Classical Euclidean field configurations and charge confinement

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The possibility of the confinement of external charge in a pure Yang-Mills theory is discussed. Only field configurations which almost obey the classical field equations are considered. The two- and three-dimensional models of Callan, Dashen, and Gross and of Polyakov are reviewed. If confinement occurs in a four-dimensional pure Yang-Mills theory it is concluded that field configurations which are far from obeying the Euclidean classical equations of motion must be predominant.

I. INTRODUCTION

There are two very interesting models of confinement wherein confinement of charge comes about predominantly through semiclassical field configurations of a vector potential in Minkowski space or through classical configurations in the corresponding Euclidean theory. In such cases confinement can be found in weak coupling, and a linear force law emerges. The above two models are the two-dimensional vortex of Callan, Dashen, and Gross[1] and the three-dimensional monopole plasma of Polyakov.[2] The possibility exists that the same is true in a more realistic four-dimensional non-Abelian gauge theory. The analog of the vortices and monopoles might be instantons[3] or merons.[1]

The purpose of this paper is to analyze such a possibility without actually attempting to exhibit the detailed mechanism of confinement. The model considered is a pure SU(2) Yang-Mills theory without scalar mesons of any sort. The Euclidean theory is considered in a classical approximation, where only configurations which "almost" obey the classical Yang-Mills equations are considered. The present paper has nothing to say about confinement in situations where quantum effects play an important role in the Euclidean theory. In addition, the assumption is made that a Lorentz gauge exists where the momentum-space Green's functions, at nonzero momentum, exist in the infinite-volume limit. Whether this is a trivial or an important assumption is not completely clear yet.

In Sec. II the two- and three-dimensional models are discussed in terms of the field configurations which are important in causing confinement. In each of these cases the confining field configurations obey the classical field equations in the low-density limit. In each case the momentum-space Green's functions are well-defined and confinement comes about through strong singularities in $A(k)$ as $k\to 0$.

In Sec. III the types of configurations of $A_\mu^a(x)$ in a four-dimensional non-Abelian gauge theory which are necessary for confinement are discussed. It is then shown that such field configurations do not obey the classical Yang-Mills equations. In fact, in the low-k part of $A(k)$, the violation of the equation of motion is as large as can be for the magnitude of $A(k)$. Such configurations must be considered strongly quantum mechanical.

Finally, there is nothing in the present paper which forbids one from obtaining confinement from special A_μ configurations, as for example from merons. All that is being said is that such configurations, at least in their long-wavelength structure, do not obey classical Euclidean field equations.

II. TWO- AND THREE-DIMENSIONAL MODELS OF CONFINEMENT

In this section a brief review of two- and three-dimensional models of confinement will be given. The object of this discussion is to compare and contrast the mechanism causing confinement in these two cases. In Sec. III it will then be argued that mechanisms like that which causes confinement in the two- or three-dimensional models cannot be the cause of confinement in a four-dimensional Yang-Mills theory.

A. Two-dimensional model

This model has been discussed by Callan, Dashen, and Gross.[1] The model is a two-dimensional Euclidean electrodynamics of a scalar field with Lagrangian density

$$\mathcal{L}=(\partial_\mu+igA_\mu)\phi^*(\partial_\mu-igA_\mu)\phi+m^2\phi^*\phi-\tfrac{1}{2}\lambda(\phi^*\phi)^2-\tfrac{1}{4}F_{\mu\nu}{}^2.$$

The sign of the $m^2\phi^*\phi$ term has been taken to correspond to spontaneous symmetry breaking for $m^2>0$, which is assumed in the following. The classical minima of $-S(\phi,A)=\int\mathcal{L}(x)dx$ correspond to $|\phi|=(m^2/\lambda)^{1/2}=v$. ϕ may be chosen to be real. The classical zero mode corresponding to changes of phase of ϕ becomes a massive mode of A_μ with $\text{mass}^2 = 2g^2v^2$ quantum mechanically. The ϕ field has a mode of $\text{mass}^2 = 2m^2$.

In addition to the quantum mechanics generated perturbatively, there are quantum effects corresponding to nonminimal, but classically stable, configurations of A and ϕ. Such configuration are instantons, and in two-dimensional scalar electrodynamics they are formally identical to the Nielsen-Olesen[4] vortices. For large x and y, in the Lorentz gauge, these vortices look like

$$A_\mu(\underline{x}) = \pm\frac{1}{g}\frac{(-y,x)}{x^2+y^2} = (A_x, A_y),$$

and

$$\phi(\underline{x}) = ve^{\pm i\theta}$$

with $\theta = \tan^{-1}(y/x)$. This corresponds to a vortex located at $\underline{x} = (x,y) = \underline{0}$. As shown by Callan, Dashen, and Gross the dominant semiclassical configurations contributing to $\int \mathfrak{D}[A]\mathfrak{D}[\phi]e^{-S(A,\phi)}$ correspond to simple superpositions of vortices when g is small. Callan, Dashen, and Gross then show that these semiclassical configurations correspond to A_μ fields which confine charges that are not integral multiples of g. A variant of this argument will now be given which will be in a form to illustrate the general discussion to be given in Sec. III.

Consider the quantity

$$I_P = \frac{\int \mathfrak{D}[A]\mathfrak{D}[\phi]\exp[-S(A,\phi) + iqg\int_P A\cdot dx]}{\int \mathfrak{D}[A]\mathfrak{D}[\phi]\exp[-S(A,\phi)]}$$

where P is a square path in the (x,y) plane of side length l. l is assumed to be large compared to both vortex sizes and intervortex spacings. In the thermodynamic limit one may break the (x,y) plane into two parts corresponding to the interior and exterior of the square path P. Then we obtain

$$\begin{aligned} Z &= \int \mathfrak{D}[A]\mathfrak{D}[\phi]e^{-S(A,\phi)} \\ &= \int \prod_{x\in P}\mathfrak{D}[A(x)]\mathfrak{D}[\phi(x)]e^{-S(A,\phi)}\int\prod_{x\notin P}\mathfrak{D}[A(x)]\mathfrak{D}[\phi(x)]e^{-S(A,\phi)} \\ &= Z_P Z_{\overline{P}}, \end{aligned}$$

while

$$\begin{aligned} Z' &= \int \mathfrak{D}[A]\mathfrak{D}[\phi]\exp\left(-S + iqg\int_P A\cdot dx\right) \\ &= \int\prod_{x\in P}\mathfrak{D}[A(x)]\mathfrak{D}[\phi(x)]\exp\left(-S+iqg\int_P A\cdot dy\right)\int\prod_{x\notin P}\mathfrak{D}[A(x)]\mathfrak{D}[\phi(x)]e^{-S} = Z'_P Z_{\overline{P}}. \end{aligned}$$

Thus, we obtain

$$I_P = \frac{Z'_P}{Z_P}$$

and

$$Z_P = \sum_{n_+,n_-=0}^{\infty}\frac{1}{n_+!}\frac{1}{n_-!}e^{-(n_++n_-)S_0}W^{n_++n_-},$$

and also

$$Z'_P = \sum_{n_+,n_-=0}^{\infty}\frac{1}{n_+!}\frac{1}{n_-!}e^{-(n_++n_-)S_0}\times W^{n_++n_-}e^{2\pi iq(n_+-n_-)}.$$

n_+ (n_-) is the number of vortices (antivortices). S_0 is the action of one vortex, and $W=(l/\xi)^2$ is a two-dimensional volume normalized to the size of the vortex, ξ.

Now write

$$Z'_P = \sum_{\sigma=-\infty}^{\infty} e^{2\pi iq\sigma}\mu(\sigma,P)$$

with

$$\mu(\sigma,P) = \sum_{n_+,n_-=0}^{\infty}\frac{1}{n_+!}\frac{1}{n_-!}e^{-(n_++n_-)S_0}\times W^{n_++n_-}\delta_{\sigma,n_+-n_-}.$$

Then we obtain

$$\mu(\sigma,P) = \sum_{N=0}^{\infty}\frac{W^N e^{-NS_0}}{[(N+\sigma)/2]!\,[(N-\sigma)/2]!}.$$

If $\overline{N} = 2We^{-S_0}$ is the average number of vortices in P, then

$$\frac{\mu(\sigma,P)}{Z_P} = \frac{1}{(2\pi\overline{N})^{1/2}}e^{-\sigma^2/2\overline{N}}$$

for $|\sigma/\overline{N}|$ small, while for $|\sigma/\overline{N}|$ large we obtain

$$\ln\left[\frac{\mu(\sigma,P)}{Z_P}\right] = -|\sigma|\ln\sigma.$$

Then we find that

$$I_P = \sum_{\sigma=-\infty}^{\infty} \mu(\sigma,P)\, e^{2\pi i q\sigma} = e^{-(1-\cos 2\pi q)\bar{N}}$$

as found by Callan, Dashen, and Gross.

The rapid decrease of I_P comes about because $\mu(\sigma, P)$ is an entire function of σ which decreases with real σ on a scale set by $\sqrt{\bar{N}}$. In terms of potentials this is equivalent to saying that essentially all configurations that are important in evaluating Z have a coherence length of l or longer. This latter statement can be seen directly by computing the two-point function in the presence of vortices. The $1/g^2$ contribution is

$$\langle A_i(\underline{x})A_j(\underline{x}')\rangle = \frac{\epsilon_{ip}\epsilon_{jq}}{g^2}\sum_{\gamma,\gamma'}\frac{(x-x_\gamma)_p}{(\underline{x}-\underline{x}_\gamma)^2} \times \frac{(x'-x_{\gamma'})_q}{(\underline{x}-\underline{x}_{\gamma'})^2}(\pm)_\gamma(\pm)_{\gamma'},$$

where γ and γ' label vortices. $(\pm)_\gamma$ tells whether a vortex or antivortex is present, and $\underline{x}_\gamma$ is the position of the vortex γ. It is assumed that $(\underline{x}-\underline{x}')^2$ is much greater than the typical intervortex spacing. Then

$$\langle A_i(\underline{x})\,A_j(\underline{x}')\rangle = \frac{2\epsilon_{ip}\epsilon_{jq}}{g^2}\sum_\gamma \frac{(x-x_\gamma)_p}{(\underline{x}-\underline{x}_\gamma)^2}\,\frac{(x'-x_\gamma)_q}{(\underline{x}'-\underline{x}_\gamma)^2}.$$

Suppose all vortices are limited to a volume V with an average density ρ. Then $\sum_\gamma \to \rho\int d^2\underline{x}_\gamma$, and

$$\langle A_i(\underline{x})\,A_j(\underline{x}')\rangle = \frac{2\rho}{g^2}\int d^2\underline{x}_\gamma\,\frac{\epsilon_{ip}\epsilon_{jq}(x-x_\gamma)_p(x'-x_\gamma)_q}{(\underline{x}-\underline{x}_\gamma)^2(\underline{x}'-\underline{x}_\gamma)^2} \approx \frac{\delta_{ij}\rho}{g^2}\ln\left[\frac{V}{(\underline{x}-\underline{x}')^2}\right].$$

In summary, then, confinement comes about because every important A_μ configuration has[5]

$$\int_P A\cdot dx = O(w^{1/2}),$$

whre w is the area enclosed by P.

B. Three-dimensional model

The three-dimensional model of confinement discussed by Polyakov[2] is a three-Euclidean-dimensional Georgi-Glashow[6] model. The Lagrangian density is

$$\mathcal{L}(x) = [(\partial_\mu\delta_{ab} + g\epsilon_{acb}A_\mu^c)\phi^b]^2 + \frac{m^2}{2}\phi^a\phi^a - \frac{\lambda}{4}(\phi^a\phi^a)^2 - \tfrac{1}{4}(F_{\mu\nu}^a)^2,$$

where the sign is chosen so that $m^2 > 0$ corresponds to spontaneous symmetry breaking. Choose $\langle\phi^a\rangle = \eta\delta_{a3}$, then A_μ^3 remains a massless vector meson. The instantons of this theory are the 't Hooft[7]-Polyakov[8] monopoles. In the gauge where $\langle\phi^a\rangle$ remains oriented in the $a=3$ direction there is a singularity in A^3 and in $\nabla\times\vec{A}^3$ parallel to the z axis. In order to calculate Euclidean Green's functions in the weak-coupling limit one must include field configurations involving monopoles and antimonopoles much as in the two-dimensional model.

However, there is an essential difference from the two-dimensional model: In the two-dimensional model the large field fluctuations causing $\langle A(\underline{x})\,A(\underline{x}')\rangle \to \rho\ln[V/(\underline{x}-\underline{x}')^2]$, for example, come about from a statistical placement of vortex and antivortex positions. Indeed, in a volume $V=L^2$ an estimate, missing only the logarithm, for $\langle A(\underline{x})\,A(\underline{x}')\rangle$ can be made. Consider a typical configuration. In this typical configuration the excess of vortices over antivortices is $\Delta N = \pm O((\rho V)^{1/2})$. Then $A(\underline{x})$ is equal to ΔN divided by the distance of x from a typical vortex. Thus we obtain $A(\underline{x}) = O(\sqrt{\rho})$, and $A(\underline{x}')\,A(\underline{x}') = O(\rho)$. In the three-dimensional model a random placement of monopole and antimonopole positions would give a typical excess of monopoles over antimonopoles as $\Delta N = \pm O((\rho V)^{1/2})$, and typical *long-wavelength* values of $\vec{H} = \nabla\times\vec{A}^3$ as

$$|\vec{H}| = O\left((\rho V)^{1/2}\frac{1}{L^2}\right) = O((\rho/L)^{1/2}).$$

However, now these long-wavelength contributions carry action, and the typical action in the long-wavelength part of the fluctuation is

$$\Delta S = O((\rho/L)V) = O(\rho L^2).$$

This ΔS is too large to allow such fluctuations. In fact monopole density and $\vec{H}$-field fluctuations are both short-range correlated. This is just the statement that the Polyakov model really is a plasma while the vortex model has a random distribution of vortices.

One can see roughly how confinement comes about by the following argument: (A little later a precise calculation will be outlined.) Take the gauge where a monopole at the origin has the form

$$(A_x, A_y, A_z) = \frac{1}{g}\,\frac{(-y,x,0)}{x^2+y^2}\left(1+\frac{z}{(x^2+y^2+z^2)^{1/2}}\right).$$

Consider $g\int_P A_\mu(x)dx^\mu$, where P is a square path of side length l centered about the origin in the x-y plane. Only those monopoles or antimonopoles lying in or near the cube of side length l centered about the origin contribute to $\exp(ig\int_P A\cdot dx)$. (Note that monopoles lying far down on the negative z axis give a contribution of $e^{4\pi i}=1$.) We must then estimate the excess of monopoles over antimonopoles in this cube. Call this excess ΔN. Then

$$\Delta N = \int_s \vec{H} \cdot d\vec{S},$$

where the integral goes over the surface of the cube mentioned above, and where the string-like singularities of $\nabla \times \vec{A}$ are ignored in computing ΔN. ΔN is estimated, then, simply by calculating $\int_{s_1} \vec{H} \cdot d\vec{S}$, where S_1 is a surface the size of one of the faces of the above cube. Let us take S_1 to be a square of side length l in the (x, y) plane. Then

$$(\Delta N)^2 \propto \left\langle \left(\int_{s_1} H_z(x, y, 0) dx\, dy \right)^2 \right\rangle$$
$$= \int d^2\underline{x}\, d^2\underline{x}' \langle H_z(\underline{x}, 0) H_z(\underline{x}', 0)\rangle .$$

If ϕ is a magnetic potential such that $\vec{H} = -\nabla\phi$, one can write

$$(\Delta N)^2 \propto \int d^2\underline{x}\, d^2\underline{x}' \frac{\partial^2}{\partial z^2} \langle \phi(\underline{x}, z) \phi(\underline{x}', z)\rangle \bigg|_{z=0},$$

where $\langle \phi(\vec{x})\phi(\vec{x}')\rangle$ is the correlation function of the magnetic potential in a monopole-antimonopole plasma. Owing to Debye screening $\langle \phi(\vec{x})\phi(\vec{x}')\rangle$ is short-range correlated and so $\Delta N^2 \propto l^2$ and $|\Delta N| \propto l$. Thus the typical excess of monopoles over antimonopoles is of order l inside the cube, and so $\exp(ig \int_P A \cdot dx) = e^{iO(l)}$. This is a sufficiently large phase to allow confinement.

One can actually see that confinement *must* take place by realizing that the expectation of $\exp(ig \int_P A \cdot dx)$ in a monopole plasma is the same as the expectation of $\exp(i \int \mathcal{L}_I d^3x)$ in a charged plasma, where $\mathcal{L}_I$ is the interaction Lagrangian for a closed monopole current loop interacting with an electric field of the plasma. The closed monopole current loop can be described by a scalar potential $\vec{E} = -\nabla\phi$, if a singular surface bounded by P is included. Because of the Debye screening, the surface carries action and the total action is minimized by taking the surface to be the minimal one attached to P.[9,10] The interaction action is proportional to the area of this minimal surface and one has a finite-size-string model.[11]

III. FOUR-DIMENSIONAL PURE YANG-MILLS THEORY

In this section it will be argued that a pure SU(2) Yang-Mills theory cannot confine charge by means of long-range fluctuations in a semiclassical approximation.

A. Nature of the A_μ^a fluctuations necessary for confinement

Consider

$$I = \frac{\int \mathfrak{D}[A] e^{-S(A)} \frac{1}{2} \mathrm{Tr} \mathcal{O} \exp(ig \int_P A \cdot dx)}{\int \mathfrak{D}[A]\, e^{-S(A)}}, \tag{1}$$

where P is a square path of side length l, $A_\mu = \frac{1}{2}\tau^a A_\mu^a$, and $\mathcal{O}$ indicates a path ordering of the integral in (1). The theory is discussed in the Euclidean region of x_μ. One can write

$$I = \int_{-\infty}^{\infty} d\sigma\, \mu(\sigma, P) e^{i\sigma},$$

where

$$\mu(\sigma, P) = \frac{\int \mathfrak{D}[A] e^{-S(A)} \delta(\sigma - \theta(A, P))}{\int \mathfrak{D}[A] e^{-S(A)}}. \tag{2}$$

and

$$\mathcal{O} \exp\left(ig \int_P A \cdot dx \right) = \cos\theta(A, P) + i\, \vec{\tau} \cdot \hat{\theta}(A, P) \sin\theta(A, P).$$

μ is a positive-semidefinite even function of σ normalized so that $\int_{-\infty}^{\infty} d\sigma\, \mu(\sigma, P) = 1$.[12]

If the pure Yang-Mills theory confines, $\ln I \sim l^{2-2\gamma}$ as $l \to \infty$, where $\gamma < \frac{1}{2}$. Constant factors and logarithmic factors of l have been ignored. Thus confinement means

$$I = \int_{-\infty}^{\infty} d\sigma\, \mu(\sigma, P) e^{i\sigma} \underset{l\to\infty}{\sim} \exp(-l^{2-2\gamma}) .$$

Define $\sigma = \xi l^{1-\gamma}$, and $\tilde{\mu}(\xi, P) = l^{1-\gamma} \mu(\xi l^{1-\gamma}, P)$. Then we obtain

$$I = \int_{-\infty}^{\infty} d\xi\, \tilde{\mu}(\xi, P) e^{i\xi l^{1-\gamma}} \underset{l\to\infty}{\sim} \exp(-l^{2-2\gamma}),$$

and the normalization is

$$\int_{-\infty}^{\infty} d\xi\, \tilde{\mu}(\xi, P) = 1.$$

It can be concluded then that $\tilde{\mu}(\xi, P)$ is analytic at $\xi = 0$ for large values of l and hence that essentially all configurations have values of $\theta(A, P)$ on the order of $l^{1-\gamma}$.

Let $A(x)$ be a field configuration such that $\theta(A, P)$ is of size $l^{1-\gamma}$ for almost every square path of side length l. Further, suppose that the theory is defined in a volume of size $V = L^4$. Fourier coefficients of $A(x)$ are defined by

$$A(k) = \frac{1}{\sqrt{V}} \int dx\, e^{-ikx} A(x),$$

and the Fourier expansion is

$$A(x) = \frac{1}{\sqrt{V}} \sum_k e^{ikx} A(k).$$

We suppose that a Lorentz gauge can be found such that momentum-space Green's functions approach a finite limit for nonexceptional momenta as $V \to \infty$, though this may require singularities of $A(x)$ at isolated points. We also suppose that very-high-frequency contributions are convergent and can be ignored in this gauge. The magnitude of $|A_\mu^a(k)|$ can be established by the following estimate: for $k \neq 0$,

$$A_\mu^a(k)A_\nu^b(-k) = \frac{1}{V}\int dx\,dy\,e^{-ikx+iky}A_\mu^a(x)A_\nu^b(y)$$
$$\approx \int dx\,e^{-ikx}\langle A_\mu^a(x)A_\nu^b(0)\rangle$$
$$= \Delta_{\mu\nu}^{ab}(k) = \delta_{ab}\left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)\Delta(k^2).$$

(When $a \neq b$ the above equation must be interpreted as the left-hand side having a rapidly varying phase as k varies over scales of the inverse length of the universe.)

The above relation uses

$$\langle A_\mu^a(x)A_\nu^b(0)\rangle = \frac{\int \mathfrak{D}[A]e^{-S(A)}A_\mu^a(x)A_\nu^b(0)}{\int \mathfrak{D}[A]e^{-S(A)}}$$
$$\approx \frac{1}{V}\int dy\,A_\mu^a(x+y)A_\nu^b(y),$$

where the $A_\mu^a(x)$ in the final expression can be any reasonable contribution to the functional integral. It will be further assumed that $\Delta(k^2) \sim k^{-4+2\gamma}$ as $k \to 0$ (see Appendix) where nonpower dependences are neglected.

Define a mixed representation having only one component, say x_0, Fourier transformed:

$$A(\vec{x}, k_0) = \frac{1}{\sqrt{L}}\int dx_0 e^{-ik_0x_0}A(x)$$

and

$$A(x) = \frac{1}{\sqrt{L}}\sum_{k_0} e^{ik_0x_0}A(\vec{x}, k_0).$$

Then we obtain

$$A_\mu^a(\vec{x}, k_0)A_\nu^b(\vec{x}, -k_0)$$
$$= \frac{1}{L}\int dx_0dy_0e^{-ik_0x_0+ik_0y_0}A_\mu^a(\vec{x}, x_0)A_\nu^b(\vec{y}, y_0)$$
$$= \int dx_0e^{-ik_0x_0}\langle A_\mu^a(\vec{0}, x_0)A_\nu^b(\vec{0}, 0)\rangle.$$

Define $A_{\bar{k}}(x)$ by

$$A_{\bar{k}}(x) = \frac{1}{\sqrt{V}}\sum_{|k_\mu| \geq \bar{k}} e^{ikx}A(k).$$

Consider the integral over a large square box of side length l having $L/l \gg 1$. Then we obtain

$$\int_l A_{\bar{k}}(x)d^4x = \frac{1}{\sqrt{V}}\sum_{|k_\mu| \geq \bar{k}} A(k)\int_l d^4x e^{ikx}$$
$$= \frac{1}{\sqrt{V}}\sum_{|k_\mu| \geq \bar{k}} A(k)\prod_{\mu=0}^{4}\frac{2\sin k_\mu l}{k_\mu}.$$

As k varies on a scale of $1/l$, $A(k)$ takes on random phases in the above sum, so, for example,

$$\sum_{k_\mu = k}^{\bar{k}+1/l} A(k) = O((V/l^4)^{1/2}).$$

Thus we obtain

$$\int_l A_{\bar{k}}(x)d^4x = O(1). \tag{3a}$$

We also obtain

$$\int_l A_{\bar{k}}^2(x)d^4x = O(l^4) \tag{3b}$$

and

$$\int_l A_{\bar{k}}^3(x)d^4x = O(1). \tag{3c}$$

Analogously, we define

$$A_{\bar{k}}(\vec{x}, x_0) = \frac{1}{\sqrt{L}}\sum_{|k_0| > \bar{k}} e^{ik_0x_0}A(\vec{x}, k_0).$$

Then one easily finds

$$\int_l A_{\bar{k}}(\vec{x}, x_0)dx_0 = O(1), \tag{4a}$$

$$\int_l A_{\bar{k}}^2(\vec{x}, x_0)dx_0 = O(l), \tag{4b}$$

and

$$\int_l A_{\bar{k}}^3(\vec{x}, x_0)dx_0 = O(1). \tag{4c}$$

Equations (3) and (4) show that large Fourier components give no sizable contribution when $A(x)$ or $A^3(x)$ is integrated over a large volume. All long-wavelength coherences are in the small Fourier components. Consider now the ordered integral

$$\mathcal{O}\exp\left(ig\int_P A\cdot dx\right)$$

over a "square" path of side l. (Since A may have point singularities, the path may involve small detours so as to avoid the singularities.) From Eq. (4) an integral such as $\int_P A\cdot dx$ receives contributions of $O(1)$ from the large Fourier components. In the Appendix the contributions of large Fourier components to $\mathcal{O}\exp(ig\int_P A\cdot dx)$ are also shown to be much too weak to lead to confinement. [Such components only give $\theta(A,P) = O(1)$.] Thus, the small Fourier components are the ones related to confinement. In particular, as shown in the Appendix, in order that $\theta(A,P)$ be of size $l^{1-\gamma}$ it is necessary that $|A_\mu^a(k)|$ be of size $k^{-2+\gamma}$ for small k and for some a and μ.

B. The equations of motion

In part A of this section we have seen that to get $\theta(A,P)$ to be of order $l^{1-\gamma}$ it is necessary that $|A_\mu^a(k)|$ be of order $k^{-2+\gamma}$ for small k. It will now be shown that such a behavior, for $\gamma<1$, is incompatible with the classical equations of motion, and that the violation of the classical equations of motion is large. The equations of motion are

$$D_\nu^{ab}F_{\mu\nu}^b=0.$$

In the Lorentz gauge we obtain

$$\Box A_\mu^a=g\epsilon_{abc}(A_\nu^b\partial_\mu A_\nu^c-2A_\nu^b\partial_\nu A_\mu^c)+g^2(A_\mu^aA_\nu^bA_\nu^b-A_\nu^aA_\mu^bA_\nu^b).$$

The integral equation corresponding to this differential equation is

$$A_\mu^a(x)=-\frac{g}{4\pi^2}\epsilon_{abc}\int\frac{dy}{(x-y)^2}[A_\nu^b(y)\partial_\mu A_\nu^c(y)-2A_\nu^b(y)\partial_\nu A_\mu^c(y)]-\frac{g^2}{4\pi^2}\int\frac{dy}{(x-y)^2}[A_\mu^a(y)A_\nu^b(y)A_\nu^b(y)-A_\nu^a(y)A_\mu^b(y)A_\nu^b(y)\}\,. \tag{5}$$

An estimate of the size of the two integrals on the right-hand side of Eq. (5) can easily be made in momentum space, where one has

$$A_\mu^a(k)=-\frac{ig}{\sqrt{V}}\frac{\epsilon_{abc}}{k^2}\sum_{k_1}[A_\nu^b(k-k_1)A_\nu^c(k_1)k_{1\mu}-2A_\nu^b(k-k_1)A_\mu^c(k_1)k_{1\nu}]-\frac{g^2}{V}\frac{1}{k^2}\sum_{k_1,k_2}[A_\mu^a(k_1)A_\nu^b(k_2)A_\nu^b(k-k_1-k_2)-A_\nu^a(k_1)A_\mu^b(k_2)A_\nu^b(k-k_1-k_2)]. \tag{6}$$

The integral equations are illustrated in Fig. 1.

Now consider the first term on the right-hand side of Eq. (6). The large-k_1 contribution is

$$\left|\epsilon_{abc}\sum_{|k_1|>\bar{k}}A_\alpha^b(k-k_1)A_\beta^c(k_1)k_{1\gamma}\right|\le O(\sqrt{V}),$$

since $\epsilon_{abc}\langle A_\alpha^b(x)A_\beta^c(x)\rangle=0$, and the absence of long-range fluctuations means that a random approximation should be a good estimate of the k sum. The small k_1 contribution is, for small k,

$$\left|\epsilon_{abc}\sum_{k_1<\bar{k}}A_\alpha^b(k-k_1)A_\beta^c(k_1)k_{1\gamma}\right|=\mathcal{O}\left|\epsilon_{abc}\sum_{k_1=k+O(k)}A_\alpha^b(k-k_1)A_\beta^c(k_1)k_{1\gamma}\right|\le O(k^{-3+2\gamma}(Vk^4)^{1/2})=\sqrt{V}k^{-1+2\gamma}.$$

The $(Vk^4)^{1/2}$ comes from a random approximation for the Vk^4 modes in the region k to $k+O(k)$.

For the second term on the right-hand side of (6) we obtain

$$\left|\sum_{\substack{|k_1|,|k_2|>\bar{k}\\|k_1\pm k_2|>\bar{k}}}A_\mu^a(k_1)A_\nu^b(k_2)A_\nu^b(k-k_1-k_2)\right|\le O(V),$$

while

$$\left|\sum_{\substack{k_1=k+O(k)\\k_2=k+O(k)}}A_\mu^a(k_1)A_\nu^b(k_2)A_\nu^b(k-k_1-k_2)\right|<O(k^{-6+3\gamma}Vk^4)=Vk^{-2+3\gamma},$$

and regions where, say, $k_1=O(k)$ with k_2 large contribute

$$\left|\sum_{\substack{|k_1|=O(k)\\|k_2|>\bar{k}}}A_\mu^a(k_1)A_\nu^b(k_2)A_\nu^b(k-k_1-k_2)\right|=O(Vk^{\gamma-2}).$$

Thus the dominant term on the right-hand side of (6) is of order $k^{-4+\gamma}$ and this is not consistent with the classical equations of motion. The $k^{-4+\gamma}$ term on the right-hand side of (6) cancels only if $|A_\mu^a(k)|=v_a a_\mu$, that is, in the Abelian situation. However, a dominant small-k Abelian configuration is not consistent in nonleading order. In order to get

$$A_\mu^a(k)\underset{k\to 0}{=}O(k^{-2+\gamma})$$

one must add a source term, $J_\mu^a(k)$. Then one gets the integral equation illustrated in Fig. 2. In order to get $|A_\mu^a(k)|=O(k^{-2+\gamma})$ for small k it is necessary

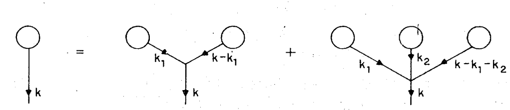

FIG. 1. An illustration of the classical Yang-Mills equation where the circles represent $A_\mu^a(k)$.

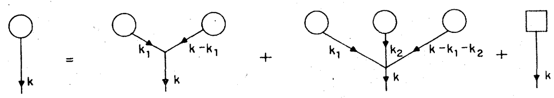

FIG. 2. An illustration of the classical Yang-Mills equation with a source. The circles represent $A_\mu^\alpha(k)$ and the square represents $J_\mu^\alpha(k)$.

that $|J_\mu^a(k)| = O(k^{-2+\gamma})$. In fact, the source is completely determining the long-wavelength structure of the theory in such a case. This might correspond to a situation of confinement in Yang-Mills theory with strong quantum effects, but is outside of a semiclassical approximation. This is to be compared to the three-dimentional Polyakov model, where the field equations become exact in the low-density limit[13] although the collective coordinates do not have values which make $-S$ a minimum.

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APPENDIX

Write

$$\mathcal{O}\exp\left(ig\int_0^{\lambda v_\mu} A(x)dx\right) = \cos\theta(\lambda) + i\vec{\tau}\cdot\hat{\theta}(\lambda)\sin\theta(\lambda) = M(\lambda),$$

where v_μ is a fixed unit vector, and the path between 0 and λv_μ is taken to be a straight line. Then,

$$\frac{dM(\lambda)}{d\lambda} = igA_\mu(\lambda v)v_\mu M(\lambda).$$

This means that

$$\frac{d\theta}{d\lambda} = A_\mu^a(\lambda v)\hat{\theta}^a v_\mu$$

or

$$\frac{1}{2}\frac{d\vec{\theta}^{\,2}}{d\lambda} = \vec{A}(\lambda)\cdot\vec{\theta}(\lambda).$$

In the above $\vec{A}(\lambda)\cdot\vec{\theta}(\lambda) = A_\mu^a(\lambda v)v_\mu\theta^a(\lambda)$. This can be integrated to give

$$\tfrac{1}{2}\vec{\theta}^{\,2}(\lambda) = \int_0^\lambda \vec{A}(\lambda')\cdot\vec{\theta}(\lambda')d\lambda'.$$

Now let us take v along the x_0 direction so that $\vec{\theta} = \vec{\theta}(x_0)$, and

$$\tfrac{1}{2}\vec{\theta}^{\,2}(x_0) = \int_0^{x_0} \vec{A}(\vec{x}, x'_0)\cdot\vec{\theta}(x'_0)dx'_0.$$

Writing

$$\vec{\theta}(x_0) = \frac{1}{\sqrt{L}}\sum_{k_0} e^{ik_0x_0}\vec{\theta}(k_0),$$

one obtains

$$\sum_{k_0,k'_0} e^{i(k_0+k'_0)x_0}\vec{\theta}(k_0)\cdot\vec{\theta}(k'_0) = \sum_{k_0k'_0} \frac{e^{i(k_0+k'_0)\cdot x}-1}{i(k_0+k'_0)}\vec{A}(\vec{x},k_0)\cdot\vec{\theta}(k'_0).$$

Now if

$$|\vec{\theta}(x_0)| \underset{x_0\to\infty}{=} O(x_0^{1-\gamma})$$

then

$$|\vec{\theta}(k_0)| \underset{k_0\to 0}{=} O(k_0^{-3/2+\gamma}).$$

For $|\vec{\theta}(k_0^{-3/2+\gamma})$ the above equation requires that

$$A(\vec{k},k_0) \underset{\substack{k_0\to 0\\ \vec{k}\to 0}}{=} O(k^{-2+\gamma}).$$

If only high-frequency parts of $A(\vec{k},k_0)$ are included in the above equation, then one would obtain

$$|\vec{\theta}(k_0)| \underset{k_0\to 0}{=} O(k^{-1/2})$$

and

$$|\vec{\theta}(x_0)| \underset{x_0\to\infty}{=} O(1).$$

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[11]The above argument is somewhat different from the one given by Polyakov in Ref. 2.
[12]For a given configuration, $A(x)$, $\theta(A,P)$ is determined by using

$$\mathcal{O}\exp\left(ig\int_{\alpha}^{\beta}A\,dx\right)=\cos\theta_{\alpha\beta}+i\vec{\tau}\cdot\hat{\theta}_{\alpha\beta}\sin\theta_{\alpha\beta},$$

which gives a unique $\theta_{\alpha\beta}$ for a direct path when β is near α, to define $\theta_{\alpha\beta}$ for a *continuous* motion of β from α around the path P and back to α again. This definition of $\theta(A,P)$ is precise although, as pointed out to me by C. Rebbi, not gauge invariant.

[13]This is not apparent in the singular gauge where it *might appear* that a magnetic source exists as the center of the monopole.