Im}F_{\mathrm{I}}(s',0)$$
$$=\frac{1}{s^2}\int_4^\infty ds'\frac{s'^2}{s'-s}\partial_t^m \mathrm{Im}F_{\mathrm{I}}(s',0)\quad\left(\equiv\frac{H(s)}{s^2}\right), \qquad (20)$$

where

$$\lim_{s\to\infty}|H(s)|<\mathrm{const}\,. \qquad (21)$$

The inequality (21) can be obtained from (11) and (12) by applying the technique described in the Appendix of our previous paper.[1]

If we assume that

$$d_1\equiv\frac{1}{\pi}\int_4^\infty du'u'^{-2}\partial_t^m \mathrm{Im}F_{\mathrm{II}}(u',0) \qquad (22)$$

is infinite, we have

$$\lim_{s\to\infty}\int_4^\infty du'\frac{s-4}{u'^2(u'-4+s)}\partial_t^m \mathrm{Im}F_{\mathrm{II}}(u',0)=\infty\,. \qquad (23)$$

However, (23) contradicts with (11)–(13), (15), (16), and (19)–(21). Therefore we conclude

$$d_1<\infty\,. \qquad (24)$$

Then (24) gives[6]

$$\frac{1}{\pi}\int_4^\infty du'\frac{(s-4)^2}{u'^2(u'-4+s)}\partial_t^m \mathrm{Im}F_{\mathrm{II}}(u',0)-d_1 s$$
$$=-\frac{1}{\pi}\int_4^\infty du'\frac{s(u'+4)-16}{u'^2(u'-4+s)}$$
$$\times\partial_t^m \mathrm{Im}F_{\mathrm{II}}(u',0)\ [\equiv G(s)]\,,$$

$$\lim_{s\to\infty} s^{-1}G(s)=0\,, \qquad (25)$$

and

$$\int_4^\infty du'u'^{-3}\partial_t^{m-1}\mathrm{Im}F_{\mathrm{II}}(u',0)<\infty\,. \qquad (26)$$

In obtaining (26), we have used the inequality

$$0<\partial_t{}^m \operatorname{Im}F_{\rm II}(u,0)-\frac{1}{m(u-4)}\left[\partial_t{}^{m-1}\operatorname{Im}F_{\rm II}(u,0)-\frac{16\pi}{(m-1)!}\left(\frac{u}{u-4}\right)^{1/2}\frac{(2m-1)!}{(u-4)^{m-1}}\right], \tag{27}$$

which results from (7). Since the integral

$$\begin{aligned}J(s)&\equiv\int_4^\infty du'\frac{s-4+2u'}{u'^2(u'-4+s)^2}\,\partial_t{}^{m-1}\operatorname{Im}F_{\rm II}(u',0)\\&=\int_4^\infty du'\frac{1}{u'^3}\,\frac{s-4+2u'}{s-4+u'}\left(1-\frac{s-4}{s-4+u'}\right)\partial_t{}^{m-1}\operatorname{Im}F_{\rm II}(u',0)\end{aligned} \tag{28}$$

is found to be uniformly convergent on account of (26), we have[6]

$$\lim_{s\to\infty}J(s)=0\,. \tag{29}$$

Equation (29) together with (11)–(16), (19)–(22), (25), (28), and (29) leads to

$$c_1+d_1=0 \tag{30}$$

and

$$\begin{aligned}\partial_t{}^m F_{\rm I}(s,0)&=c_2+\frac{1}{\pi}\int_4^\infty ds'\frac{\partial_t{}^m\operatorname{Im}F_{\rm I}(s',0)}{s'-s}\\&\quad+\frac{1}{\pi}\int_4^\infty du'\left[-\frac{s(u'+4)-16}{u'^2(u'-4+s)}\,\partial_t{}^m\operatorname{Im}F_{\rm II}(u',0)+m\,\frac{(s-4)(s-4+2u')}{u'^2(u'-4+s)^2}\,\partial_t{}^{m-1}\operatorname{Im}F_{\rm II}(u',0)\right]\\&\quad+\sum_{i=2}^m I_i(s)\,.\end{aligned} \tag{31}$$

With the help of (7), the integrand $I(u';s,m)$ of the second integral on the right-hand side of (31) is divided into two parts, $I(u';s,m)_{l<2m}$ and $I(u';s,m)_{l\geq 2m}$. [Hereafter we attach to I the suffix $l<2m$ ($l\geq 2m$), in order to denote that I is composed of contributions from partial waves $l<2m$ ($l\geq 2m$).] Then we easily find by using (7)

$$\lim_{s\to\infty}\int_4^\infty du' I(u';s,m)_{l<2m}=\int_4^\infty du'\left[-\frac{u'+4}{u'^2}\,\partial_t{}^m\operatorname{Im}F_{\rm II}(u',0)+\frac{m}{u'^2}\,\partial_t{}^{m-1}\operatorname{Im}F_{\rm II}(u',0)\right]_{l<2m}$$

$$\left(\equiv\int_4^\infty du' I(u';\,m)_{l<2m}\right). \tag{32}$$

Since $I(u';s,m)_{l\geq 2m}$ is shown by (7) to be negative definite, the assumption

$$d_2\equiv\int_4^\infty du' I(u';m)_{l\geq 2m}=-\infty \tag{33}$$

leads to

$$\lim_{s\to\infty}\int_4^\infty du' I(u';s,m)_{l\geq 2m}=-\infty\,. \tag{34}$$

However, (34) contradicts (11), (12), (14)–(16), (19)–(21), (31), and (32), so that d_2 should be finite. Then we find[6]

$$c_2+\frac{1}{\pi}\int_4^\infty du' I(u';m)=0 \tag{35}$$

and

$$\begin{aligned}\partial_t{}^m F_{\rm I}(s,0)&=\frac{1}{\pi}\int_4^\infty ds'\frac{\partial_t{}^m\operatorname{Im}F_{\rm I}(s',0)}{s'-s}+\frac{1}{\pi}\int_4^\infty du'\left[\frac{1}{u'-4+s}\,\partial_t{}^m\operatorname{Im}F_{\rm II}(u',0)-\frac{m}{(u'-4+s)^2}\,\partial_t{}^{m-1}\operatorname{Im}F_{\rm II}(u',0)\right.\\&\quad\left.+\frac{m(m-1)}{(u'-4+s)^3}\,\partial_t{}^{m-2}\operatorname{Im}F_{\rm II}(u',0)\right]+\sum_{i=3}^m I_i(s)\,.\end{aligned} \tag{36}$$

By applying the same logic to the second integral on the right-hand side of (36), we find (for $m \geq 3$)

$$\int_4^\infty ds'\,\partial_t{}^m \mathrm{Im}F_\mathrm{I}(s',0) = \int_4^\infty du'\,\partial_t{}^m \mathrm{Im}F_\mathrm{II}(u',0) \tag{37}$$

and

$$\begin{aligned}\partial_t{}^m F_\mathrm{I}(s,0) = &-\frac{1}{s^2}\frac{1}{\pi}\int_4^\infty ds' s'\,\partial_t{}^m \mathrm{Im}F_\mathrm{I}(s',0) + \frac{H(s)}{\pi s^2}\\ &+\frac{1}{\pi}\int_4^\infty du'\Big[-\frac{1}{s}\frac{u'-4}{u'-4+s}\,\partial_t{}^m \mathrm{Im}F_\mathrm{II}(u',0) - \frac{m}{(u'-4+s)^2}\,\partial_t{}^{m-1}\mathrm{Im}F_\mathrm{II}(u',0)\\ &+\frac{m(m-1)}{(u'-4+s)^3}\,\partial_t{}^{m-2}\mathrm{Im}F_\mathrm{II}(u',0) - \frac{m(m-1)(m-2)}{(u'-4+s)^4}\,\partial_t{}^{m-3}\mathrm{Im}F_\mathrm{II}(u',0)\Big] + \sum_{i=4}^m I_i(s)\,.\end{aligned} \tag{38}$$

Similarly treating the second integral [denoted by $K(s)$] on the right-hand side of (38), we obtain by noticing (21) and $m \geq 3$

$$\lim_{s\to\infty} s^2K(s) = -\frac{1}{\pi}\int_4^\infty du'[(u'-4)\partial_t{}^m \mathrm{Im}F_\mathrm{II}(u',0) + m\partial_t{}^{m-1}\mathrm{Im}F_\mathrm{II}(u',0)]\,. \tag{39}$$

Then (11), (12), (38), and (39) lead to

$$\lim_{s\to\infty} H(s) = \int_4^\infty ds' s'\partial_t{}^m \mathrm{Im}F_\mathrm{I}(s',0) + \int_4^\infty du'[(u'-4)\partial_t{}^m \mathrm{Im}F_\mathrm{II}(u',0) + m\partial_t{}^{m-1}\mathrm{Im}F_\mathrm{II}(u',0)]\,. \tag{40}$$

On the other hand, the integral $H(s)$ defined by (20) gives[6]

$$\lim_{s_I\to\infty,\, s_R=2} H(s) = 0\,, \tag{41}$$

with

$$s \equiv s_R + is_I\,.$$

Then (40), (41), and the polynomial upper boundedness of $H(s)$ [which results from (9) and (38)] make it possible to apply the Phragmén-Lindelöf theorem[7] to $H(s)$. Thus we find that the right-hand side of (40) should be zero, so that

$$\partial_t{}^m \mathrm{Im}F_\mathrm{I}(s',0) \equiv \partial_t{}^{m-1}\mathrm{Im}F_\mathrm{II}(u',0) \equiv 0\,. \tag{42}$$

Therefore, there never occurs the case when both (11) and (12) hold simultaneously, provided that we exclude the case (42).[8]

Our conclusion is that we have for any fixed m (≥ 3) a nonvanishing constant c_m and a sequence $s_n \to \infty$ such that

$$\lim_{n\to\infty} s_n{}^5(\ln s_n)^{2m+2}\,\partial_t{}^m \mathrm{Im}F_I(s_n,0) = c_m(>0)\,. \tag{43}$$

The elastic scattering amplitude $F_I(s,t)$ among two spinless particles A and B (of equal and unit mass) satisfies

$$\partial_t{}^m \mathrm{Im}F_I(s,0) = \frac{16\pi}{m!}\left(\frac{s}{s-4}\right)^{1/2}\frac{1}{(s-4)^m}\times\sum_{l=m}^\infty (2l+1)\frac{(l+m)!}{(l-m)!}\,\mathrm{Im}f_l^I(s)\,, \tag{44}$$

with

$$0 \leq \mathrm{Im}f_l^I(s) \leq 1. \tag{45}$$

From (43) we find[9] that $\partial_t{}^m \mathrm{Im}F_I(s,0)$, at sufficiently high energy $s=s_n$, can well be estimated by the partial waves up to $L=K\sqrt{s}\ln s$, provided K is taken sufficiently large. Therefore, we have *for* $m \geq 3$

$$\partial_t{}^m \mathrm{Im}F_I(s_n,0) \simeq \frac{16\pi}{m!}\left(\frac{s_n}{s_n-4}\right)^{1/2}\frac{1}{(s_n-4)^m}\times\sum_{l=m}^{K\sqrt{s_n}\ln s_n}(2l+1)\frac{(l+m)!}{(l-m)!}\,\mathrm{Im}f_l^I(s_n) \geq d_m s_n{}^{-5}(\ln s_n)^{-2m-2}\quad(>0)\,, \tag{46}$$

where the constant d_m is *finite* and smaller than c_m (which might be infinite). On the other hand, (45) gives *for* $i \leq m$

$$\partial_t{}^i \mathrm{Im}F_I(s_n,0) \geq \frac{16\pi}{i!}\left(\frac{s_n}{s_n-4}\right)^{1/2}\frac{1}{(s_n-4)^i}\times\sum_{l=m}^{K\sqrt{s_n}\ln s_n}(2l+1)\frac{(l+i)!}{(l-i)!}\,\mathrm{Im}f_l^I(s_n)\,. \tag{47}$$

When we have the inequality (46), the smallest value of the right-hand side of (47) takes place when

$$\mathrm{Im}f_l^I(s_n) \neq 0 \quad \text{only for } l(\sqrt{s_n}\,\ln s_n)^{-1} \simeq O(1)\,, \tag{48}$$

so that

$$\partial_t{}^i \,\mathrm{Im}F_{\mathrm{I}}(s_n, 0) \geq c_{(i)} s_n{}^{-5} (\ln s_n)^{-4m-2+2i} . \qquad (49)$$

It should be noticed that the lower bound (49) holds for the *same* sequence $s_n \to \infty$ as in (46). Taking $m = 3$ in (49), we obtain

$$\lim_{s\to\infty} \sup s^5 (\ln s)^{12} \partial_t \,\mathrm{Im}F_{\mathrm{I}}(s, 0) > 0 \qquad (50)$$

and

$$\lim_{s\to\infty} \sup s^5 (\ln s)^{10} \partial_t{}^2 \,\mathrm{Im}F_{\mathrm{I}}(s, 0) > 0. \qquad (51)$$

It is easy to see[2] that (50) leads to the lower bound on the *s-channel*[10] slope parameter

$$\lim_{s\to\infty} \sup s^5 (\ln s)^{12} \frac{\partial_t \,\mathrm{Im}F_I(s, 0)}{\mathrm{Im}F_{\mathrm{I}}(s, 0)} > 0 , \qquad (52)$$

while the lower bound on $\partial_t{}^2 \,\mathrm{Im}F_{\mathrm{I}}(s, 0)$ has never been investigated previously, so that (51) is an entirely new result.

If we use the Froissart bound[4]

$$\mathrm{Im}F_{\mathrm{I}}(s, 0) \leq \mathrm{const} \times s (\ln s)^2 \quad \text{as} \quad s \to \infty , \qquad (53)$$

the inequality (10) for any fixed m (≥ 3) (Ref. 11) leads to

$$\lim_{s\to\infty} \sup s^6 (\ln s)^{2m+4} \frac{\partial_t{}^m \,\mathrm{Im}F_{\mathrm{I}}(s, 0)}{\mathrm{Im}F_{\mathrm{I}}(s, 0)} > 0 , \qquad (54)$$

which can be rewritten into

$$\lim_{s\to\infty} \sup s^{6/m} (\ln s)^{2+4/m} B_1(s) > 0, \qquad (55)$$

in the special cases when we have the scaling (3). Our result (55) can also be interpreted in the following way: Cases where the scaling (3) holds occur only when the slope parameter is large enough to satisfy (55).

---

[1]S. Naito, Phys. Rev. D 13, 2884 (1976). [Some remarks about this reference: We have given (3.11) and (3.20) for the special $M_A = M_B = 1$. Generalization of them is straightforward and does not essentially change results. Incidentally the first and the second ~ symbols in (3.11) should be replaced with = and ≤, respectively.]

[2]B. K. Chung, Nucl. Phys. B105, 178 (1976). Chung asserts in this paper that my derivation of (3.1) and (3.2) in Ref. 1 involves unnecessary complications, and he shows the way he can rederive (3.1) and (3.2). However, his method does not succeed in deriving our result: Even if

$$I_i(s) \equiv \int_4^\infty dx \, \frac{x^i A_s'(x, t)}{x - s} \to 0 \quad \text{as} \quad s \to \infty$$

(for $i = 0, 2$) is not true, we have

$$I_i(s) \to c_i \quad (\neq 0)$$

for *at least* one sequence of $s \to \infty$, and c_i might even be *infinite*. Therefore, his logic does not permit applying the Phragmén-Lindelöf theorem to $I_i(s)$ [especially in the case when $\int_4^\infty dx \, A_u(x, t) = \infty$].

[3]T. Uchiyama asserts in his paper [Prog. Theor. Phys. 55, 1871 (1976)] that both $B_1(s)$ and $B_2(u)$ satisfy (1.1) and (1.2). Although his logic is useful in some points, his derivation contains an essential error [his (3.26)], so that his conclusion is not only wrong but also his method does not succeed in deriving (3.1) and (3.2) in Ref. 1.

[4]Y. S. Jin and A. Martin, Phys. Rev. 135, B1375 (1964).

[5]M. Froissart, Phys. Rev. 123, 1053 (1961); A. Martin, *ibid.* 129, 1432 (1963).

[6]E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis* (Cambridge Univ. Press, New York, 1953), p. 70.

[7]E. C. Titchmarsh, *The Theory of Functions* (Oxford Univ. Press, New York, 1939), second edition, p. 177 (5.61) and p. 179 (5.64).

[8]The condition (42) might be physically unrealizable for the following reason: (42) means that we have

$$\mathrm{Im} f_l^{\mathrm{I}}(s') \equiv \mathrm{Im} f_{l-1}^{\mathrm{II}}(u') \equiv 0 \qquad (\mathrm{I})$$

for any s', u' and l ($\geq m$). Then (I) leads to the fact that interactions between two particles A and B should not take place at any high energies s', so far as the angular momentum l is larger than m. At this stage, it is useful to notice the two facts that the relation $l \sim b(s' - 4)^{1/2}/2$ (where b is the impact parameter) holds at high energies and that there usually exist extended pion clouds around A and B. Therefore, interactions between A and B are expected to take place when b is about the pion Compton wavelength, so that (I) cannot be satisfied.

[9]R. J. Eden, *High Energy Collisions of Elementary Particles* (Cambridge Univ. Press, New York, 1967), p. 170.

[10]In the *s-u symmetric* case, (1) gives the stronger lower bound

$$\lim_{s\to\infty} \sup s^5 (\ln s)^4 B_1(s) > 0$$

than (2).

[11]These results can be improved by the technique in Ref. 2: Improved results are given by the substitutions ($6 \to 5$ and $2m + 4 \to 2m + 2$ for $m = 3, 4, 5$) and ($6 \to 6$ and $2m + 4 \to 0$ for $m = 6$).