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New compact form of a multipseudoparticle configuration and other solutions of Yang-Mills theory

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A new compact form of a multipseudoparticle configuration is obtained. It has all its singularities located at points different from those of the known solution and related to the latter by a gauge transformation. The two together may be used with advantage in the calculation of winding number. We discuss also the classical solutions of $\lambda\phi^4$ theory, which are relevant to the problem at hand, by using the method of characteristic equations.

I. INTRODUCTION

Classical solutions with finite action of SU(2) Yang-Mills theory have recently drawn much interest. Belavin, Polyakov, Schwartz, and Tyupkin[1] (BPST) have constructed a regular one-pseudoparticle solution in Euclidean four-space. Subsequently, multipseudoparticle solutions with increasing degrees of generality were given by Witten,[2] 't Hooft,[3] and Jackiw, Nohl, and Rebbi.[4]

We present here an alternative compact form for the multipseudoparticle configuration, obtained by making use of a single singular gauge transformation of the solutions in Refs. 3 and 4. The new form has all its singularities at locations different from those of the solutions just cited, and at infinity it behaves as a "pure gauge" term $\sim U\partial_\mu U^{-1}$, vanishing like $\sim 1/x$. The two forms together may be used to advantage in the calculation of the winding number (Sec. III).

We give in Sec. IV a simple treatment for obtaining classical solutions of massless $\lambda\phi^4$ theory, which is relevant for obtaining solutions of the problem at hand, by using the method of characteristic equations. We find a new solution in closed form. We also discuss briefly how the form invariant under conformal transformations may be used to generate, starting from a simple solution, a family of solutions dependent on many more parameters.

II. YANG-MILLS EQUATIONS: NOTATION

The SU(2) Yang-Mills fields[5] over Euclidean four-space E_4 are given by

$$F^i_{\mu\nu}=\partial_\mu A^i_\nu-\partial_\nu A^i_\mu+4g\epsilon_{ijk}A^j_\mu A^k_\nu \ , \tag{1}$$

where A^i_μ are gauge potentials, $i=1,2,3$ are SU(2) indices, and $\mu,\nu=1,2,3,4$ are E_4 indices. Introducing the matrices $F_{\mu\nu}=4F^i_{\mu\nu}L_i$ and $A_\mu=4A^i_\mu L_i$, where $L_i=\sigma_i/2$ are SU(2) generators, we obtain

$$F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu-ig[A_\mu,A_\nu] \ . \tag{2}$$

The equations of motion for the sourceless case are

$$\partial_\mu F_{\mu\nu}-ig[A_\mu,F_{\mu\nu}]=0 \ , \tag{3}$$

as derived from the action density $S(A)=-\frac{1}{4}F^i_{\mu\nu}F^i_{\mu\nu}$. Under an SU(2) gauge transformation $U(x)$, the potential and the field strengths transform as

$$\begin{aligned} &A_\mu\to UA_\mu U^{-1}+\frac{i}{g}U\partial_\mu U^{-1}\,,\\ &F_{\mu\nu}\to UF_{\mu\nu}U^{-1}\ . \end{aligned} \tag{4}$$

The potentials A_μ are gauge connections appearing in the covariant derivative $D_\mu=(\partial_\mu-igA_\mu)$, and the gauge curvature tensor $F_{\mu\nu}$ satisfies Bianchi identities[6] $\epsilon_{\mu\nu\lambda\rho}D_\lambda F_{\mu\nu}=0$, which are integrability conditions for the existence of the potential. They may be reexpressed as

$$\partial_\mu\tilde F_{\mu\nu}-ig[A_\mu,\tilde F_{\mu\nu}]=0 \ , \tag{5}$$

where $\tilde F_{\mu\nu}=\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F_{\rho\sigma}$ is the dual of $F_{\mu\nu}$. Thus a field configuration for which $F_{\mu\nu}$ and $\tilde F_{\mu\nu}$ are proportional provides a solution of the equations of motion. Of special interest are the solutions which correspond to finite action, the so-called pseudoparticle solutions. They give rise to local minima of the action functional different from that of the trivial case corresponding to vanishing field strengths and are relevant for the structure of quantum theory.[1] It is worth mentioning that solutions with infinite action in Euclidean space may have finite action in Minkowski space.

It is convenient to introduce the generators $M_{\alpha\beta}$, where $\alpha,\beta=1,2,3,4$, of the O(4) group, which satisfy the Lie algebra

$$[M_{\alpha\beta},M_{\gamma\rho}]=i(\delta_{\alpha\gamma}M_{\beta\rho}+\delta_{\beta\rho}M_{\alpha\gamma}-\delta_{\alpha\rho}M_{\beta\gamma}-\delta_{\beta\gamma}M_{\alpha\rho}) \tag{6}$$

and define two sets of SU(2) generators $M^\pm$ corresponding to the two SU(2) subgroups of O(4) by

$$M^{\pm}_{\alpha\beta}=\tfrac{1}{2}(M_{\alpha\beta}\pm\tilde{M}_{\alpha\beta})=\pm\tilde{M}^{\pm}_{\alpha\beta}\ .$$

We may give a representation for them in terms of Pauli matrices,

$$L_i=\tfrac{1}{2}\epsilon_{ijk}M^{\pm}_{jk}=\pm M^{\pm}_{i4},$$

where

$$\epsilon_{ijk}=\epsilon_{ijk4}\ \text{ and }\ \epsilon_{1234}=1\ .$$

They also satisfy Eq. (6) while the anticommutator is given by

$$\{M^{\pm}_{\alpha\beta},M^{\pm}_{\gamma\rho}\}=\tfrac{1}{2}(\delta^{\alpha\beta}_{\gamma\rho}\pm\epsilon_{\alpha\beta\gamma\rho})\ , \tag{7}$$

where $\delta^{\alpha\beta}_{\gamma\rho}=(\delta_{\alpha\gamma}\delta_{\beta\rho}-\delta_{\alpha\rho}\delta_{\beta\gamma})$. For any SU(2) vector A_i, we have $A=4A_iL_i=A_{\alpha\beta}M^{\pm}_{\alpha\beta}=A^{\pm}_{\alpha\beta}M^{\pm}_{\alpha\beta}$, where

$$A_i=\tfrac{1}{2}\epsilon_{ijk}A^{\pm}_{jk}=\pm A^{\pm}_{i4}$$

and (8)

$$A^{\pm}_{\alpha\beta}=\tfrac{1}{4}(\delta^{\alpha\beta}_{\gamma\rho}\pm\epsilon_{\alpha\beta\gamma\rho})A_{\gamma\rho}=\pm\tilde{A}^{\pm}_{\alpha\beta}\ .$$

The Chern-Pontryagin topological or winding number is defined by

$$q=\frac{g^2}{8\pi^2}\int S^*d^4x\ , \tag{9}$$

where

$$S^*=\tfrac{1}{2}\mathrm{Tr}(F_{\mu\nu}\tilde{F}_{\mu\nu})\ .$$

At the points where the potential and the field are regular we have

$$S^*=\partial_\mu I_\mu\equiv\tfrac{1}{2}\epsilon_{\mu\nu\alpha\beta}\partial_\mu\,\mathrm{Tr}[A_\nu F_{\alpha\beta}+\tfrac{2}{3}igA_\nu A_\alpha A_\beta]\ . \tag{10}$$

III. POSSIBLE SOLUTIONS: NEW COMPACT FORM OF MULTIPSEUDOPARTICLE CONFIGURATION

We make now the ansatz that the O(4) gauge indices $\alpha,\beta,\ldots$ behave like tensor indices $\mu,\nu,\ldots$ of Euclidean four-space. We may then, for example, introduce four-vector fields $a_\alpha(x), b_\alpha(x)$ at each point of E_4 and seek, having in view Eq. (8), solutions of the form

$$A^{\pm\,\alpha\beta}_{\mu}=-\tfrac{1}{4}(\delta^{\alpha\beta}_{\mu\gamma}\pm\epsilon_{\alpha\beta\mu\gamma})a_\gamma+(\delta^{\alpha\beta}_{\nu\rho}\pm\epsilon_{\alpha\beta\nu\rho})b_\nu\partial_\mu b_\rho+\cdots\ . \tag{11}$$

The simplest ansatz[7] is obtained if we retain only the first term. Nonlinearity of equations of motion makes it difficult to handle the general case. The potential then takes the form $A_\mu=a_\alpha M^{\pm}_{\alpha\mu}$. Further simplification is obtained by assuming that a_α is irrotational[8] so that

$$A_\mu=2u_{,\alpha}(x)M^{\pm}_{\alpha\mu}\ , \tag{12}$$

where $u(x)$ is a scalar superpotential and $\partial_\mu A_\mu=0$. This form for the potential was used in Refs. 3, 4, and 7. Substituting it in Eq. (3) we are led to the differential equation

$$[u_{,\mu\mu}+2g(u_{,\nu})^2]_{,\alpha}-4g[u_{,\mu\mu}+2g(u_{,\nu})^2]u_{,\alpha}=0\ , \tag{13}$$

while

$$F_{\mu\nu}=F^{\pm}_{\mu\nu}=2(u_{,\alpha\mu}M^{\pm}_{\alpha\nu}-u_{,\alpha\nu}M^{\pm}_{\alpha\mu})-4igu_{,\alpha}u_{,\beta}[M^{\pm}_{\alpha\mu},M^{\pm}_{\beta\nu}]\ , \tag{14}$$

where $F^{+}_{\mu\nu}$ ($F^{-}_{\mu\nu}$) corresponds to the solution with upper (lower) sign in Eq. (12).

We find easily, say by considering $F^{\mp}_{12}=\pm F^{\mp}_{34}$, that for the field to satisfy $F^{\mp}_{\mu\nu}=\pm\tilde{F}^{\mp}_{\mu\nu}$ requires that

$$u_{,\mu\mu}+2g(u_{,\nu})^2=0\ . \tag{15}$$

This equation takes the form[7] $(\Box\phi/\phi)=0$ if we write $u=(1/2g)\ln\phi+\text{const}$. Equation (13) leads to

$$(\Box\phi/\phi^3)_{,\alpha}=0\ . \tag{16}$$

The self-duality conditions $F^{\pm}_{\mu\nu}=\pm\tilde{F}^{\pm}_{\mu\nu}$ require, in addition to Eq. (16), that $\phi\partial_\mu(\partial_\nu\phi/\phi^2)=\delta_{\mu\nu}f$ where $f(x)$ is arbitrary. This equation can easily be solved[9] to give the one-pseudoparticle solution with finite action; the other simple solutions carry infinite action.

To fix our notation, we rederive the multipseudoparticle solution[3,4] corresponding to Eq. (15). Introduce the invariant $z=\sum_{i=1}^{N}(\lambda_i^2/y_i^2)$, where $(y_i)_\alpha=x_\alpha-(a_i)_\alpha$ and λ_i, a_i are constants with a_i all distinct. We set $u=u(z)$ so that $u_{,\alpha}=f(z)\eta_\alpha$ where $2\eta_\alpha=\partial z/\partial x_\alpha$. Equation (15) then gives

$$\left(1+\frac{f'}{gf^2}\right)+\frac{1}{4g}\frac{\Box z}{f\eta^2}=0\ . \tag{17}$$

Here $f'(z)=\partial f/\partial z$ and $\Box z=-2\pi^2\sum\lambda_i^2\delta^4(y_i)$. For $x\neq a_i$, $i=1,2,\ldots,N$ we obtain $f(z)\sim1/(z+b)$. This form for $f(z)$ also makes the second term in Eq. (17) vanish when x approaches the singular point a_i. We thus obtain for the multipseudoparticle solution[3,4]

$$A_\mu=2f(z)\eta_\alpha M^{\pm}_{\alpha\mu}\ , \tag{18}$$

where

$$f(z)=\frac{1}{g}\frac{1}{z+b}$$

and

$$\eta_\alpha=-\sum_{i=1}^{N}\lambda_i^2\frac{(y_i)_\alpha}{(y_i^2)^2}\ .$$

For x approaching a_i, the solution takes the form of a pure gauge term $U\partial_\mu U^{-1}$. The singularities may be shifted to other locations by gauge transformations. The existence of a regular form for the multipseudoparticle solution obtained by a product of successive gauge transformations was shown by Giambiagi and Rothe[10] and independently by Sciuto.[9] The topological interpretation of winding number is then restored. The resulting form, though, is not compact for practial calculations.

We may, however, obtain[11] an alternative compact form which has all of its singularities displaced to points other than $x=a_i$ by using a single singular gauge transformation. The new form and the solution in Eq. (18) may be used together to advantage, for example, in the calculation of winding number. We define two overlapping regions R and R' in which A_μ and A'_μ are regular, respectively, excluding the pertinent singular points. Following the prescription of Wu and Yang,[12] whenever the integration point approaches a singularity of the solution, say A_μ, we pass from one region to another by a gauge transformation and use A'_μ. An appropriate gauge transformation is found to be

$$U=\frac{1}{(\eta^2)^{1/2}}\,(\eta_4 \mp i\vec{\eta}\cdot\vec{\sigma})\ , \tag{19}$$

where the upper (lower) sign corresponds to the upper (lower) sign in Eq. (18). The new form thus obtained is given by

$$A'_\mu=-2f(z)\eta_\alpha M^{\mp}_{\alpha\mu}-\frac{2}{g}\frac{1}{\eta^2}\,\eta_\lambda\eta_{\mu\rho}\,M^{\mp}_{\lambda\rho}\ , \tag{20}$$

where

$$\eta_{\mu\rho}=-\sum_{i=1}^{N}\frac{\lambda_i^{\,2}}{(y_i^{\,2})^2}\left[\delta_{\mu\rho}-4\,\frac{(y_i)_\mu(y_i)_\rho}{y_i^{\,2}}\right]\ . \tag{21}$$

In deriving it we made use of the identity

$$U\eta_\alpha M^{\pm}_{\alpha\mu}U^{-1}=-\eta_\alpha M^{\mp}_{\alpha\mu}\ . \tag{22}$$

The form in Eq. (20) could not be obtained with the simple ansatz used above, and in this context $M^{+}_{\alpha\beta}$ and $M^{-}_{\alpha\beta}$ play complementary roles. We readily verify that A'_μ is no longer singular at $x=a_i$; in fact, it vanishes there, simplifying calculations. The field strengths $F'^{\pm}_{\mu\nu}$ are shown to be finite, continuous, and self-dual at these points. The new form is now singular at the points where $\eta^2=0$, but at these locations the first term in Eq. (20) drops out. For $|x|\to\infty$ the first term, which falls as $1/x^3$ is small compared to the second term which vanishes as $1/x$. The latter, moreover, takes a "pure gauge" form $U\partial_\mu U^{-1}$.

For the purpose of illustration we calculate the winding number using the form A'_μ. The two alternative forms show that the Pontryagin density S^* is regular everywhere. Hence we may write, making use of Eq. (10) and Gauss's theorem,

$$\begin{aligned}\int_V S'^*d^4x&=\lim_{\Delta_j\to 0}\int_{V-\Sigma\Delta_j}S'^*d^4x\\&=\left(\lim_{R\to\infty}\int_{S_3}-\sum_{(\eta_\lambda=0)}\lim_{\epsilon_j\to 0}\int_{S_3'^j}\right)I'_\mu d\sigma_\mu\ .\end{aligned} \tag{23}$$

Here Δ_j indicates an infinitesimal spherical region of radius ϵ_j in E_4 with three-dimensional surface $S_3'^j$ enclosing a zero of $\eta^2=0$. S_3 indicates a large spherical surface of radius R, and the directed surface element, say on S_3, has the form $d\sigma_\mu=x_\mu x^2 d\Omega$, where $d\Omega=\sin^2\chi\sin\theta\,d\chi d\theta d\phi$ with $0\le\chi\,,\theta\le\pi$ and $0\le\phi\le 2\pi$. From the asymptotic behavior of A'_μ discussed above, we conclude that we may in Eq. (23) replace I'_μ by I^G_μ, the contribution coming from the second (pure gauge) term A^G_μ of A'_μ. From the fact that a potential of the form $U\partial_\mu U^{-1}$, $U(x)\in \mathrm{SU}(2)$, gives vanishing field strength at the points where it is regular, we deduce that

$$\begin{aligned}0&=\int_{V-\Sigma\Delta_j-\sum_{i=1}^{N}\delta_i}S^{*G}d^4x\\&=\left(\lim_{R\to\infty}\int_{S_3}-\sum_{(\eta_\lambda=0)}\lim_{\epsilon_j\to 0}\int_{S_3'^j}\right.\\&\qquad\left.-\sum_{i=1}^{N}\lim_{\epsilon_i\to 0}\int_{S_3^i}\right)I^G_\mu d\sigma_\mu\ .\end{aligned} \tag{24}$$

Here δ_i is the spherical region with surface S_3^i enclosing the point $x=a_i$, and S^{*G} is the Pontryagin density calculated from the pure gauge term A^G_μ and A'_μ. In deriving Eq. (24) we have used the fact that A^G_μ, S^{*G}, and I^G_μ are singular not only at the points where $\eta^2=0$ but also at the point $x=a_i$ as shown from the Eqs. (10) and (20). From Eqs. (23) and (24) it follows that

$$\int S'^*d^4x=\sum_{i=1}^{N}\lim_{\epsilon_i\to 0}\int_{S_3^i}I^G_\mu d\sigma_\mu\ . \tag{25}$$

The right-hand side is easily calculated to give the winding number, say, for $b\neq 0$ in $f(z)$ of Eq. (18), the value $q=\mp N$. The calculation using the expression in Eq. (18) is also straightforward. Using arguments similar to the above we find that

$$\int S^*d^4x=\left(\lim_{R\to\infty}\int_{S_3}-\sum_{i=1}^{N}\lim_{\epsilon_i\to 0}\int_{S_3^i}\right)I_\mu d\sigma_\mu\ . \tag{26}$$

The right-hand side then may be easily calculated. An essentially similar procedure was used, though in a different way, in the first calculation in Ref. 4.

IV. CLASSICAL SOLUTIONS OF MASSLESS $\lambda\phi^4$ THEORY

The simple ansatz made above permits still other solutions corresponding to Eq. (16) which for $\lambda \neq 0$ reads

$$\Box\phi + 4\lambda\phi^3 = 0 \ . \tag{27}$$

This nonlinear equation has been studied by many authors[13]. An alternative simpler treatment using the method of characteristic equations will be presented here. We find a new closed solution. We remark that Eq. (27) is form invariant under special conformal transformations $\overline{x}_\mu = (x_\mu - c_\mu x^2)\sigma^{-1}$ with $\overline{\phi}(\overline{x}) = \sigma\phi(x)$, where $\sigma(x) = 1 - 2\,c\cdot x + c^2x^2$ as well as under dilatations $\overline{x} = e^\rho x$ with $\overline{\phi}(\overline{x}) = e^{-\rho}\phi(x)$. This symmetry allows us to find, from a particular simple solution, a corresponding family of solutions which involve more parameters. It also allows us to use the method of characteristic equations.[14]

For simplicity we consider only the case $\phi = \phi(z)$, where $z = (x-a)^2$. Writing $z\phi(z) = y(z)$ we obtain

$$F \equiv y'' + \lambda(y/z)^3 = 0 \ , \tag{28}$$

where F is form invariant under scale transformation $\overline{z} = \alpha^2 z, \overline{y} = \alpha y$. For infinitesimal transformations this implies that

$$\delta z\,\frac{\partial F}{\partial z} + \delta y\,\frac{\partial F}{\partial y} + \delta y'\,\frac{\partial F}{\partial y'} + \delta y''\,\frac{\partial F}{\partial y''} = 0 \ , \tag{29}$$

where $\delta z = 2\epsilon z$, $\delta y = \epsilon y$, $\delta y' = -\epsilon y'$, $y'' = -3\epsilon y'' = 3\lambda\epsilon(y/z)^3$. Hence the characteristic equations to be solved are

$$\frac{dz}{2z} = \frac{dy}{y} = -\frac{dy'}{y'} \ . \tag{30}$$

From the first and the second and the first and the third equalities we find $v(z,y) = y/\sqrt{z} = \text{const}$ and $w(z,y,y') = \sqrt{z}\;y' = \text{const}$. Working now with the scale-invariant variables v and w we obtain

$$\frac{dw}{dv} = \frac{\frac{1}{2}w - \lambda v^3}{w - \frac{1}{2}v} \ , \tag{31}$$

which leads to the first integral $w^2 - wv + \frac{1}{2}\lambda v^4 = \text{const} = -b$, that is,

$$2w = 2\sqrt{z}\;y' = v \pm \{v^2 - 4[b + (\lambda/2)v^4]\}^{1/2} \ , \tag{32}$$

which is also scale invariant. From $dv/v = dy/y - \frac{1}{2}(dz/z)$ we find

$$2z\,dv = \pm\{v^2 - 4[b + (\lambda/2)v^4]\}^{1/2}dz \ , \tag{33}$$

whose solution gives a solution of the problem.

Before discussing this equation we note that Eq. (28) may be written in terms of v as follows:

$$\sqrt{z}\;v'' + \frac{v'}{\sqrt{z}} - \frac{1}{4}\frac{v}{z\sqrt{z}}(1 - 4\lambda v^2) = 0 \ . \tag{34}$$

This leads clearly to the parameter-independent solution[15] $v = \pm(1/4\lambda)^{1/2}$ or

$$\phi(z) = \pm\left(\frac{1}{4\lambda}\right)^{1/2}\frac{1}{\sqrt{z}} \ .$$

From Eq. (33) for $b = 0$ we find easily that

$$\phi(z) = B[1 + \tfrac{1}{2}B^2\lambda z]^{-1} \ , \tag{35}$$

which contains the BPST solution. We mention another closed solution ($b = 1/32\lambda$)

$$\phi(z) = \pm\left(\frac{1}{-4\lambda}\right)^{1/2}\frac{1}{\sqrt{z}}\tan\left[\frac{1}{\sqrt{2}}\ln(B\sqrt{z})\right] \ . \tag{36}$$

For arbitrary b we have in general an elliptic integral which can be cast in canonical form by substitutions $t = \frac{1}{2}\ln z$ and $y = e^t h(t)$. This case was discussed in detail by Cervero, Jacobs, and Nohl.[13]

We now illustrate by considering the solution in Eq. (35), i.e., how the form invariant mentioned above may be used to generate solutions with more parameters. It is clear that

$$\overline{\phi}(\overline{x}) = \sigma(x)\phi(x) = \frac{B}{1 + \frac{1}{2}B^2\lambda(\overline{x} - a)^2}$$

gives, in view of conformal covariance, a solution of Eq. (27). This may be written as

$$\phi(x) = \frac{1}{\sigma(x)}\,\frac{B}{1 + \frac{1}{2}B^2\lambda[\tilde{\sigma}(a)/\sigma(x)](x - \tilde{a})^2} \ ,$$

where

$$\tilde{\sigma}(a) = 1 + 2\,a\cdot c + a^2c^2$$

and

$$\tilde{a}_\mu = (a_\mu + c_\mu a^2)\tilde{\sigma}(a)^{-1} \ .$$

Since a_μ and c_μ are arbitrary, we may write the solution in the form

$$\phi(x) = \frac{B[(u-v)^2]^{1/2}}{(x-u)^2 + \frac{1}{2}\lambda B^2(x-v)^2} \ , \tag{37}$$

where u_μ, v_μ, and B are constant parameters. As for the gauge potential, nothing significantly new is obtained by using this solution. Treating similarly the parameter independent solution described above we find that

$$\phi(x) = \pm\left(\frac{1}{4\lambda}\right)^{1/2}\left[\frac{(u-v)^2}{(x-u)^2(x-v)^2}\right]^{1/2} \ , \tag{38}$$

which was used in Ref. 16 to show that it leads to gauge field having finite action in Minkowski space.

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