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Meson wave functions in two-dimensional quantum chromodynamics

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We consider 't Hooft's eigenvalue problem for the meson spectrum in two-dimensional quantum chromodynamics by defining some alternative formulations whose equivalence we prove. Hence we are able to prove that the spectrum is discrete and of finite multiplicity and to derive bounds (upper and lower) for the eigenvalues (ground state, nth state, and $n \to \infty$ state). We prove that the functions are analytic and use this to carry out explicit numerical calculations of the wave functions for various values of the quark masses and to recalculate the meson spectrum.

I. INTRODUCTION

The aim of this paper is to analyze the mathematical structure of 't Hooft's two-dimensional quantum-chromodynamic (QCD) model for mesons and to present the numerical solutions of the problem. Sections II and III of this paper contain the mathematical results of 't Hooft's eigenvalue problem (HEP) which will be proved elsewhere.[1] The main feature is the use of Bardeen and Einhorn's[2] equivalent eigenvalue problem for harmonic functions (BEP). Applying a variational approach for quadratic forms we obtain the whole spectrum and the complete spectral decomposition of HEP and BEP. The eigenfunctions are harmonic functions in the upper half plane which are continuous on its closure $\overline{\mathcal{H}}$, real analytic on $\mathcal{H} = \overline{\mathcal{H}} - \{0, 1\}$, and vanish on the real axis outside the interval $(0,1)$. The spectrum is purely discrete and consists of denumerably many *positive* eigenvalues of *finite* multiplicity which tend to infinity.[3]

In particular, the lowest eigenvalue λ_1 is simple and positive, and the corresponding eigenfunction of HEP has strictly one sign on $(0,1)$. The number $k(n)$ of nodes of the nth eigenfunction of HEP is bounded by $1 \leq k \leq 2n-1$ for $n \geq 1$. The meson masses depend continuously on the quark masses and the mass of the ground state tends to zero with the quark masses. Finally, the eigenfunctions of HEP behave at the boundary points $x=0$ and $x=1$ like powers x^{β_1} and $(1-x)^{\beta_2}$, where the powers β_1 and β_2 tend to zero with the quark masses.

In Sec. IV we describe another variational approach to the problem which will be the basis of our explicit calculations. We derive upper and lower bounds on the first eigenvalue and also on the higher ones in terms of the quark masses and the order n. In particular, we give the bounds for the asymptotic behavior of λ_n as $n \to \infty$.

Finally, in Sec. V we describe our numerical procedure and display our results for the spectrum and the eigenfunctions. (A complete set of numerical results together with some other calculations of physical interest will appear in Ref. 4.)

II. VARIOUS EQUIVALENT EIGENVALUE PROBLEMS

By performing the $1/N$ expansion[5] in the U(N) gauge field theory in one space plus one time dimension for fermions, 't Hooft[6] obtained the eigenvalue equation (HEP),

$$\lambda\phi(x) = \left(\frac{\alpha_1}{x} + \frac{\alpha_2}{1-x}\right)\phi(x) - \mathrm{P}\int_0^1 \frac{\phi(y)}{(x-y)^2}\,dy\,,$$
$$0 < x < 1 \quad \phi(0)=0, \quad \phi(1)=0\,, \tag{2.1}$$

for the meson wave function as a bound state of the quark-antiquark pair. The eigenvalue parameter λ is essentially the squared mass of the bound state,

$$\lambda = \frac{\pi m^2}{g^2 N}\,, \tag{2.2}$$

where m is the meson mass, N is the number of colors, and g denotes the coupling constant. If the meson consists of the quarks a and b with the masses m_a and m_b, then $\alpha_1 = \gamma_a - 1$, $\alpha_2 = \gamma_b - 1$, where

$$\gamma_{a,b} = \frac{\pi m_{a,b}{}^2}{g^2 N}\,. \tag{2.3}$$

The light-cone scaling variable is $x = p_-/r_-$, p_- and r_- being the minus components of the quark a and meson momenta, respectively. The principal value in (2.1) is defined by

$$\mathrm{P}\int_0^1 \frac{\phi(y)}{(x-y)^2}\,dy = \lim_{\epsilon\to 0}\int_0^1 \frac{1}{2}\left(\frac{1}{(x-y+i\epsilon)^2} + \frac{1}{(x-y-i\epsilon)^2}\right)\phi(y)\,dy\,. \tag{2.4}$$

It has been observed by Bardeen and Einhorn[2] that HEP is equivalent to another eigenvalue problem called BEP, at least, if we take some sufficient precautions. BEP is defined as follows. Determine a real-valued function $v(x,y)$ which is harmonic in the upper half plane $\mathcal{H}=\{(x,y):\ y>0\}$, continuous on its closure $\bar{\mathcal{H}}$, and satisfies the boundary conditions

$$v(x,0)=0 \quad \text{for } x\notin[0,1] \tag{2.5}$$

and

$$-\pi v_y(x,0)+\left(\frac{\alpha_1}{x}+\frac{\alpha_2}{1-x}\right)v(x,0)=\lambda v(x,0)\ . \tag{2.6}$$

The eigenvalue parameter λ is to be determined as well, and $v(x,y)$ must not be identically zero.

The eigenvalue problems HEP and BEP are related as follows. If $\phi(\xi)$ is a solution of HEP possessing Hölder continuous first derivatives on $(0,1)$, then

$$v(x,y)=\operatorname{Im}\frac{1}{\pi}\int_0^1\frac{\phi(\xi)}{\xi-x-iy}\,d\xi \tag{2.7}$$

is a solution of BEP. Conversely if $v(x,y)$ is a solution of HEP which is of the class $C^{1+\alpha}$ on $\mathcal{H}\cup(0,1)$ for some $\alpha\in(0,1)$, then $\phi(x)=v(x,0)$ is an eigenfunction of BEP.

The basis of our mathematical discussion is a result about simultaneous diagonalization of two quadratic forms on a Hilbert space which is essentially due to Courant and Hilbert.[7] It can be proved by the classical variational technique of Courant, translated into Hilbert-space language.

For this purpose, consider an infinite-dimensional real Hilbert space H with the scalar product (u,v) and the norm $\|u\|=(u,u)^{1/2}$. Let $B(u,v)$ and $K(u,v)$ be bounded, symmetric bilinear forms such that the quadratic form $K(u)=K(u,u)$ is completely continuous and non-negative, and the $K(u)>0$ on an infinite-dimensional subset of H, while $B(u)=B(u,u)$ satisfies the inequality

$$B(u)\geq c\|u\|^2-c^*K(u),\quad \text{for all } u\in H \tag{2.8}$$

for suitable fixed numbers $c>0$ and $c^*\geq 0$. Let us consider the following *abstract eigenvalue* problem (AEP).

Determine real numbers λ and vectors $u\in H$ with $K(u)>0$ such that

$$B(u,\phi)=\lambda K(u,\phi)\quad \text{for all } \phi\in H\ .$$

We call u an eigenvector and λ an eigenvalue of AEP. Then, the following holds: There are infinitely many eigenvalues λ_n with the associated eigenvectors e_n of AEP, $n=1,2,\ldots$, such that $\lambda_1\leq\lambda_2\leq\lambda_3\leq\cdots$, $\lim_{n\to\infty}\lambda_n=\infty$, and

$$\lambda_1=\inf\{B(u):\ K(u)=1\}=B(e_1)\ ,$$
$$\lambda_n=\inf\{B(u):\ K(u)=1,\ K(u,e_j)=0,1\leq j\leq n-1\}$$
$$=B(e_n)\ . \tag{2.9}$$

The eigenvalues λ_n and the eigenvectors e_n of AEP satisfy

$$B(e_k,\phi)=\lambda K(e_K,\phi)\quad \text{for all } \phi\in H \tag{2.10}$$

and the completeness relation

$$K(u)=\sum_{n=1}^{\infty}|c_n|^2,\quad \text{where } c_n=K(u,e_n)\ . \tag{2.11}$$

Therefore, all eigenvalues of AEP appear among the λ_n, every eigenvector u is a linear combination of finitely many e_n, and H is the span of $e_1,e_2,\ldots,e_n,\ldots$. Moreover, λ_n can also be characterized by the well-known Courant-Weyl maximum-minimum principle.

III. PROPERTIES OF THE EIGENVALUES AND EIGENFUNCTIONS

Let us apply these results to BEP choosing H as the Hilbert space of real-valued, locally square-integrable functions $u(x,y)$ on the upper half plane $\mathcal{H}$ having generalized first-order derivatives u_x and u_y on $\mathcal{H}$ with

$$D(u)=\iint_{\mathcal{H}}(|u_x|^2+|u_y|^2)\,dx\,dy<\infty \tag{3.1}$$

and such that $u(x,0)=0$ for $x\notin[0,1]$. We define H with the scalar product

$$(u,v)=\pi D(u,v)+K(u,v)\ , \tag{3.2}$$

where $K(u,v)$ is defined by

$$K(u,v)=\int_0^1 u(x,0)v(x,0)dx\ . \tag{3.3}$$

Finally, we define $B(u,v)$ on H by

$$B(u,v)=\pi\iint_{\mathcal{H}}(u_xv_x+u_yv_y)dx\,dy$$
$$+\int_0^1\left(\frac{\alpha_1}{x}+\frac{\alpha_2}{1-x}\right)u(x,0)v(x,0)dx\ . \tag{3.4}$$

It is not difficult to check that the triple H,B,K satisfies the assumptions stated before. In particular, the proof of (2.8) and of the boundedness of $B(u,v)$ rests on the inequality

$$\int_0^1\left(\frac{1}{x}+\frac{1}{1-x}\right)|u(x,0)|^2dx+\tfrac{1}{2}\int_0^1\int_0^1\left|\frac{u(x,0)-u(y,0)}{x-y}\right|^2dx\,dy\leq\pi\iint_{\mathcal{H}}[\,|u_x(x,y)|^2+|u_y(x,y)|^2]\,dx\,dy\ , \tag{3.5}$$

which holds for all $u \in H$. Hence, the results for AEP hold for the triple H, B, K. Thus we obtain "eigenvalues" $\lambda_1, \lambda_2, \ldots$ and eigenfunctions $e_1, e_2, \ldots$ satisfying (2.9)–(2.11).

However, one can prove that the functions $e_n(x,y)$ are harmonic in the upper half plane $\mathcal{H}$, continuous on its closure $\bar{\mathcal{H}}$, and real analytic on $\hat{\mathcal{H}} = \bar{\mathcal{H}} - \{0,1\}$. Furthermore, the $e_n(x,y)$ satisfy the boundary conditions

$$e_n(x,0) = 0 \quad \text{for } x \notin (0,1) \tag{3.6}$$

and

$$-\pi \frac{\partial}{\partial y} e_n(x,y) + \left(\frac{\alpha_1}{x} + \frac{\alpha_2}{1-x}\right) e_n(x,0) = \lambda_n e_n(x,0) \quad \text{for } 0 < x < 1. \tag{3.7}$$

Therefore, λ_n and $e_n(x,y)$ are the eigenvalues and eigenfunctions of BEP, and λ_n and $\phi_n(x) = e_n(x,0)$, $0 < x < 1$, are the eigenvalues and eigenfunctions of HEP. In this way, we have constructed the complete spectrum and the complete spectral decomposition of HEP and BEP. In particular, it follows that the spectrum of HEP and BEP is *purely discrete and consists of denumerably many eigenvalues of finite multiplicity*.

For the eigenvalues λ_n and the eigenfunctions $e_n(x,y)$ and $\phi_n(x) = e_n(x,0)$, $0 < x < 1$, of BEP and HEP, one can prove the following additional properties:

1. The lowest eigenvalue λ_1 is *simple* and *positive*, i.e.,

$$0 < \lambda_1 < \lambda_2 \,, \tag{3.8}$$

and $\phi_1(x) = e_1(x,0)$ does not vanish in $0 < x < 1$, that is $\phi_1(x)$ has strictly one sign in $(0,1)$. Moreover, let $k(n)$ be the number of zeros ("nodes") of $\phi_n(x) = e_n(x,0)$ in $0 < x < 1$, and denote by $d(n)$ the number of (connected) components of the set

$$\{(x,y) \in \mathcal{H}:\ e_n(x,y) \neq 0\}.$$

Then

$$1 \leq k(n) \leq 2n-1 \quad \text{for } n \geq 2 \tag{3.9}$$

and

$$d(n) \leq n \quad \text{for } n \geq 1 \,. \tag{3.10}$$

2. As we have already stated, every eigenfunction $e_n(x,y)$ behaves continuously at each of the two singular points $x=0$, $y=0$, and $x=1$, $y=0$, whence

$$e_n(x,y) \to 0 \quad \text{as} \quad (x,y) \to (0,0) \text{ or } (1,0) \,. \tag{3.11}$$

Moreover, there are numbers $\beta_1(\gamma_a) > 0$ and $\beta_2(\gamma_b) > 0$ such that

$$\lim_{\gamma_a \to 0} \beta_1(\gamma_a) = 0, \quad \lim_{\gamma_b \to 0} \beta_2(\gamma_b) = 0 \,,$$

and that

$$\begin{aligned} |e_n(x,y)| &\leq K_1 r^{\beta_1} \\ &\quad \text{for } 0 \leq r \leq r_1, \quad r = (x^2+y^2)^{1/2} \\ |e_n(x,y)| &\leq K_2 r^{\beta_2} \\ &\quad \text{for } 0 \leq r \leq r_2, \quad r = [(1-x)^2 + y^2]^{1/2} \end{aligned} \tag{3.12}$$

holds for appropriate positive constants K_1, K_2, r_1, r_2 depending only on γ_a, γ_b, and n.

In particular, we have

$$\begin{aligned} |\phi_n(x)| &\leq K_1 x^{\beta_1} \quad \text{for } 0 \leq x \leq r_1 \\ \text{and} \\ |\phi_n(x)| &\leq K_2 (1-x)^{\beta_2} \quad \text{for } 1 - r_2 \leq x \leq 1 \,. \end{aligned} \tag{3.13}$$

For any pair $\gamma_a, \gamma_b > 0$, let us denote the associated nth eigenvalue λ_n of HEP (or BEP) by

$$\lambda_n = \lambda_n(\gamma_a, \gamma_b) \,.$$

The function $\lambda_n(\gamma_a, \gamma_b)$ depends continuously on γ_a and γ_b, and $\lambda_1(\gamma_a, \gamma_b) \to 0$ as $\gamma_a \to 0$ and $\gamma_b \to 0$.

IV. ANOTHER VARIATIONAL APPROACH AND BOUNDS ON THE EIGENVALUES

One can treat HEP by still another approach. For this purpose, we define the quadratic forms

$$C(\phi) = \int_0^1 |\phi(x)|^2 \, dx \,, \tag{4.1a}$$

$$Q(\phi) = \tfrac{1}{2} \int_0^1 \int_0^1 \left| \frac{\phi(x) - \phi(y)}{x - y} \right|^2 dx\, dy \,, \tag{4.1b}$$

$$V(\phi) = \int_0^1 \left(\frac{1}{x} + \frac{1}{1-x}\right) |\phi(x)|^2 dx \,, \tag{4.1c}$$

$$E(\phi) = Q(\phi) + \int_0^1 \left(\frac{\gamma_a}{x} + \frac{\gamma_b}{1-x}\right) |\phi(x)|^2 dx \,, \tag{4.1d}$$

for real-valued function $\phi(x)$ on $0 < x < 1$, and a Hilbert space $\mathfrak{H}$ which is defined as the completion of $C_C^\infty(0,1)$ in the norm $[Q(\phi) + V(\phi)]^{1/2}$. Then, $\mathfrak{H}$ is a proper linear submanifold of the space $W_2^{1/2}((0,1))$ of L_2 functions $\phi(x)$ on $(0,1)$ with half a derivative which is square integrable.

If we now replace H, B, K in AEP by the triple $\mathfrak{H}$, E, C, the procedure described before yields all the eigenfunctions $\phi_n(x)$ ($= e_n$ in AEP) and eigenvalues λ_n of HEP. This variational problem is completely equivalent to the one described in Sec. II. However, the present approach is much more appropriate for explicit calculations and for deriving bounds for the eigenvalues.

One could try to calculate the critical points of $E(\phi)$ on $\mathfrak{H}$ with respect to the constraint condition

$C(\phi)=1$ approximately by the simple polynomial ansatz

$$\phi(x)=x(1-x)\sum_{k=0}^{N} a_k x^k \ . \tag{4.2}$$

This means that we replace $\mathfrak{H}$ by an $(N+1)$-dimensional subspace $\mathfrak{H}_{N+1}$, and (1.1) by the eigenvalue problem

$$E(\phi,\psi)=\lambda C(\phi,\psi) \text{ for all } \psi\in\mathfrak{H}_{N+1} \ , \tag{4.3}$$

for $\phi\in\mathfrak{H}_{N+1}$ with $C(\phi)=1$. This turns out to be equivalent to the matrix eigenvalue problem,

$$(\gamma_a A+\gamma_b B+C)a=\lambda Da \ , \tag{4.4}$$

for eigenvalues λ and eigenvectors $a=(a_0, a_1, \dots, a_N)$ with $a\cdot Da=1$, where the $(N+1)\times(N+1)$ matrices A, B, C, D with the matrix elements $A_{ij}, B_{ij}, C_{ij}, D_{ij}$ are given by

$$A_{ij}=\frac{4}{(i+j+2)(i+j+3)(i+j+4)} \ , \tag{4.5a}$$

$$B_{ij}=\frac{2}{(i+j+3)(i+j+4)} \ , \tag{4.5b}$$

$$C_{ij}=\frac{1}{i+j+3}-\sum_{k=0}^{j}\frac{1}{(k+1)(k+2)[(i+j+2)-k]}-\sum_{k=0}^{i}\frac{1}{(k+1)(k+2)[(i+j+2)-k]} + \sum_{k=0}^{i}\sum_{m=0}^{j}\frac{2}{(k+m+1)(k+m+2)(k+m+3)(i+j-k-m+1)} \ , \tag{4.5c}$$

$$D_{ij}=\frac{4}{(i+j+3)(i+j+4)(i+j+5)} \ . \tag{4.5d}$$

We were not able to derive an explicit solution of Eq. (4.4) for every N and to carry out the limiting procedure $N\to\infty$, and for numerical purposes, the Fourier ansatz (5.1) proved to be more efficient. However, for $N=1$, the ansatz (4.2) yields a good upper bound for the lowest eigenvalue λ_1 of HEP. Together with a lower bound for λ_1 derived in Ref. 1, we obtain

$$(\sqrt{\gamma_a}+\sqrt{\gamma_b})^2\le\lambda_1(\gamma_a,\gamma_b)\le 3(\gamma_a+\gamma_b)+\tfrac{13}{2} - [2(\gamma_a^2+\gamma_b^2)-3\gamma_a\gamma_b + 4(\gamma_a+\gamma_b)+16]^{1/2} \ . \tag{4.6}$$

For $N=0$, we obtain the estimate

$$\lambda_1(\gamma_a,\gamma_b)\le\tfrac{5}{2}(1+\gamma_a+\gamma_b) \ , \tag{4.7}$$

which is less sharp but handier.

For $\gamma_a=\gamma_b=\gamma$ the upper bounds in (4.6) and (4.7) are identical, and we find

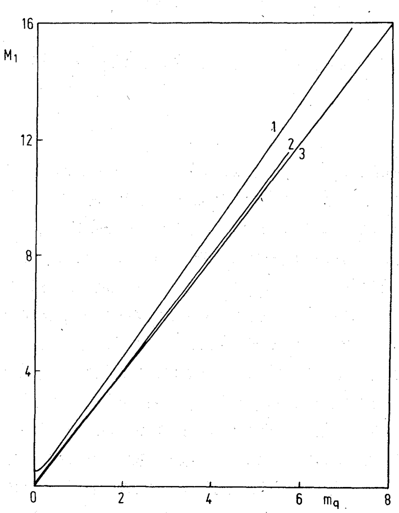

FIG. 1. The mass of the ground state as a function of the quark mass ($m_q=m_{\bar q}$). (1) and (3) are the upper and lower bounds. (2) is the result of the numerical calculation. All masses are in units $1/\sqrt{\alpha'}$.

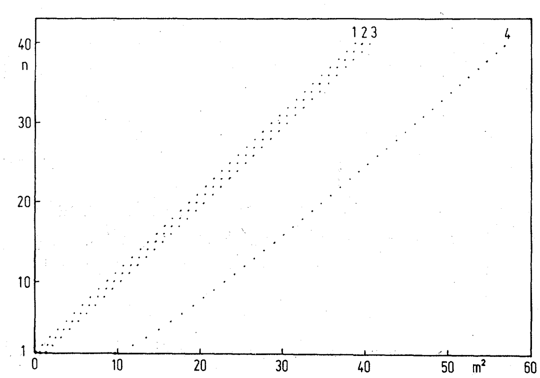

FIG. 2. The "Regge trajectories" for the mesons with $m_q=m_{\bar q}=0$ (1), 0.3 (2), 0.5 (3), and 1.5 (4) in units $1/\sqrt{\alpha'}$.

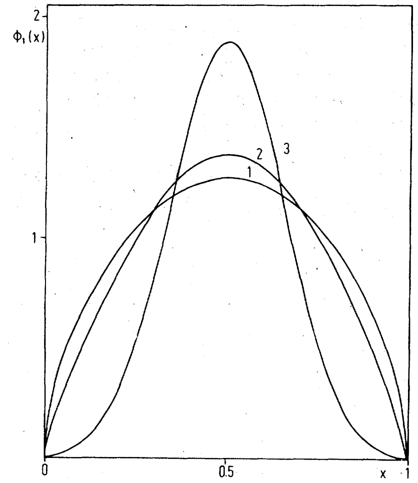

FIG. 3. The ground-state wave functions for the mesons with $m_q = m_{\bar{q}} = 0.3$ (1), 0.5 (2), 1.5 (3) in $1/\sqrt{\alpha'}$.

$$4\gamma \leq \lambda_1(\gamma,\gamma) \leq \tfrac{5}{2} + 5\gamma \ . \tag{4.8}$$

The higher eigenvalues can be estimated as follows:

$$2(n-1) + (\sqrt{\gamma_a} + \sqrt{\gamma_b}\,)^2 \leq \lambda_n(\gamma_a, \beta_b) \leq \left(1 - \frac{2\epsilon}{\pi}\right)^{-1}(n-1)\,\pi^2 + \frac{2\gamma\pi^2}{\epsilon}\ , \tag{4.9}$$

where $\gamma \equiv \max(\gamma_a, \gamma_b)$, and the upper bound is valid for every $\epsilon \in (0, \pi/2)$.

In certain cases, the following estimates from below are slightly better:

$$2(1+\beta)\pi\nu_n \leq \lambda_n(\gamma_a, \gamma_b)\ , \tag{4.10}$$

where $\{\nu_n\}$ denotes the sequence $0, 1, 1, 2, 2, \ldots$, and $\beta = \min\{0, \alpha_1, \alpha_2\} \in (-1, 0]$. Therefore,

$$(1+\beta)\pi n + \cdots \leq \lambda_n(\gamma_a, \gamma_b) \leq \theta\pi^2 n + \cdots \quad \text{for } n \to \infty\ , \tag{4.11}$$

for every number $\theta > 1$. The bounds (4.8) are displayed in Fig. 1.

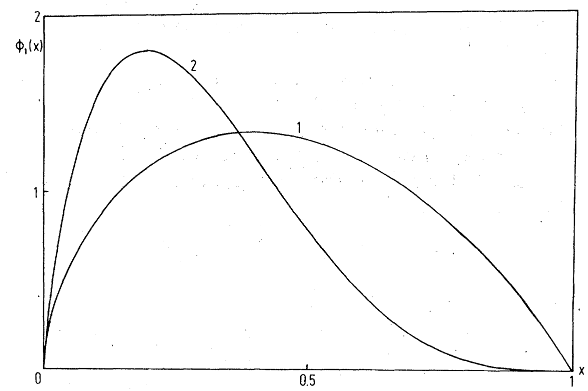

FIG. 5. The ground-state wave functions for the mesons with $m_q = 0.3$, $m_{\bar{q}} = 0.5$, and $m_q = 0.3$, $m_{\bar{q}} = 1.5$ in $1/\sqrt{\alpha'}$.

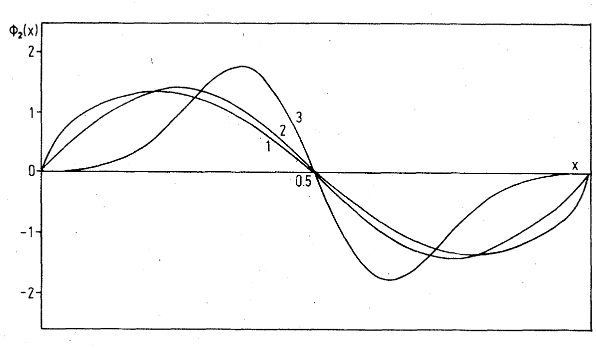

FIG. 4. The first-excited-state wave functions for the same values as Fig. 3.

V. NUMERICAL RESULTS

To get numerical solutions to the variational problem (4.1), the Fourier ansatz

$$\phi(x) = \sum a_k \sin k\pi x \tag{5.1}$$

is more suitable than the polynomial one (3.3), but in this case the integrals in (4.1) cannot be computed analytically. We approximate $\mathfrak{H}$ by a subspace $\mathfrak{H}_N$ of dimension $N \simeq 170$. This leads to an order of accuracy of $< 10^{-5}$ for the approximate eigenvalues compared with the exact ones.

Figure 1 shows how the mass of the ground state increases with the quark mass for quark-antiquark pairs of the same mass. The spectra and the eigenfunctions for various quark masses are shown in Figs. 2, 3, 4, and 5. The numerical procedure did not reveal any sign of degeneracy.

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