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Note on the Abelian Higgs-Kibble model on a lattice: Absence of spontaneous magnetization

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We study the magnetization (expectation value of the scalar field in the presence of a symmetry-breaking term) for the Abelian Higgs-Kibble model quantized on a Euclidean space-time lattice according to Wilson's procedure and show that it vanishes when the external field goes to zero. More explicitly, we show that the gauge invariance of the infinite-volume system is recovered when the symmetry-breaking term is removed.

I. INTRODUCTION

In a series of recent papers[1,2,3,4] we discussed the existence of the infinite-volume limit for a class of Abelian lattice gauge models ranging from a gauge-invariant version of the *XY* model[5,6] to scalar electrodynamics and the Abelian Higgs-Kibble model[7,8,9] quantized on a Euclidean space-time lattice according to Wilson's[10] procedure requiring no gauge-fixing terms.

In this paper we take the first steps towards the study of the possibility of critical behavior in such models. We examine the possibility that an external field, coupled in the most natural way to the system, may drive it into an ordered state: We not only rule out this possibility in the sense that we prove that there is no residual magnetization after the external field has been sent to zero, but also give strong mathematical support to the intuitive idea that the local character of the gauge symmetry provides an effective decoupling between the degrees of freedom localized in different space-time regions, which makes the onset of collective phenomena extremely difficult.

This same line of reasoning led in Ref. 11 to the conclusion that, for the gauge-invariant *XY* model, it is impossible to enforce a situation in which any local quantity having a vanishing mean value on its orbit under the action of the gauge group has a nontrivial expectation value. We give a new version of this argument for the same model in Sec. II, where we sharpen its consequences into the statement that the removal of the term explicitly breaking the symmetry from the *infinite-volume* state leads to a gauge-invariant state.

The model in Sec. II, while incorporating the main effects due to gauge symmetry, is in one respect unsatisfactory: It corresponds to a situation in which the modulus of the charged scalar field is not only bounded but also constant; it is completely insensitive to the sign of the coefficient of the mass term in the action which is responsible for the breaking of gauge symmetry in the classical Higgs-Kibble model. While the restriction to a constant modulus for the scalar field can be removed by essentially the same argument as in Sec. II, the study of the unbounded field case, which is necessary to recover the full Higgs-Kibble model, requires particular care and new arguments, to which Sec. III is devoted.

For the sake of brevity we will refer to Refs. 1–4 for the proof of the existence of the relevant limits (in the cutoff volume and in the external field). We only recall that, as the full apparatus of the Griffiths-Kelly-Sherman (GKS) inequalities is available for the models considered here, the existence proofs follow from monotonicity arguments supplemented, in the unbounded field case, by suitable upper bounds.

II. THE GAUGE-INVARIANT *XY* MODEL

On the lattice Z^d we consider the system described, in the cutoff volume Λ, by the action

$$U_\Lambda = -\beta_l \sum_{n,n'\in\Lambda} \cos[\theta(n) - \theta(n') - A(n,n')] - \beta_p \sum_{P\subseteq\Lambda} \cos[\partial A(P)],$$

where β_l and β_p are non-negative constants, $\sum_{n,n'\in\Lambda}$ extends to the links [ordered pairs (n, n') of nearest-neighbor sites] in Λ, and $\sum_{P\subseteq\Lambda}$ extends to the plaquettes (elementary squares on the lattice) in Λ,

$$-\pi \le \theta(n) \le \pi, \quad -\pi \le A(n,n') \le \pi .$$

$\partial A(P)$, if the plaquette P has the consecutive vertices n_1, n_2, n_3, n_4, stands for

$$A(n_1,n_2)+A(n_2,n_3)+A(n_3,n_4)+A(n_4,n_1).$$

The expectation of any function f of the configuration $\{\theta, A\}$ of the system in the cutoff volume Λ is defined by

$$\langle f\rangle_\Lambda = Z_\Lambda^{-1}\int d\sigma_\Lambda[\exp(-U_\Lambda)]f,$$

where

$$d\sigma_\Lambda = \prod_{n\in\Lambda}\frac{d\theta(n)}{2\pi}\prod_{(n,n')\in\Lambda}\frac{dA(n,n')}{2\pi}$$

and

$$Z_\Lambda = \int d\sigma_\Lambda \exp(-U_\Lambda).$$

The field-theoretical interest of this model will later result from its close connection with the full model in Sec. III. As a statistical-mechanical model, it can be described as a planar classical Heisenberg model whose global rotational symmetry has been promoted to a local one by the introduction of the gauge field A.

We wish to study the effect of the introduction of an external field $\vec{h}$ coupled to the spins

$$\vec{\sigma}(n) = (\cos\theta(n), \sin\theta(n))$$

through a term

$$-\vec{h}\cdot\sum_{n\subset\Lambda}\vec{\sigma}(n)$$

to be added to U_Λ. We denote the volume cutoff expectations in the presence of the external field $\vec{h}$ by

$$\langle f\rangle_\Lambda^{\vec{h}} = Z_\Lambda^{-1}(\vec{h})\int d\sigma_\Lambda\left\{\exp\left[-U_\Lambda + \sum_{n\subset\Lambda}\vec{h}\cdot\vec{\sigma}(n)\right]\right\}f,$$

with

$$Z_\Lambda(\vec{h}) = \int d\sigma_\Lambda \exp\left[-U_\Lambda + \sum_{n\in\Lambda}\vec{h}\cdot\vec{\sigma}(n)\right].$$

First of all we estimate the effect of a gauge transformation on the infinite-volume state

$$\langle\,\rangle^{\vec{h}} = \lim_{\Lambda\nearrow Z^d}\langle\,\rangle_\Lambda^{\vec{h}}.$$

If f is a function of the field configuration in a fixed bounded region $M\subset Z^d$, which means a periodic[12] function of the variables θ and A in M, and χ is a real function on the lattice giving the gauge transformation

$$\theta(n)\to\theta_\chi(n) = \theta(n) + \chi(n),$$

$$A(n,n')\to A_\chi(n,n') = A(n,n') + \chi(n) - \chi(n'),$$

we set

$$f_\chi(\theta, A) = f(\theta_{-\chi}, A_{-\chi}).$$

Notice that

$$f_\chi = f_{\chi_M},$$

where

$$\chi_M(n) = \begin{cases}\chi(n), & n\in M\\ 0, & n\notin M.\end{cases}$$

If in the integral defining $\langle f_\chi\rangle_\Lambda^{\vec{h}}$, $\Lambda\supseteq M$, we perform the changes of variables

$$\theta'(n) = \theta(n) - \chi_M(n),$$

$$A'(n,n') = A(n,n') - \chi_M(n) + \chi_M(n'),$$

as U_Λ is invariant under this change of variables, having the form of a gauge transformation, and the symmetry-breaking term reads, in the new variables,

$$-\vec{h}\cdot\sum_{n\in\Lambda}\vec{\sigma}'(n) - \vec{h}\cdot\sum_{n\in M}\delta\vec{\sigma}'(n),$$

where

$$\delta\vec{\sigma}'(n) = (\cos[\theta'(n) + \chi_M(n)], \sin[\theta'(n) + \chi_M(n)]) - (\cos\theta'(n), \sin\theta'(n)),$$

exploiting the periodicity in θ and A, we conclude that

$$\langle f_\chi\rangle_\Lambda^{\vec{h}} = \left\langle f\exp\left[\vec{h}\cdot\sum_{n\in M}\delta\vec{\sigma}(n)\right]\right\rangle_\Lambda^{\vec{h}}.$$

From this inequality we obtain the bound

$$|\langle f_\chi\rangle_\Lambda^{\vec{h}} - \langle f\rangle_\Lambda^{\vec{h}}| \le [\exp(|\vec{h}|k) - 1]\langle|f|\rangle_\Lambda^{\vec{h}},$$

where use has been made of the obvious inequality

$$|\exp(ax) - 1| \le \exp(|ax|) - 1$$

and

$$k = \sup_\theta\sum_{n\in M}|\delta\vec{\sigma}(n)|.$$

As this bound goes through to the thermodynamic limit, and as $[\exp(|\vec{h}|k) - 1]$ goes to zero as $\vec{h}$ goes to zero, we can conclude that, in the state

$$\langle\,\rangle' = \lim_{h\to 0}\lim_{\Lambda\nearrow Z^d}\langle\,\rangle_\Lambda^{\vec{h}},$$

$$\langle f_\chi\rangle' = \langle f\rangle',$$

for any local function of the configurations: the state is, therefore, gauge invariant. In particular,[13] from gauge invariance, it follows that

$$\langle\vec{\sigma}\rangle' = \vec{0}.$$

III. THE FULL MODEL

In this section we consider models corresponding to a Euclidean action of the form

$$U_\Lambda = \sum_{n\in\Lambda}P(\rho(n)) - \beta_l\sum_{n,n'\in\Lambda}\rho(n)\rho(n')\cos[\theta(n) - \theta(n') - A(n,n')] - \beta_p\sum_{P\subseteq\Lambda}\cos[\partial A(P)]. \quad (1)$$

Refer to Refs. 3, 4, 10 for more details and motivations; we only recall here that this action stems from Wilson's procedure for promoting to a local symmetry the global SO(2) symmetry of a theory describing a two-component Euclidean field,

$$\vec{\varphi}(n) = (\rho(n)\cos\theta(n), \rho(n)\sin\theta(n)),$$

on a lattice self-interacting through a polynominal self-interaction P.

The proofs of the existence of the infinite-volume limit in Refs. 3, 4 refer to any fourth-degree even polynomial bounded below. In particular, if the coefficient of the second-degree term is negative, we are dealing with a lattice version of the Higgs-Kibble model.

We use the same notations as in Sec. II to indicate expectations for this model, with the only obvious difference that here $d\sigma_\Lambda$ stands for

$$\prod_{n\in\Lambda}\frac{d\theta(n)}{2\pi}\prod_{n\in\Lambda}\rho(n)d\rho(n)\prod_{(n,n')\in\Lambda}\frac{dA(n,n')}{2\pi},$$

where $d\rho$ is the Lebesgue measure on $[0,+\infty)$.

Notice that the model in Sec. II corresponds to considering only those configurations in which the moduli $\rho(n)$ are "frozen" to the value $\rho(n)=1$, so that, while taking into account most of the effects due to the phases, it neglects those effects which are due to the additional couplings $\rho(n)\rho(n')$ and to the unbounded character of the "spins" $\vec{\varphi}(n)$.

The main tool for obtaining an estimate, for the full model, of the effect of a gauge transformation on the state $\langle\,\rangle^{\vec{h}}$ is the chessboard estimate of Refs. 3, 4, 14, 15. This estimate describes the effect on the free-energy density (the "pressure" in field-theoretical language[16]) of the system due to a perturbation term

$$-\sum_{n\in\Lambda}k(n)\rho(n),\quad k(n)\geq 0$$

in the Gibbs exponent.

This estimate simply expresses the fact that the free-energy density in an external field $k(n)$ varying with the site n is smaller than the average over n of the free-energy densities corresponding to the situations in which the field k is everywhere fixed at the value it has at point n. Explicitly, if we set[17]

$$\alpha_\Lambda(k,\vec{h})=|\Lambda|^{-1}\ln\int d\sigma_\Lambda\exp\Big[-U_\Lambda+\vec{h}\cdot\sum_{n\in\Lambda}\vec{\varphi}(n)+k\sum_{n\in\Lambda}\rho(n)\Big],\qquad(2)$$

it states that

$$\Big\langle\exp\Big[\sum_{n\in\Lambda}k(n)\rho(n)\Big]\Big\rangle_\Lambda^{\vec{h}}\leq\exp\Big\{\sum_{n\in\Lambda}[\alpha_\Lambda(k(n),\vec{h})-\alpha_\Lambda(0,\vec{h})]\Big\}.\qquad(3)$$

If now $f\geq 0$ is a function of the field configuration in a bounded region M, we can estimate, using the Griffiths-Kelly-Sherman inequalities, that for $\Lambda\supset M$

$$\langle f\rangle_\Lambda^{\vec{h}}\leq Z_M^{-1}(\vec{h})\int d\sigma_M f\exp\Big[-U_M+\vec{h}\cdot\sum_{n\in M}\vec{\varphi}(n)\Big]\exp[\beta_p n(\partial M)]Z_{\Lambda\setminus M}^{-1}(\vec{h})\times\int d\sigma_{\Lambda\setminus M}\exp\Big[-U_{\Lambda\setminus M}+\vec{h}\cdot\sum_{n\in\Lambda\setminus M}\vec{\varphi}(n)+\beta_l\sum_{n,n'}\rho(n')\rho(n)\Big],\qquad(4)$$

where $n(\partial M)$ is the number of plaquettes on the outer boundary of M, and $\sum_{n,n'}$ extends to the nearest-neighbor sites $n\in\Lambda\setminus M$ and $n'\in M$. The last term in the previous inequality is exactly of the form

$$\Big\langle\exp\Big[\beta_l\sum_{n\in\Lambda\setminus M}k(n)\rho(n)\Big]\Big\rangle_{\Lambda\setminus M}^{\vec{h}},$$

which can be estimated by the chessboard inequality.

For each configuration $\rho'(n')$, $n'\in\partial M$ of the ρ's on ∂M, it can be bounded above by the GKS inequalities, by

$$\Big\langle\exp\Big[\beta_l\sum_{n\in\Lambda\setminus M}k_{\rho'}(n)\rho(n)\Big]\Big\rangle_\Lambda^{\vec{h}},$$

which by the chessboard inequality can be bounded above by

$$\exp\Big[\sum_{n\in\Lambda\setminus M}[\alpha_\Lambda(k_{\rho'}(n),\vec{h})-\alpha_\Lambda(0,\vec{h})]\Big],$$

where in the exponent only those $n\in\Lambda\setminus M$ which have at least one nearest neighbor in M appear. We will call $\partial M'$ the set of such n's.

It helps at this point to have a bound on $\alpha_\Lambda(k,\vec{h})$ for a constant field $k\geq 0$ and for $\vec{h}$ small enough, say $|\vec{h}|\leq a$.

Such a bound can be obtained by direct inspection of the integral appearing in the definition of α_Λ and reads as follows:

$$\exp[\alpha_\Lambda(k,\vec{h})]\leq A\exp(Bk^{4/3});\qquad(5)$$

A and B are positive constants *independent of* Λ *and of* $\vec{h}$ for $|\vec{h}|\leq a$.

Combining these results with the observation that

$$\alpha_\Lambda(0,\vec{h})\geq 0$$

and

$$Z_M(\vec{h})\geq Z_M(0)\quad\text{(by GKS inequalities)},$$

we conclude that

$$\langle f\rangle_\Lambda^{\vec{h}} \leq C\left\langle f \exp\left[a\sum_{n\in M}\rho(n) + B\sum_{n\in\partial M'} k_\rho(n)^{4/3}\right]\right\rangle_M, \quad (6)$$

where C is a non-negative constant independent of Λ and $\vec{h}$.

If now f is any bounded continuous function of the configuration in M, we get from the previous inequality

$$\begin{aligned} |\langle f_\chi\rangle_\Lambda^{\vec{h}} - \langle f\rangle_\Lambda^{\vec{h}}| &\leq \left\langle |f| \left[\exp\left(|\vec{h}|\sum_{n\in M}\left|\delta\vec{\varphi}(n)\right|\right) - 1\right]\right\rangle_\Lambda^{\vec{h}} \\ &\leq C\left\langle |f| \left[\exp\left(|\vec{h}|\sum_{n\in M}\left|\delta\vec{\varphi}(n)\right|\right) - 1\right] \right. \\ &\quad \left. \times \exp\left[a\sum_{n\in M}\rho(n) + B\sum_{n\in\partial M'} k_\rho^{(n)4/3}\right]\right\rangle_M. \quad (7)\end{aligned}$$

As the right-hand side is independent of Λ, this inequality goes through to the infinite-volume limit, and as

$$\exp\left(|\vec{h}|\sum_{n\in M}\left|\delta\vec{\varphi}(n)\right|\right) - 1$$

is pointwise convergent to zero as $\vec{h}\to\vec{0}$, we can conclude using Lebesgue's dominanted convergence theorem that

$$\lim_{\vec{h}\to 0}\langle f_\chi\rangle^{\vec{h}} = \lim_{\vec{h}\to 0}\langle f\rangle^{\vec{h}},$$

namely, that the state

$$\langle\ \rangle' = \lim_{\vec{h}'\to 0}\langle\ \rangle^{\vec{h}}$$

is gauge invariant in the sense that

$$\langle f_\chi\rangle' = \langle f\rangle' \quad (8)$$

for any localized f and for any gauge transformation χ.

From this result in particular, it can be immediately seen that the state $\langle\ \rangle'$ does not depend on the direction along which $\vec{h}$ goes to zero.

Owing to the gauge invariance of the state $\langle\ \rangle'$, its generating functional

$$W'_\infty(m,\vec{J}) = \left\langle \exp\left\{i\left[\sum_{n<n'} m(n,n')A(n,n') + \sum_n \vec{J}(n)\cdot\vec{\varphi}(n)\right]\right\}\right\rangle',$$

which, by the same techniques as in Ref. 4 can be shown to be an entire analytic function of $\{\vec{J}(n)\}$, is an even function of $\vec{J}$, so that, in particular, its derivative with respect to $\vec{J}(n)$ at the point $\vec{J}=\vec{0}$ vanishes, leading to the announced result that

$$\langle\vec{\varphi}(n)\rangle' = \lim_{\vec{h}\to\vec{0}}\langle\vec{\varphi}(n)\rangle^{\vec{h}} = 0\,.$$

A brief comment is in order concerning the physical meaning of the previous result. We are not claiming that the Higgs model cannot, for suitable values of the parameters, exhibit a phenomenon of dynamical mass generation for the gauge field. We only point out that, in the framework of Wilson's quantization, in which gauge invariance is strictly preserved at each stage of the argument, such a phenomenon proves to be much more subtle than expected and, though very plausible,[18] is not, in such a framework, related to the spontaneous breaking of the gauge invariance of the *quantized* theory.

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[12] The configuration space of the system is, in fact, the product over the sites and over the links of copies of the circle group.

[13] $\langle\vec{\sigma}\rangle^{\vec{h}}$ can, indeed, be explicitly evaluated, by the change of variables $B(n,n') = A(n,n') - \theta(n) + \theta(n')$ in the A integrations. An interesting result was that it coincided with the magnetization of a *single* spin in the external field $\vec{h}$ and drew our attention to the decoupling effects of gauge invariance. We preferred, however, the longer proof just discussed in order to show that the result does not depend on the particular observable considered and, though this point has not been explicitly stressed, is largely independent of the form of the symmetry-breaking term.

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[16]The term "pressure" better corresponds to the interpretation of $|\Lambda|^{-1}\ln Z_\Lambda = \alpha_\Lambda$, via the Feynman-Kac-Nelson formula, as the ground-state energy density of the Hamiltonian theory underlying the Euclidean theory considered here.

[17]Some care is necessary as to the boundary conditions to be considered here. For more details we refer to Ref. 4.

[18]K. Osterwalder and E. Seiler, Ann. Phys. (N.Y.) (to be published). In this paper it is shown that for models of the Higgs type exhibiting a spontaneous breaking of gauge symmetry at the *classical* level the cluster expansion converges for suitable values of the parameters, leading to a mass gap for the lattice theory. The very interesting problem is whether this mass gap survives in the continuum limit.