Some exact dyon solutions for the classical Yang-Mills field equation*

Ilmun Ju
Institute of Theoretical Science, University of Oregon, Eugene, Oregon 97403
(Received 28 February 1977)

Many exact dyon solutions for the classical Yang-Mills field equation with Higgs scalar fields are obtained in Minkowski space. We accomplish this by solving the field equation for a sourceless Yang-Mills theory using the self-duality condition and a specific ansatz in Euclidean space. Each of our sourceless solutions in SU(2) can be converted into one of the real solutions in SL(2, C) previously discussed by Wu and Yang.

I. INTRODUCTION

Recently 't Hooft[1] has shown that one may find a static solution for the spontaneously broken Yang-Mills field equations that corresponds to the monopole predicted by Dirac in his pioneering work. Julia and Zee[2] found an explicit realization of Schwinger's dyon[3] which possesses both electric and magnetic charge by analyzing a set of the coupled nonlinear differential equations on a computer. Subsequently Prasad and Somerfield[4] discovered an exact regular solution by "shimmying" rather than systematically solving the field equations. Hsu and Mac[5] found a complex solution by "a stroke of luck" which is very similar to that of Prasad and Sommerfield; their solution was later analyzed and discussed in the context of SL(2, C) by Wu and Yang.[6] Even though other finite-energy solutions are obtained in larger gauge groups,[7,8] all the other solutions obtained in SU(2) so far lead to an infinite energy.

It is the purpose of this paper to present a systematic method to obtain the other exact dyon solutions; in this scheme the above-mentioned two exact solutions emerge naturally. The coupled nonlinear differential equations for monopoles and dyons are very difficult to solve analytically. In Sec. II we outline our method of avoiding this difficulty and obtain exact analytic solutions in a certain version of these models. Using a specific ansatz we present solutions closely related to the pseudoparticle solutions[9] in Secs. III and IV. In the conclusions we discuss the masses and electromagnetic fields of the dyons that we obtained and also we discuss the SL(2, C) aspects of our complex solutions.

II. SELF-DUALITY CONDITION IN EUCLIDEAN SPACE

We are interested in a classical SU(2) non-Abelian gauge-field theory with a Higgs triplet,[1,2] i.e.,

$$\mathcal{L} = -\tfrac{1}{4} F^a_{ij} F^a_{ij} + \tfrac{1}{2} F^a_{0i} F^a_{0i} - \tfrac{1}{2} D_i\phi^a D_i\phi^a + \tfrac{1}{2} D_0\phi^a D_0\phi^a + V(\phi)\,, \tag{1}$$

where

$$V(\phi) = \tfrac{1}{2}\mu^2\phi^a\phi^a - \tfrac{1}{4}\lambda\phi^2\phi^a\,,$$
$$D_\mu\phi^a = \partial_\mu\phi^a + e\epsilon_{abc}A^b_\mu\phi^c\,,$$
$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + e\epsilon_{abc}A^b_\mu A^c_\nu\,.$$

The field equations are[10]

$$\partial^\mu F^a_{\mu i} + e\epsilon_{abc}A^{b\mu}F^c_{\mu i} + e\epsilon_{abc}\phi^b D_i\phi^c = 0\,, \tag{2}$$
$$\partial^\mu D_\mu\phi^a + e\epsilon_{abc}A^{b\mu}D_\mu\phi^c - \mu^2\phi^a + \lambda\phi^2\phi^a = 0 \tag{3}$$

and the constraint equation is

$$\partial^\mu F^a_{\mu 0} + e\epsilon_{abc}A^{b\mu}F^c_{\mu 0} + e\epsilon_{abc}\phi^b D_0\phi^c = 0\,. \tag{4}$$

Consider the static case where

$$A^a_0(r) = 0\,, \quad \mu^2 = \lambda = 0\,. \tag{5}$$

Equations (2), (3), and (4) reduce to

$$\partial_j F^a_{ji} + e\epsilon_{abc}A^b_j F^c_{ji} = e\epsilon_{abc}\phi^b D_i\phi^c\,, \tag{6}$$
$$\partial_j D_j\phi^a + e\epsilon_{abc}A^b_j D_j\phi^c = 0\,, \tag{7}$$
$$\partial_j F^a_{j0} + e\epsilon_{abc}A^b_j F^c_{j0} \equiv 0\,. \tag{8}$$

Under the change of variable

$$\phi^a = B^a_0\,, \quad A^a_i \equiv B^a_i\,, \tag{9}$$

Eqs. (6) and (7) become

$$\begin{aligned} &\partial_j F^a_{ji} + e\epsilon_{abc}B^b_j F^c_{ji} + e\epsilon_{abc}B^b_0 F^c_{0i} = 0\,,\\ &\partial_j F^a_{j0} + e\epsilon_{abc}B^b_j F^c_{j0} = 0\,. \end{aligned} \tag{10}$$

The relations are exactly the field equations for a static sourceless Yang-Mills theory in Euclidean space. If one uses the Wu-Yang-'t Hooft-Julia-Zee ansatz, i.e.,

$$\begin{aligned} A^a_i &= \frac{\epsilon_{aij}x^j[1-K(r)]}{er^2}\,,\\ A^a_0 &= \frac{J(r)}{er^2}x^a\,,\\ \phi^a &= \frac{H(r)}{er^2}x^a\,, \end{aligned} \tag{11}$$

Eqs. (2), (3), and (4) reduce to

$$r^2K''=K(K^2+H^2-J^2-1),$$
$$r^2H''=2HK^2+\frac{\lambda}{e^2}(H^3-\frac{e^2}{\lambda}\mu^2H), \tag{12}$$
$$r^2J''=2JK^2.$$

Therefore, if B_i^a and B_0^a solve Eq. (10) in Euclidean space, then

$$A_i^a=B_i^a,$$
$$A_0^a=\pm B_0^a\sinh\gamma, \tag{13}$$
$$\phi^a=\pm B_0^a\cosh\gamma$$

is the solution of Eqs. (6) and (7) in Minkowski space as already noted in the Ref. 4.

In order to find the Euclidean-space solution B^a of Eq. (10), one can use the self-duality condition of Belavin, Polyakov, Schwartz, and Tyupkin.[11] Consider a dual-field strength

$$\tilde{F}^a_{\mu\nu}\equiv\tfrac{1}{2}\epsilon_{\mu\nu\alpha\beta}F^a_{\alpha\beta}$$

and calculate

$$\tilde{F}^a_{\mu\nu}\tilde{F}^a_{\mu\nu}=\tfrac{1}{2}(\epsilon_{\nu\alpha\beta}\epsilon_{\nu\gamma\delta}F^a_{\alpha\beta}F^a_{\gamma\delta}+\epsilon_{ijk}F^a_{k0}\epsilon_{ijl}F^a_{l0})$$
$$=F^a_{ij}F^a_{ij}+2F^a_{i0}F^a_{i0}.$$

Now

$$F^a_{\mu\nu}F^a_{\mu\nu}=F^a_{ij}F^a_{ij}\pm2F^a_{i0}F^a_{i0},$$

where the upper sign is for the Euclidean space while the lower sign is for the Minkowski space. Therefore, it is clear that the only nontrivial self-dual fields are to be found in Euclidean space. The only self-dual Minkowski-space solution is the gauge-rotated vacuum.

One of the most interesting feature of our Euclidean space solution is that, as can be seen from Eq. (10), the solution

$$C_i^a\equiv B_i^a,\quad C_0^a\equiv iB_0^a \tag{14}$$

is a complex solution in Minkowski space. This aspect will be discussed later.

Bearing all these in mind we now proceed to find Euclidean-space solutions B^a of the sourceless gauge-field equation (10).

III. SOLUTIONS

We seek the solution of the form

$$B_i^a=\frac{1}{e}\epsilon_{ai\alpha}\psi^\alpha,$$
$$B_0^a=-\frac{\delta}{e}\psi^a. \tag{15}$$

The self-duality condition is

$$\tfrac{1}{2}\epsilon_{\mu\nu\alpha\beta}F^a_{\alpha\beta}=\delta F^a_{\mu\nu}. \tag{16}$$

Equations (15) and (16) imply that

$$\sum_{i=1}^{3}[\partial_i\psi^i-(\psi^i)^2]=0.$$

Thus we have

$$\psi^\nu=-\partial^\nu\ln\left[\sum_{i=1}^{n}\left(b_i+\frac{\lambda_i}{|\vec{X}-\vec{X}_i|^{1/2}}\right)\right], \tag{17}$$

where b_i, λ_i, and X_i are arbitrary constants. When $b_i=0$, $\lambda_{i\neq1}=0$, $\vec{X}_i=0$, we then have

$$B_i^a=\frac{1}{e}\frac{\epsilon_{ai\alpha}x^\alpha}{r^2},$$
$$B_0^a=-\frac{\delta}{e}\frac{x^a}{r^2}. \tag{18}$$

This Euclidean-space solution corresponds to the dyon solution which Hsu and Mac[5] discovered in Minkowski space, with

$$G_0=\frac{\delta}{e},\quad G_1=0,\quad \gamma=0.$$

More explicitly, in Minkowski space

$$A_i^a=\frac{1}{e}\frac{\epsilon_{ai\alpha}x^\alpha}{r^2},$$
$$A_0^a=-\frac{\delta}{e}\sinh\gamma\frac{x^a}{r^2}, \tag{19}$$
$$\phi^a=-\frac{\delta}{e}\cosh\gamma\frac{x^a}{r^2}.$$

In order to see the meaning of the solution (6) we follow 't Hooft and define a gauge-invariant electromagnetic field strength[12]

$$\overline{F}_{\mu\nu}=\partial_\mu(\hat{\phi}^aA^a_\nu)-\partial_\nu(\hat{\phi}^aA^a_\mu)-\frac{1}{e}\epsilon_{abc}\hat{\phi}^a(\partial_\mu\hat{\phi}^b)(\partial_\nu\hat{\phi}^c), \tag{20}$$

where

$$\hat{\phi}^a=\phi^a/(\phi^b\phi^b)^{1/2}.$$

The electromagnetic field is

$$E_i=\overline{F}_{i0},\quad H_i=\tfrac{1}{2}\epsilon_{ijk}\overline{F}_{jk}. \tag{21}$$

The electric charge and energy of the solution are, respectively,

$$Q=\int d^3x\,\partial_i\overline{F}_{0i},$$
$$M=\int d^3x\,T^{00}, \tag{22}$$

where

$$T^{\mu\nu}=F^{a\mu\lambda}F^{a\nu}_{\lambda}+D^{a\mu}D^{a\mu}+g^{\mu\nu}\mathfrak{L}.$$

As an example we pick the following solutions from solution (17):

$$A_i^a = \frac{\lambda}{e}\frac{\epsilon_{ai\alpha}x^\alpha}{(r+\lambda)r^2},$$
$$A_0^a = -\frac{\delta\lambda}{e}\sinh\gamma\frac{x^a}{(r+\lambda)r^2}, \qquad (23)$$
$$\phi^a = -\frac{\delta\lambda}{e}\cosh\gamma\frac{x^a}{(r+\lambda)r^2}.$$

One can easily show that solutions (18) and (23) are not gauge equivalent. The electromagnetic field and electric charge due to solution (18) are

$$H_i = -\frac{1}{e}\frac{x_i}{r^3},$$
$$E_i = \frac{\delta}{e}\sinh\gamma\frac{x_i}{r^3}, \qquad (24)$$
$$Q = -\frac{e}{\alpha}\delta\sinh\gamma,$$

where α is the fine-structure constant. Equation (23) gives

$$H_i = -\frac{1}{e}\frac{x_i}{r^3},$$
$$E_i = \frac{\delta\lambda}{e}\sinh\gamma\frac{(2r-\lambda)x_i}{(r+\lambda)^2r^3}, \qquad (25)$$
$$Q = e\frac{\delta\lambda}{\alpha}\sinh\gamma\frac{2r-\lambda}{(r+\lambda)^2}.$$

The total charge observed near the origin is $-e(\delta/\alpha)\sinh\gamma$, and it is zero if observed at infinity. This is due to the spherically symmetric charge distribution of the opposite sign

$$\rho(r) = e\frac{\delta\lambda\sinh\gamma}{4\pi\alpha}\left(-\frac{2}{r^2(r+\lambda)^2} - \frac{\delta(r)}{\lambda r^2} + \frac{6\lambda}{r^2(r+\lambda)^3}\right). \qquad (26)$$

This is qualitatively similar to the charge distribution of a neutral atom. It is easy to see that the energies of the system implied by solution (17) diverge.

IV. OTHER SOLUTIONS

One can make another ansatz, instead of Eq. (15):

$$B_i^a = \frac{\phi_1(r)+1}{er^2}\epsilon_{ai\alpha}x^\alpha + \frac{\phi_2(r)}{er^3}(r^2\delta_{ia} - x_ix_a),$$
$$B_0^a = -\delta\frac{\phi_3(r)}{er}x^a. \qquad (27)$$

The self-duality condition (16) implies that

$$\left(\frac{d}{dr} - \delta\phi_3\right)\phi_1 = 0,$$
$$\left(\frac{d}{dr} - \delta\phi_3\right)\phi_2 = 0, \qquad (28)$$
$$-\delta r^2\frac{d\phi_3}{dr} = 1 - \phi_1^2 - \phi_2^2.$$

Equation (28) has two trivial solutions, i.e.,

$$\phi_1 = \phi_2 = 0, \quad \phi_3 = \frac{1}{r} \qquad (29)$$

and

$$\phi_3 = 0, \quad \phi_1 = \cos\beta, \quad \phi_2 = \sin\beta, \qquad (30)$$

where β is an arbitrary constant. The solution (29) is the exact colution (18) previously obtained, while solution (30) is merely a gauge-rotated vacuum, because, for a self-dual static gauge field with $A_0^a = 0$, the field strength $F_{i0}^a = 0$; thus $F_{ij}^a = 0$.

If we choose $\phi_i \neq 0$, Eq. (28) then reads

$$-r^2\frac{d^2\psi}{dr^2} = 1 - (c_1^2 + c_2^2)e^{2\psi}, \qquad (31)$$

where we have assumed

$$\phi_3 = \delta\, d\psi/dr, \quad \phi_1 = c_1e^\psi, \quad \phi_2 = c_2e^\psi \qquad (32)$$

where c_1 and c_2 are arbitrary constants. Contrary to the pseudoparticle case, Eq. (31) reduces to the one-dimensional Liouville equation

$$\frac{d^2\rho}{dr^2} = e^{2\rho} \qquad (33)$$

if we put $\psi = \ln r - \frac{1}{2}\ln(c_1^2 + c_2^2) + \rho$. Equation (33) has three distinct solutions, i.e.,

$$\rho = -\ln(r+b), \qquad (34a)$$
$$\rho = -\ln\left(\frac{\sin\lambda(r+b)}{\lambda}\right), \qquad (34b)$$
$$\rho = \lambda r + \ln|2\lambda|\eta|^{1/2}| - \ln(1+\eta e^{2\lambda r}), \qquad (34c)$$

where b, λ, and η are arbitrary constants. More explicitly we have

$$B_i^a = \left(1 + \frac{r\cos\theta}{(r+b)}\right)\frac{\epsilon_{ai\alpha}x^\alpha}{er^2} + \frac{\sin\theta}{e(r+b)r^2}(r^2\delta_{ia} - x_ix_a),$$
$$B_0^a = -\frac{\delta b}{e}\frac{x^a}{(r+b)r^2} \qquad (35a)$$

and

$$B_i^a = \left(1 + \frac{\cos\theta\lambda r}{\sin\lambda(r+b)}\right)\frac{\epsilon_{ai\alpha}x^\alpha}{er^2} + \frac{\sin\theta\lambda r}{\sin\lambda(r+b)er^3} \times (r^2\delta_{ia} - x_ix_a),$$
$$B_0^a = -\frac{\delta}{e}[1 - \lambda r\cot\lambda(r+b)]\frac{x^a}{r^2} \qquad (35b)$$

and

$$B_i^a = \left(1 + \frac{\cos\theta 2\lambda|\eta|^{1/2}e^{\lambda r}}{1+\eta e^{2\lambda r}}\right)\frac{\epsilon_{ai\alpha}x^\alpha}{er^2} + \frac{2\lambda|\eta|^{1/2}e^{\lambda r}}{(1+\eta e^{2\lambda r})er^3} \times (r^2\delta_{ia} - x_ix_a),$$
$$B_0^a = -\frac{\delta}{e}\left(1 + \lambda r - \frac{2r\lambda\eta e^{\lambda r}}{1+\eta e^{2\lambda r}}\right)\frac{x^a}{r^2}, \qquad (35c)$$

where $\tan\theta = c_2/c_1$. It is very interesting to note that if we choose $c_1=-|c|$, $c_2=0$, $b=0$, and $\lambda=i\beta$ in Eq. (35b) we have precisely the Prasad and Sommerfield solution

$$A_i^a=\left(1-\frac{\beta r}{\sinh\beta r}\right)\frac{\epsilon_{ai\alpha}x^\alpha}{er^2},$$
$$A_0^a=-\frac{\delta}{e}\sinh\gamma(1-\beta r\coth\beta r)\frac{x^a}{r^2}, \qquad (36)$$
$$\phi^a=-\frac{\delta}{e}\cosh\gamma(1-\beta r\coth\beta r)\frac{x^a}{r^2}.$$

One of the complex solutions is the Hsu and Mac Minkowski-space solution

$$c_i^a=\left(1-\frac{\beta r}{\sinh\beta r}\right)\frac{\epsilon_{ai\alpha}x^\alpha}{er^2},$$
$$c_0^a=i(1-\beta r\coth\beta r)\frac{x^a}{er^3}. \qquad (37)$$

Moreover, it easy to show by calculating $\mathrm{Tr}F_{\mu\nu}F_{\mu\nu}$, that the solutions with different values of θ in Eqs. (35) are not gauge equivalent.

V. CONCLUSION

All the exact dyon solutions we found so far are singular at the origin, except the Prasad and Sommerfield solution (36). In fact we proved that this one is the only possible regular, and thus finite-energy, solution as long as we use the ansatz (11) and $\mu=\lambda=0$; this follows since the ansatz (11) is a special case of ansatz (27) while the solution (36) is the only regular solution among the solutions (35). However, the divergence we encounter here may not be a fundamental problem to the extent that we are in the same situation as for an electron. A peculiar property of these dyon solutions is that while magnetic unit charge is concentrated at the center, the electric charges of both sign are spread all over the space.

Even though the complex-field solution in Minkowski space in the SU(2) case given by the prescription (14) has complex isotopic spin, it is all real in SL(2, C). As pointed out by Wu and Yang, their physical meaning is not clear at this moment. We conclude with the remark that it is rather straightforward to prove that any solution with ansatz (27) and Eq. (16) indeed satisfies the field equation in Euclidean space.

Finally, I deeply appreciate the many discussions with Professor R. C. Hwa and kindnesses he extended to me. I also thank Professor N. Deshpande, Professor J. Leahy, and K. Choi for discussions.

---

*Research supported in part by U. S. Energy Research and Development Administration under Contract No. AT(45-1)-2230.

[1]G. 't Hooft, Nucl. Phys. B79, 276 (1974).
[2]B. Julia and A. Zee, Phys. Rev. D 11, 2227 (1975).
[3]J. Schwinger, Science 165, 757 (1969).
[4]M. K. Prasad and C. M. Sommerfield, Phys. Rev. Lett. 35, 760 (1975).
[5]J. P. Hsu and E. Mac, J. Math. Phys. 18, 100 (1977).
[6]T. T. Wu and C. N. Yang, in *Properties of Matter under Unusual Conditions*, edited by H. Mark and S. Fernbach (Interscience, New York, 1969), pp. 349-354; T. T. Wu and C. N. Yang, Phys. Rev. D 13, 3233 (1976).
[7]W. Marciano, H. Pagels, and Z. Parsa, Phys. Rev. D 15, 1044 (1977).
[8]D. Toussaint and F. Wilczek (unpublished).
[9]E. Witten, Phys. Rev. Lett. 38, 121 (1977); G. 't Hooft (unpublished).
[10]Our metric is $g_{00}=-g_{11}=-g_{22}=-g_{33}=1$.
[11]A. A. Belavin *et al*., Phys. Lett. 59B, 85 (1975).
[12]The fact that the definition of the electromagnetic field strength is not determined by dynamics and thus another definition is possible is emphasized by J. P. Hsu.