Explicit currents in SU(5) and E_7 unified gauge theories

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Explicit currents generating the SU(5) and E_7 Lie Algebras are constructed out of the quark and lepton fields appearing in the appropriate representations of these groups. The diquark-leptoquark structure of the currents involved in baryon-lepton transitions is exhibited, and it is pointed out that the diquark currents are related to color and flavor currents through Pauli-Gürsey transformations. In the SU(5) case, the u and d quarks are found to have charges $+2/3$ and $-1/3$, respectively, as an inevitable result of the way $SU(2)\times U(1)$ is embedded in SU(5). The E_7 currents are obtained by using Majorana leptons which can be related to Weyl leptons through an SU(6) generalization of the Pauli-Gürsey transformations. The dependence of diquark charges and algebra upon the assumed statistics for the basic quark and lepton fields is also discussed.

I. INTRODUCTION

The bold proposal of unifying strong, electromagnetic, and weak interactions with a single gauge group was initially put forward by Pati and Salam[1] and by Georgi and Glashow.[2] In the first of these, SU(4) is the color gauge group, with leptons representing the fourth color. In the second approach, color is described by SU(3) only and leptons are classified as color singlets. In both schemes, leptons and quarks are placed together in representations of the unifying gauge group, making transitions between baryons and leptons inevitable in principle, albeit sufficiently rare in practice through the choice of extremely large masses for the intermediate bosons involved. If the second alternative is adopted with the usual assumptions of permanent color confinement and fractional quark charges, new currents which have diquark as well as leptoquark components are needed to realize a baryon-lepton-number-violating process. In contrast, diquark currents are not mandatory in the Pati-Salam approach, where the quarks have integer charges as well as the leptons. However, it is interesting that an independent theoretical motivation for the existence of diquark currents has been given by Nambu,[3] who observed that a system of massless fermions automatically admits Pauli-Gürsey[4] transformations as an additional internal symmetry and that these are generated by diquark currents.

In this article, which is largely a continuation of an earlier work,[5] we will not concern ourselves with the Pati-Salam alternative since our interest mainly centers around diquark currents. In Ref. 5 unifying vectorlike simple supergroups of the SO(k), SU(k), and Sp(k) for all k as well as the exceptional types are sought such that the generators of the supergroup consist of those of the SU(n) $\times$SU(3) subgroup, plus a set of diquark-leptoquark charges whose number can be fixed by the assumption that the diquarks are generated by Pauli-Gürsey transformations. The only solution to the Diophantine equations (quadratic in both n and k; $n\leqslant 30$) thus set up is found to be $n=7$, leading to the supergroup SU(15). This model suffers from the fact that it involves at least 84 four-component leptons. On the other hand, as shown in Ref. 5, the exceptional groups G_2, F_4, and E_7 have the correct number of generators for the cases $n=1$, $n=3$, and $n=6$, respectively. Naturally, obtaining the correct number of currents is a necessary but not sufficient condition for establishing that these currents are the generators of the corresponding exceptional groups: it must be verified in addition that they generate the appropriate Lie commutator algebra. In Ref. 5 this was done for the cases of G_2 and F_4. In the present article we construct charges bilinear in quark and lepton fields belonging to low representations of SU(5) and E_7 and prove that these charges (some of which are of the diquark-leptoquark variety) satisfy the commutation relations of the above groups. The SU(5) example of Sec. II is interesting in that it does not fit into the general treatment of Ref. 4 by virtue of not being vectorlike. We also show that the commutation relations of SU(5) and the standard Weinberg[6] model currents for the leptons inevitably result in charges $\frac{2}{3}$ and $-\frac{1}{3}$ for the u and d quarks, respectively. *The factor of one third in the quark charges is a direct consequence of choosing the color group as* SU(3).

We treat E_7 in Sec. III. E_7 charges and their commutation relations have recently been obtained by Gürsey and Sikivie,[7] who take the fundamental 56-dimensional representation of E_7 to consist of left-handed spinors. In contrast, in our approach which uses Pauli-Gürsey transformations to generate the $(15^*,3)+(15,3^*)$ parts of the E_7 generators, it is more natural to use a Majorana representation and Majorana leptons. With our E_7 charges the commutation relations are satisfied only if the Majorana leptons obey a somewhat unusual anticommutation relation. This is easily shown to result if the Majorana leptons are given by a particular Pauli-Gürsey combination of Weyl leptons. In Sec. IV we draw attention to an observation of Nambu[3] that an algebra involving diquark charges qq or $q^\dagger q^\dagger$ depends crucially on the choice of Bose or Fermi statistics for the quarks and leptons, unlike algebras generated by normal $q^\dagger q$ charges which are insensitive to such a choice. We show in particular that the SU(5) charges and commutators are unchanged when Bose statistics is assumed while the changes resulting in the algebras of the exceptional groups can be compensated for by additional applications of the chiral $[\exp(i\alpha\gamma_5)]$ Pauli-Gürsey transformations. Finally, we present some concluding remarks in Sec. V.

II. SU(5) CHARGES AND COMMUTATION RELATIONS

The SU(5) algebra can be generated by charges built of only electron-type leptons (ν_e, e) and a single isodoublet of tricolored quarks (u_i, d_i; $i = 1, 2, 3$). If a more physically realistic spectrum is desired, the doublets (ν_μ, μ) and (c_i, s_i) can be trivially incorporated by analogy to the previous set of particles. Hence, we will stick to the minimal set,

$$\underline{5} \sim \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ e^c \\ \nu^c \end{bmatrix}_R , \quad \underline{10} \sim \begin{bmatrix} 0 & u_3^c & -u_2^c & -u_1 & -d_1 \\ -u_3^c & 0 & u_1^c & -u_2 & -d_2 \\ u_2^c & -u_1^c & 0 & -u_3 & -d_3 \\ u_1 & u_2 & u_3 & 0 & -e^c \\ d_1 & d_2 & d_3 & e^c & 0 \end{bmatrix}_L , \tag{1}$$

for the purpose of illustrating how the algebra is constructed. In the above, $L, R \equiv (1 \pm \gamma_5)$ and $\psi^c \equiv \gamma_2 \psi^*$ in the representation that we will employ. The representations $\underline{5}^*$ and $\underline{10}^*$ are obtained by $\psi \leftrightarrow \psi^c$, $L \leftrightarrow R$. The charges Q_λ ($\lambda = 1, \ldots, 24$) of SU(5) are then combinations of the entries of the 5×5 matrices, which we can indicate symbolically by

$$Q_\lambda \subset \{\underline{5}^* \gamma_2 \underline{5},\ \ \underline{10}^* \gamma_2 \underline{10}\} . \tag{2}$$

To ensure commutation between the color SU(3) charges and those of $SU_L(2) \times U(1)$, the matrices Q_λ must have the block-diagonal form

$$Q_\lambda \sim \begin{bmatrix} 3 \times 3 = (1, 8) & \begin{matrix} 3 \times 2 \\ \text{diquark-} \\ \text{leptoquarks} \end{matrix} \\ \begin{matrix} 2 \times 3 \text{ diquark-} \\ \text{leptoquarks} \end{matrix} & 2 \times 2 = (1, 1) + (3, 1) \end{bmatrix} , \tag{3}$$

where the first and second entries in (,) denote $SU(2)_L$ and $SU(3)_{\rm color}$ transformation properties, respectively. From (1) and (2) and the definitions of $\underline{5}^*$ and $\underline{10}^*$ in the above paragraph, the construction of the charges is now straightforward. The color charges are trivially given by (note the distinction in covariant and contravariant color indices)

$$T^i_j = \int dv \{(u^i u_j + d^i d_j) - \tfrac{1}{3} \delta^i_j (u^k u_k + d^k d_k)\} , \tag{4}$$

while the $SU(2)_L$ charges are obviously

$$W_L^+ = \int dv (\nu^\dagger e_L + u_L^i d_{iL}) , \tag{5a}$$

$$W_L^3 = \tfrac{1}{2} \int dv (\nu^\dagger \nu - e_L^\dagger e_L + u_L^i u_{iL} - d_L^i d_{iL}) . \tag{5b}$$

The "hypercharge" operator Y should have the form $Y = Y_{\rm leptonic} + Y_{\rm quark}$. We will take the leptonic part of this U(1) generator as in the 1967 Weinberg[6] paper,[8]

$$Y_{\rm leptonic} = \int dv (-e_R^\dagger e_R - \tfrac{1}{2} \nu^\dagger \nu - \tfrac{1}{2} e_L^\dagger e_L) , \tag{6}$$

so that the leptonic electric charges are expressed as

$$Q_{\rm leptonic} = W^3_{L\ \rm leptonic} + Y_{\rm leptonic} . \tag{7}$$

The quark charges should also be given by an equation as (7) but the generator $Y_{\rm quark}$ is not determined yet. It will emerge from a particular commutator of two SU(5) charges when W_L^3 and $Y_{\rm leptonic}$ are separated off from the right-hand side. We will then see that this operator added to $W^3_{L\,\rm quark}$ gives the charges $+\frac{2}{3}$ and $-\frac{1}{3}$ to the u_i and d_i quarks. Let us now turn to the diquark-leptoquark charges. The proper combinations out of (2) that lead to the closing of the SU(5) commutator algebra are

$$N_i \equiv \int dv (\nu \gamma_2 d_{iR} + u_{iL} \gamma_2 e_R + \epsilon_{ikj} u_R^k \gamma_2 d_L^j) , \tag{8a}$$

$$E_i \equiv \int dv (e_L \gamma_2 d_{iR} + d_{iL} \gamma_2 e_R - \epsilon_{ikj} u_R^k \gamma_2 u_L^j) , \tag{8b}$$

$$E^l \equiv \int dv (d_R^l \gamma_2 e_L^\dagger + e_R^\dagger \gamma_2 d_L^l + \epsilon^{lmn} u_{mL} \gamma_2 u_{nR}) , \tag{9a}$$

$$N^l \equiv \int dv (d_R^l \gamma_2 \nu^\dagger + e_R^\dagger \gamma_2 u_L^l - \epsilon^{lmn} d_{mL} \gamma_2 u_{nR}) . \tag{9b}$$

The minus signs in (8) and (9) come from the very well-known fact that under $SU(2)_L$, $(d_L^*, -u_L^*)$ transforms as (u_L, d_L). Note that (8) and (9) behave as (2, 3) and (2, 3*) respectively under $SU(2)_L \times SU(3)$ giving the transformation properties of all 24 SU(5) generators under this group,

$$Q_\lambda \sim \{(1,8)+(3,1)+(1,1)+(2,3)+(2,3^*)\}. \quad (10)$$

The diquark-leptoquark structure is clearly evident in (8) and (9). The diquark parts of the above charges are a subset of the diquark operators that can be derived by applying the Pauli-Gürsey transformations

$$\psi \to a\psi + b\gamma_5\psi^C = a\psi + b\gamma_5\gamma_2\psi^* \quad (|a|^2+|b|^2=1)\,, \quad (11a)$$

$$\psi \to \exp(i\alpha\gamma_5)\psi\,, \quad (11b)$$

on the quark fields in Eqs. (4) and (5). In particular, using (11b) together with (11a) cancels the γ_5 matrices, giving

$$d^i_L(u^i_L) \leftrightarrow \gamma_2 d_{iR}(u_{iR})\,. \quad (12)$$

When (4) and (5) are subjected to (12), the resulting diquarks have color content $\underline{3}^*+\underline{6}+\underline{3}+\underline{6}^*$, of which only the $\underline{3}$ and $\underline{3}^*$ parts are physically relevant since they transform the same way as the leptoquarks. We will see in Sec. III that Pauli-Gürsey transformations are more suitable for a derivation of the E_7 charges than they are in this case.

The SU(5) commutators of (4), (5), (8), and (9) now follow in a straightforward way from the anticommutation relations of the spinor fields,

$$[T^i_j, W^\pm_L] = [T^i_j, W^3_L] = [T^i_j, Y] = 0\,, \quad (13)$$

$$[N_i, N_j] = [E_i, E_j] = [N_i, E_j] = 0\,, \quad (14)$$

$$[N_i, E^l] = -\delta^l_i(d^k_L u_{kL} + e^\dagger_L\nu) \equiv -\delta^l_i W^-_L\,, \quad (15)$$

$$[N_i, N^l] = -\delta^l_i W^3_L + \delta^l_i Y - T^l_i\,, \quad (16)$$

$$[E_i, E^l] = \delta^l_i(W^3_L + Y) - T^l_i \quad (17)$$

$$= \delta^l_i(-e^\dagger e + \tfrac{2}{3}u^k u_k - \tfrac{1}{3}d^k d_k) - T^l_i\,, \quad (18)$$

$$[T^i_j, E^l] = \delta^l_j E^i - \tfrac{1}{3}\delta^i_j E^l\,, \quad (19)$$

$$[T^i_j, N_l] = -\delta^i_l N_j + \tfrac{1}{3}\delta^i_j N_l\,, \quad (20)$$

$$[W^+_L, E_i] = [W^+_L, N^l] = 0\,, \quad (21)$$

$$[W^+_L, N_i] = -E_i\,, \quad (22)$$

$$[W^+_L, E^l] = N^l\,, \quad (23)$$

$$[W_3, N_i] = -\tfrac{1}{2}N_i\,, \quad (24)$$

$$[W_3, E_i] = \tfrac{1}{2}E_i\,. \quad (25)$$

The commutation relations which have been left out are derivable by Hermitian conjugation from the above. Let us briefly point out the significance of these equations. (13) ensures the familiar strong-interaction conservation laws since color and weak charges commute. The right-hand side of (14) has to vanish since the algebra would not close otherwise: $\underline{2}_L \times \underline{2}_L$ gives $\underline{3}_L + \underline{1}_L$, neither of which is present in the set of (10). In (18), the term proportional to δ^l_i is simply the charge operator, from which the charges of the u and d quarks can be read off as $+\frac{2}{3}$ and $-\frac{1}{3}$. *It should be noted that this result is obtained without the use of the Gell-Mann–Nishijima relation* $Q = T_3 + (B+S)/2$ *or the assignment of strong-interaction flavor quantum numbers to quarks. In fact, in that approach the color degree of freedom is irrelevant to the charge assignments, whereas here the factor of one third results directly from the fact that the color group is* SU(3). The equations (19)–(25) simply state that the E^i, N_i, etc. transform as (2, 3) and (2, 3*) under $SU(2)_L \times SU(3)$. Finally, the expression for Y_{quark} is obtained from Eqs. (16) and (17),

$$Y_{\text{quark}} = +\tfrac{2}{3}u^i_R u_{iR} - \tfrac{1}{3}d^i_R d_{iR} + \tfrac{1}{6}(u^i_L u_{iL} + d^i_L d_{iL})\,. \quad (26)$$

This clearly commutes with $W^\pm_L, W^3_L$ as it should.

III. E_7 CHARGES AND COMMUTATION RELATIONS

A. Derivation in the Majorana representation

E_7 has a maximal subgroup SU(6) × SU(3), where the former factor can be assumed to correspond to the flavor and the latter to the color degrees of freedom.[9] The 56-dimensional fundamental and the 133-dimensional adjoint representations can be decomposed with respect to this subgroup as[10]

$$\underline{56} = (6,3) + (6^*,3^*) + (20,1)\,, \quad (27)$$

$$\underline{133} = (35,1) + (1,8) + (15,3^*) + (15^*,3)\,. \quad (28)$$

Obviously, the three terms in (27) are respectively to be thought of as quarks, antiquarks, and leptons while (28) represents 35 color-singlet weak, electromagnetic, and superweak currents, 8 gluons, and 90 diquark-leptoquarks. The group can be taken as vectorlike, which means that there is a right-handed current for each left-handed one, but this does not necessarily imply that the weak neutral current is purely vector. As we will see below, Pauli-Gürsey transformations are especially well suited for an explicit construction of the charges when the group is vectorlike. Let us denote quarks by $q_{i\alpha}$, where α and i are flavor and color indices, and leptons by the completely antisymmetric multiplet $l_{\alpha\beta\gamma}$. Antiparticles will be shown by upper indices. Then, as a first step, we can trivially write down the color generators,

$$C^l_k \equiv \int dv(q^{l\alpha}q_{k\alpha} - \tfrac{1}{3}\delta^l_k q^{i\alpha}q_{i\alpha})\,. \quad (29)$$

We will use a Majorana representation with an antisymmetric γ_5 in this section. The transformations (11) thus become

$$q_{i\alpha} \to a q_{i\alpha} + b\gamma_5 q^{i\alpha}\,, \quad (30a)$$

$$q_{i\beta} \to \exp(i\alpha\gamma_5) q_{i\beta}, \qquad (30b)$$

since $(q_{i\alpha})^C = q^{i\alpha}$ in our representation. Applying (30b) together with (30a) allows

$$q_{i\alpha} \to q^{i\alpha}, \qquad (31)$$

in correspondence to (21). Thus we can derive two new types of color-triplet diquark charges by subjecting (29) to (30a) or (31). These are

$$D^k_{\alpha\beta} \sim \epsilon^{kij} \int dv\, q_{i\alpha}\gamma_5 q_{j\beta}, \qquad (32)$$

$$D'^k_{\alpha\beta} \sim \epsilon^{kij} \int dv\, q_{i\alpha} q_{j\beta}. \qquad (33)$$

Equations (32) and (33) correspond to different flavor multiplets: Since we are assuming that the quarks obey fermion anticommutation relations and since γ_5 is antisymmetric in this representation, the antisymmetry in color, common to both charges, forces (32) to consist of the antisymmetric part of $\underline{6} \times \underline{6}$, i.e., $\underline{15}^*$; while (33) constitutes the 21-dimensional symmetric part. Comparing these with (28), we see that (32) and its charge conjugate belong to the $(15, 3^*) + (15^*, 3)$ sector of the E_7 generators while (33) has to be excluded altogether. In fact, there is no other simple group which has $35 + 8 + 2 \times 3 \times 21$ parameters, hence, (33)-type charges cannot be used in any such theory. It should be emphasized in this context that the above argument, which links the γ matrix and flavor structure of the charges with the statistics of the basic fields, is peculiar to theories employing diquark currents. In contrast, any SU(n) flavor or color algebra can be generated by $q^i q_j$-type charges independently of whether the quarks obey Bose or Fermi statistics. This aspect of diquark charges has been first pointed out by Nambu[3] and we will analyze it more extensively in Sec. IV.

We can now turn to constructing the full diquark-leptoquark charges by adding a leptoquark part with the same $(15, 3^*)$ transformation property onto (32),

$$Q^k_{\alpha\beta} = a\epsilon^{kij} \int dv\, q_{i\alpha}\gamma_5 q_{j\beta} + b \int dv\, q^{k\gamma} l_{\alpha\beta\gamma}. \qquad (34)$$

Here a and b are coefficients that will be determined by the requirement that the algebra closes. Because of the transposition of fields in Hermitian conjugation, the $(15^*, 3)$ charges must have the form

$$Q^{\mu\nu}_l = a^* \epsilon_{lmn} \int dv\, q^{\nu n}\gamma_5 q^{\mu m} + b^* \int dv\, l^{\rho\mu\nu} q_{l\rho}. \qquad (35)$$

The commutator of the generators of (34) must either vanish or be proportional to $(15^*, 3)$ since this is the only representation common to both (28) and the decomposition of $(15, 3^*) \times (15, 3^*)$. Indeed, with $a = b = 1$, normal quark anticommutation relations, and the relation

$$l_{\alpha\beta\gamma}(\vec{0})\, l_{\mu\nu\sigma}(\vec{x}) + l_{\mu\nu\sigma}(\vec{x})\, l_{\alpha\beta\gamma}(\vec{0}) = \epsilon_{\alpha\beta\gamma\mu\nu\sigma}\gamma_5 \delta^3(\vec{x}) \qquad (36)$$

for the lepton fields, we obtain the E_7 commutation relation

$$[Q^l_{\mu\nu}, Q^k_{\alpha\beta}] = \tfrac{1}{2}\epsilon_{\mu\nu\alpha\beta\rho\sigma}\epsilon^{lkn} Q^{\rho\sigma}_n. \qquad (37)$$

The remaining E_7 commutators can be worked out similarly with the following results:

$$[Q^l_{\mu\nu}, Q^{\alpha\beta}_k] = (\delta^\alpha_\mu\delta^\beta_\nu - \delta^\beta_\mu\delta^\alpha_\nu) C^l_k + \tfrac{1}{6}\epsilon^{\alpha\beta\eta\pi\sigma\theta}\epsilon_{\mu\nu\rho\pi\sigma\theta} F^\rho_\eta, \qquad (38)$$

$$[C^i_j, F^\rho_\eta] = 0, \qquad (39)$$

$$[C^i_j, Q^l_{\mu\nu}] = \delta^l_j Q^i_{\mu\nu} - \tfrac{1}{3}\delta^i_j Q^l_{\mu\nu}, \qquad (40)$$

$$[F^\alpha_\beta, Q^l_{\mu\nu}] = \delta^\alpha_\nu Q^l_{\beta\mu} - \delta^\alpha_\mu Q^l_{\beta\nu} + \tfrac{1}{3}\delta^\alpha_\beta Q^l_{\mu\nu}. \qquad (41)$$

The commutators not shown here can be derived from the above by Hermitian conjugation. In the preceding equations F^α_β are the (35, 1) flavor charges defined by

$$F^\alpha_\beta = \int dv [q^{\alpha i} q_{\beta i} - \tfrac{1}{6}\delta^\alpha_\beta (q^{\gamma i} q_{\gamma i}) + \tfrac{1}{4} l^{\alpha\mu\nu} l_{\beta\mu\nu}], \qquad (42)$$

where

$$l^{\alpha\beta\gamma} = \tfrac{1}{6}\gamma_5 \epsilon^{\alpha\beta\gamma\mu\nu\rho} l_{\mu\nu\rho}. \qquad (43)$$

It should be noted that leptonic currents constitute part of the flavor currents just as the usual $SU(2)_L$ weak currents have quark and lepton components. The properties of (36) and (43) ensure that the last term in (42) is pure (35, 1).

B. Connection with earlier work

The relations (37)–(41) have first been given by Sikivie and Gürsey,[7] who use only left-handed fields for the fundamental 56-dimensional representation. As a result, the explicit expressions of their charges in terms of quark and lepton fields differ from ours especially for the diquark-leptoquarks of Eqs. (34) and (35). In the following, we will briefly indicate how the expressions in Ref. 7 can be recovered from (34)–(36). First of all, when passing from the Majorana representation to the Weyl representation the γ_5 in (34) and (35) must be replaced by $\gamma_5\gamma_2$. Now let us consider the diquark part of (34) as an example and decompose the fields into their left- and right-handed components,

$$D^k_{\alpha\beta} \equiv \epsilon^{kij}\int dv\, q_{i\alpha}\gamma_5\gamma_2 q_{j\beta}$$

$$= \epsilon^{kij}\int dv (q^L_{i\alpha} + q^R_{i\alpha})\gamma_5\gamma_2(q^L_{j\beta} + q^R_{j\beta}) \qquad (44)$$

$$= \epsilon^{kij}\int dv (q^L_{i\alpha}\gamma_2 q^R_{j\beta} - q^R_{i\alpha}\gamma_2 q^L_{j\beta}) \,. \qquad (45)$$

Since $\gamma_2 q^R_{j\beta}$ is the left-handed field $(\hat{q}^{j\beta}_R)^\dagger$, used in Ref. 7 it is clear that (45) is nothing but the diquark term in Eq. (A10) of that work.

Next we will show that the seemingly anomalous anticommutation relation (36) is in fact just the normal one for a Pauli-Gürsey-transformed Weyl field. Let us introduce Weyl fields $\psi^L_{\alpha\beta\gamma}$, also transforming as (20, 1) under SU(6) × SU(3) and related to the $l_{\alpha\beta\gamma}$ through

$$l_{\alpha\beta\gamma} = \frac{1}{\sqrt{2}}(\psi^L_{\alpha\beta\gamma} + \tfrac{1}{6}\gamma_5\gamma_2\,\epsilon_{\alpha\beta\gamma\mu\nu\eta}\psi^{\mu\nu\eta}_L) \,. \qquad (46)$$

It should be noted that since the Weyl representation is now being used, (36) should have $\gamma_5\gamma_2$ on the right-hand side instead of γ_5. With the relation

$$\psi^{\mu\nu\eta}_L = \tfrac{1}{6}\epsilon^{\mu\nu\eta\alpha\beta\gamma}\psi^L_{\alpha\beta\gamma} \,, \qquad (47)$$

which is Eq. (A9) of Ref. 7, and the normal anticommutation condition,

$$\psi^{\alpha\beta\gamma}_L(\vec{0})\psi^L_{\mu\nu\eta})(\vec{x}) + \psi^L_{\mu\nu\eta}(\vec{x})\psi^{\alpha\beta\gamma}_L(\vec{0}) = \tfrac{1}{6}\epsilon^{\alpha\beta\gamma\rho\sigma\theta}\epsilon_{\mu\nu\eta\rho\sigma\theta}\delta^3(\vec{x}) \,, \qquad (48)$$

it is easy to verify that (46) obeys (36) in its Weyl representation form with $\gamma_5\gamma_2$ on the right. It should be noticed that (46) is itself a special Pauli-Gürsey transformation of the form (11a) with $a = b = 1/\sqrt{2}$ properly generalized to incorporate SU(6) internal symmetry. Substituting (46) in the leptoquark part of (34), we obtain the leptoquark terms in (A10) of Ref. 7.

IV. STATISTICS AND DIQUARKS

Although the spin-statistics theroem[12] dictates the type of statistics according to the spin of a given field, it is interesting that at the level of current algebra a set of spin-$\frac{1}{2}$ fields q_i $(i = 1, 2, \ldots, n)$ and their conjugates q^i generates the same U(n) algebra,

$$[T^i_j, T^k_l] = \delta^k_j T^i_l - \delta^i_l T^k_j \,, \qquad (49)$$

through the integrated charge densities $T^i_j = \int dv\, q^i q_j$, regardless of whether the fields obey Fermi or Bose statistics. This is perhaps not too surprising since the fields q^i and q_j in T^i_j are not subject to any permutation symmetry that identical particles must obey, the result in (49) is independent of a particular exchange symmetry. Indeed, because of the same reason, $q^i q_j$ makes up the representation $(n^2 - 1) \oplus (1)$ independently of the choice of statistics. Nambu[3] has suggested that the situation in principle can be entirely different in an algebra involving diquarks $\int dv\, q^i \Gamma q^j$, $\int dv\, q_j \Gamma q_i$, as permutation symmetry now becomes relevant between two fields both carrying upper or lower indices. Here Γ denotes a combination of γ matrices and the fields can of course have more than one index corresponding to color, flavor, etc. In Ref. 5, Γ is specified by the assumption that the diquark currents originate from the application of Pauli-Gürsey transformations on normal color and flavor currents. Thus, in a Dirac-Pauli or Weyl representation, $\Gamma = \gamma_5\gamma_2$ if (11a) is applied single, or $\Gamma = \gamma_2$ if it is combined with (11b). $\gamma_5\gamma_2$ and γ_2 are respectively antisymmetric and symmetric real matrices. Considering quarks with SU(n) flavor indices α, β and SU(3) color indices i, j we can now see that

$$q_{\alpha i}\gamma_5\gamma_2 q_{\beta j} \sim \left(\frac{n(n-1)}{2}, 3^*\right) \oplus \left(\frac{n(n+1)}{2}, 6\right) \qquad (50)$$

and

$$q_{\alpha i}\gamma_2 q_{\beta j} \sim \left(\frac{n(n-1)}{2}, 6\right) \oplus \left(\frac{n(n+1)}{2}, 3^*\right) \qquad (51)$$

with Fermi statistics, while

$$q_{\alpha i}\gamma_5\gamma_2 q_{\beta j} \sim \left(\frac{n(n-1)}{2}, 6\right) \oplus \left(\frac{n(n+1)}{2}, 3^*\right) \qquad (52)$$

and

$$q_{\alpha i}\gamma_2 q_{\beta j} \sim \left(\frac{n(n-1)}{2}, 3^*\right) \oplus \left(\frac{n(n+1)}{2}, 6\right) \qquad (53)$$

with Bose statistics. These of course simply follow from the requirement that the product of interchanges connected with permutation symmetry (statistics) of the fields, the γ matrix, flavor, and color indices be symmetric, since the bilinear diquark forms vanish otherwise. Thus we see that the flavor-color representation content of diquarks does indeed depend on quark statistics, unlike the situation with $q^\dagger q$ currents. In a "realistic" unified gauge theory with Fermi statistics for spin $-\frac{1}{2}$ fields, diquarks are only present in association with leptoquarks which transform as 3 and 3* in color space. Hence in such a theory only the first term of (50), the second term of (51), and their Hermitian conjugates need be considered. We have shown in Ref. 5 that color-triplet (51)-type diquarks are parts of the generators of the

G_2 and F_4 for $n=1$ and $n=3$, respectively. On the other hand for $n=6$, the first term of (50) is merely the E_7 diquarks of Eq. (32) where the γ_2 is missing on account of the Majorana representation used there.

Now let us return to analyzing how the commutator algebras of SU(5) and E_7 are affected if Bose statistics is ascribed to the quarks and leptons. Considering SU(5) first, it is clear from the observations at the beginning of this section that the generators and the algebras connected with the $SU(2)_L \times U(1) \times SU(3)_{color}$ subgroup will be unaffected by a change of statistics. Observing that the diquark parts of the remaining diquark-leptoquarks (8) and (9) are built of two "nonidentical" fields in the sense of the fields always having opposite handedness, we may again expect statistics to be irrelevant. Indeed, it is obvious from (8) and (9) that these generators will respectively transform as (2, 3) and (2, 3*) under $SU(2)_L \times U(1) \times SU(3)$ even with Bose statistics. With some more work one can easily verify that the SU(5) commutation relations (13)–(25) are obeyed with commuting as well as anticommuting quark and lepton fields.

The situation with E_7 is somewhat different. The diquarks here must transform as (15, 3*) + (15*, 3) under $SU(6)_{flavor} \times SU(3)_{color}$, i.e., both the flavor and color representations must be antisymmetric. Thus with Fermi statistics we must use the first term of (50) and with Bose statistics the first term of (53), both with $n=6$. The former is of course the operator in Eq. (32), leading to the algebra (37)–(41) when the lepton fields obey (36). However, it is an interesting fact that the latter leads to the same algebra if the lepton fields *commute* according to

$$l_{\alpha\beta\gamma}(\vec{0})l_{\mu\nu\sigma}(\vec{x}) - l_{\mu\nu\sigma}(\vec{x})l_{\alpha\beta\gamma}(\vec{0}) = \epsilon_{\alpha\beta\gamma\mu\nu\sigma}\gamma_2\delta^3(\vec{x}), \qquad (54)$$

where the γ_2 can be replaced by the identity if a Majorana representation as in Sec. IIIA is used. Similarly, the G_2 and F_4 algebras can be generated by *commuting* quarks, generators as in the second term of (52) with $n=1$ and $n=3$ respectively. The proofs of these statements involve nothing but straightforward evaluation of commutators wi quarks and leptons obeying obvious commutation relations.

V. SUMMARY AND DISCUSSION

We have explicitly displayed all the physical currents that enter into the unified gauge theories SU(5) and E_7 in terms of the quark and lepton fields characteristic of these groups. Since the forms of color octet and color singlet currents in such theories are immediately obvious, the really novel element in this paper is that the specific diquark-leptoquark structure of the remaining generators is obtained and exhibited. It is worth stressing again some special aspects of diquark currents first observed by Nambu. (i) Given a theory with a particular color $\otimes$ flavor symmetry generated by the usual $q^{+}q$-type charges, a larger internal symmetry is automatically present if the fermions are massless—this is the Pauli-Gürsey symmetry generated by diquark charges of the form qq and $q^{+}q^{+}$. Since these charges have been seen to constitute parts of the generators of a unifying supergroup, one can view their Pauli-Gürsey origin as additional theoretical support for the necessity of combining the strong, electromagnetic, and weak interactions within a single gauge group. (ii) A theory involving diquark currents may depend intricately on the statistics of its basis fields unlike theories generated only by $q^{\dagger}q$-type currents, where the algebra is the same whether the fields obey Fermi or Bose statistics. From the results of Ref. 5 and this work we also have the conclusion that when $SU(n)_{flavor} \times SU(3)_{color}$ groups are enlarged to unifying supergroups by the addition of color-triplet diquark-leptoquark generators, the only possible such groups turn out to be G_2, F_4, E_7, and SU(15), where the last contains 7 quark flavors and 84 leptons. Since E_6 follows from similarly enlarging the group $[SU(3) \times SU(3)]_{flavor} \times SU(3)_{color}$, it appears that all exceptional groups other than E_8 involve diquarks, and their generators decompose as flavor charges $\oplus$ color octet charges $\oplus$ diquark-leptoquarks.

Finally, we will briefly compare some outstanding mathematical and physical aspects of the guage theories SU(5) and E_7. The SU(5) color triplet and antitriplet diquark-leptoquarks trivially commute among themselves as can be seen from (14), whereas the corresponding charges in E_7 generate the richer noncommutative algebra (37). This can perhaps be regarded as an aesthetic point in favor of the latter group. A somewhat more serious mathematical advantage of E_7 is that it can accommodate all the relevant quarks and leptons in its fundamental representation while even the minimal set of particles requires the use of both 5- and 10-dimensional representations with SU(5). Related to this property is the situation that the total number of quarks and leptons in E_7 is fixed at the outset,[13] in contrast to SU(5) which can admit arbitrarily many quark and lepton weak doublets as long as anomalies[14] cancel.

In regard to the physical processes described by the two groups, the main difference is the presence of flavor-changing neutral currents in E_7. While the corresponding intermediate bosons have to be made extremely heavy (with masses around 10^6 GeV), this feature also allows for a natural incorporation of a superweak[15] mechanism of *CP* violation into the theory. On the other hand, within SU(5), superweak results can be very effectively simulated via the Kobayashi-Maskawa[16] scheme by the use of six flavors of quarks and six leptons. Because of the success[17] of this simulation, it is unlikely that an experimental choice between the two alternatives will be possible in the near future. However, if no parity violation at the expected level is seen in atomic physics[18] experiments, new right-handed currents outside the SU(5) model will be necessary. E_7, being vector-like,[8,19] can naturally accommodate this phenomenon.

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