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Propagation functions in pseudoparticle fields

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The Green's functions for massless spinor and vector particles propagating in a self-dual but otherwise arbitrary non-Abelian gauge field are shown to be completely determined by the corrresponding Green's functions of scalar particles. Simple, explicit algebraic expressions are constructed for the scalar Green's functions of isospin-1/2 and isospin-1 particles in the self-dual field of a configuration of n pseudoparticles described by $5n$ arbitrary parameters.

I. INTRODUCTION

The existence of classical, pseudoparticle solutions[1] in non-Abelian gauge theory has profound implications for the structure and physical consequences of this theory. The classical pseudoparticle solution in Euclidean space provides a tunneling path which yields a finite quantum transition amplitude between different would-be vacuum states characterized by a vanishing field-strength tensor but with a nontrivial gauge field. Thus the true vacuum state in non-Abelian gauge theory has a rich structure.[2-4] The pseudoparticle solution may provide a resolution[2,3] of the "U(1) problem" by removing the unwanted Goldstone boson from a theory which has an apparent chiral-phase symmetry. Pseudoparticle solutions may also provide a mechanism for quark confinement.[5,6]

Clearly, it is important to develop the dynamical theory of fields quantized about classical pseudoparticle solutions. The first part of this program entails the determination of the nature of the small fluctuations of such quantum fields. This involves the calculation of propagators (Green's functions) for particles moving in the external field of a pseudoparticle. The small fluctuation of the non-Abelian gauge field and its associated "ghost" field correspond to the motion of spin-1 and spin-0, massless particles. Hadronic matter is presumably described by the interaction of these fields with those of spin-$\frac{1}{2}$, massless particles (quantum chromodynamics). In this paper, we shall present explicit and simple algebraic formulas for the Euclidean propagation functions of massless particles with spin 0, $\frac{1}{2}$, and 1 moving in the external, classical, non-Abelian gauge field of any pseudoparticle solution. A brief account of some of our results has already appeared.[7]

The pseudoparticle solutions that concern us are characterized by field-strength tensors which are either self-dual or anti-self-dual. We begin our development by showing that the propagation functions for massless spin-$\frac{1}{2}$ and spin-1 particles moving in a self-dual or anti-self-dual but otherwise arbitrary gauge field are determined explicitly by the propagation functions of the corresponding massless, spin-0 particles. This is a completely general result which holds for any gauge group; the only restrictions are that the field strength be self-dual (or anti-self-dual) and that the particles be massless. The spin-$\frac{1}{2}$ case will be worked out below in Sec. II and the spin-1 case in Sec. III. Thus we will need explicit expressions only for the propagation functions of massless, spin-0 particles moving in pseudoparticle fields.

The pseudoparticle solutions, and the propagation functions in these external fields, are defined in Euclidean space-time. We shall work entirely in Euclidean space in this paper. Moreover, we shall restrict our discussion of the spin-0 propagators to those in a SU(2) gauge field. The original pseudoparticle solution[1] approaches a pure SU(2) gauge transformation at infinity, with the gauge transformation covering the SU(2) group once as the field point at infinity covers the S_3 hypersphere once. The gauge field has a topological character described by a winding number (or Pontryagin index) 1. Thus the solution is referred to as the field of one pseudoparticle. Its field-strength tensor is self-dual. There is another solution giving a similar mapping of the S_3 hypersphere once onto the SU(2) group, but with the points on S_3 mapped to the inverse group elements. This is the antipseudoparticle with winding number -1; its field-strength tensor is anti-self-dual. The general solution with n pseudoparticles or n antipseudoparticles covers the SU(2) group n times as the field point at infinity covers the hypersphere S_3 once. It has winding number $\pm n$ (with a self-dual or anti-self-dual field-strength tensor). The general solution is determined by $8n$ parameters[8] (three of which designate a global gauge orientation). Of these $8n$ parameters, $4n$ determine the positions of the n pseudoparticles, n describe their sizes, and $3n$ fix their orientations in the

SU(2) gauge space. Explicit n-pseudoparticle solutions have been constructed[9] in terms of $5n$ parameters[10]; these solutions do not contain the $3n$ parameters needed to fix the gauge orientations of the pseudoparticles. We shall construct explicit and simple algebraic expressions for spin-0 propagators in this somewhat restricted n-pseudoparticle (or antipseudoparticle) field. The construction for isospin $\frac{1}{2}$ will be carried out in Sec. IV and that for isospin 1 in Sec. V.

We discuss our results in Sec. VI. Here we note that the spin-$\frac{1}{2}$ and spin-1 propagation functions decrease more slowly at large distances than do free propagators. At large distances the spin-$\frac{1}{2}$ propagator is of order $1/x^2$ in contrast to the $O(1/x^3)$ behavior of the free propagator. Similarly, at large distances the spin-1 propagation function is of order $1/x$ while the free propagator is $O(1/x^2)$. This slowly vanishing character of the propagators in pseudoparticle fields is a gauge-independent property; it may significantly affect physical processes occurring in pseudoparticle fields.

II. SPIN $\frac{1}{2}$

We turn now to the construction of the massless spin-$\frac{1}{2}$ propagator in a self-dual or anti-self-dual but otherwise arbitrary non-Abelian gauge field. We shall use simple operator techniques to express this propagator in terms of the corresponding spin-0 propagator. Before considering the spin-$\frac{1}{2}$ propagator, let us first establish our notation and conventions. We work in Euclidean space-time, and use skew-Hermitian Dirac matrices $\gamma_1, \gamma_2, \gamma_3, \gamma_4 = i\gamma^0$ obeying the anticommutator condition

$$\{\gamma_\mu, \gamma_\nu\} = -2\delta_{\mu\nu}. \tag{2.1}$$

Hermitian Dirac matrices γ_5 and $\sigma_{\mu\nu}$ are defined by

$$\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4 \tag{2.2}$$

(with $\gamma_5^2 = +1$) and

$$\sigma_{\mu\nu} = \tfrac{1}{2}i[\gamma_\mu, \gamma_\nu]. \tag{2.3}$$

The dual of a skew-symmetrical tensor $f_{\mu\nu} = -f_{\nu\mu}$ is defined by

$${}^d f_{\mu\nu} = \tfrac{1}{2}\epsilon_{\mu\nu\lambda\kappa} f_{\lambda\kappa}, \tag{2.4}$$

where $\epsilon_{\mu\nu\lambda\kappa}$ is the completely antisymmetrical tensor with $\epsilon_{1234} = +1$. A simple exercise in the Dirac matrix algebra shows that

$$\sigma_{\mu\nu}\left(\frac{1\pm\gamma_5}{2}\right) = \mp {}^d\sigma_{\mu\nu}\left(\frac{1\pm\gamma_5}{2}\right). \tag{2.5}$$

The matrices $\sigma_{\mu\nu}$ are generators of the Euclidean O(4) rotation group while the matrices $\frac{1}{2}(1\pm\gamma_5)$ project into spaces of definite chirality, $\gamma_5' = \pm 1$. The formula (2.5) shows that the spaces of definite chirality reduce the representation of the O(4) algebra generated by $\sigma_{\mu\nu}$ into the direct product, $O(4) = SU(2) \otimes SU(2)$.

We shall consider the spin-$\frac{1}{2}$ propagation function in the non-Abelian field of an arbitrary gauge group specified by the real, completely antisymmetrical structure constants f_{abc}. The spin-$\frac{1}{2}$ field belongs to some representation of the gauge group with generators specified by Hermitian matrices T_a which obey the commutation relations

$$[T_a, T_b] = if_{abc}T_c, \tag{2.6}$$

where a sum over repeated indices will always be implicit. The gauge-covariant derivative D_μ is defined by

$$D_\mu = \partial_\mu - iT_a A_{\mu a}(x), \tag{2.7}$$

where $A_{\mu a}(x)$ is the non-Abelian gauge field. By virtue of Eq. (2.6), this derivative obeys

$$[D_\mu, D_\nu] = -iT_a F_{\mu\nu a}, \tag{2.8}$$

where

$$F_{\mu\nu a}(x) = \partial_\mu A_{\nu a}(x) - \partial_\nu A_{\mu a}(x) + f_{abc}A_{\mu b}(x)A_{\nu c}(x) \tag{2.9}$$

is the non-Abelian field-strength tensor. Now, using Eqs. (2.1), (2.3), and (2.8), we find that

$$-(\gamma D)^2 = D^2 + \tfrac{1}{2}\sigma_{\mu\nu}T_a F_{\mu\nu a}. \tag{2.10}$$

Let us suppose that $F_{\mu\nu a}$ is self-dual,

$${}^d F_{\mu\nu a} = +F_{\mu\nu a}. \tag{2.11}$$

Then, according to Eqs. (2.5) and (2.10), we have

$$-(\gamma D)^2\left(\frac{1+\gamma_5}{2}\right) = D^2\left(\frac{1+\gamma_5}{2}\right), \tag{2.12}$$

since the $\sigma_{\mu\nu}F_{\mu\nu}$ term arising from Eq. (2.10), when acting on the chiral projection matrix $\frac{1}{2}(1+\gamma_5)$, is equal to $-{}^d\sigma_{\mu\nu}F_{\mu\nu} = -\sigma_{\mu\nu}{}^dF_{\mu\nu} = -\sigma_{\mu\nu}F_{\mu\nu}$ and thus vanishes. It is this simple statement [Eq. (2.12)] which will enable us to relate the massless spin-$\frac{1}{2}$ propagator to a massless spin-0 propagator in a self-dual, non-Abelian gauge field. In order to keep our notation clear, we shall write out the development for the case of a self-dual field. The anti-self-dual case is obtained simply by changing the sign of γ_5, $\gamma_5 \to -\gamma_5$.

The massless spin-$\frac{1}{2}$ propagator $S(x, y)$ has a formal representation as sum over normal modes,

$$S(x, y) \sim \sum_n \frac{\psi_n(x)\psi_n^\dagger(y)}{\lambda_n}, \tag{2.13}$$

with mode functions ψ_n of eigenvalues λ_n,

$$\gamma D\psi_n = \lambda_n\psi_n. \tag{2.14}$$

There are, however, a finite number N of zero-mode functions[11] $\psi_n^{(0)}$, satisfying

$$\gamma D\psi_n^{(0)} = 0 . \tag{2.15}$$

Since the matrix γ_5 anticommutes with γD, the zero-mode functions $\psi_n^{(0)}$ can be chosen to simultaneously diagonalize γ_5; i.e., they can be chosen to be chirality eigenstates. Now if we multiply the zero-mode equation (2.15) by γD we get, by Eq. (2.10),

$$(-D^2 - \tfrac{1}{2}\sigma_{\mu\nu}T_a F_{\mu\nu a})\psi_n^{(0)} = 0 . \tag{2.16}$$

If $\psi_n^{(0)}$ has positive chirality, $\gamma_5\psi_n^{(0)} = +\psi_n^{(0)}$, then, according to the discussion following Eq. (2.12), $\sigma_{\mu\nu}F_{\mu\nu}\psi_n^{(0)}$ vanishes. In this case there can be no zero-mode solution to Eq. (2.16) because the remaining operator, $-D^2$, is essentially a positive operator. (See Brown *et al.*, Ref. 8.) Hence the zero-mode functions must have negative chirality,

$$\gamma_5\psi_n^{(0)} = -\psi_n^{(0)} . \tag{2.17}$$

In this case, the $-\frac{1}{2}\sigma_{\mu\nu}T_a F_{\mu\nu a}$ contribution in Eq. (2.16) no longer vanishes and, in fact, presents a negative potential energy in appropriate spin-isospin states. Thus, Eq. (2.16) can appear as a Schrödinger equation at zero energy for a particle moving in a negative potential, an equation that may possess square-integrable solutions. As shown rigorously in Ref. 11, there are indeed a finite number of these solutions.

None of these zero modes can appear in the mode sum representation of the propagator, Eq. (2.13), if the propagator is to exist. Note, however, that the quantum transition amplitude with spin-$\frac{1}{2}$ fermion fields involves a factor of Det $\gamma D = \prod \lambda_n$ times the propagation function for some number, k, of spin-$\frac{1}{2}$ Fermi particles. The latter, k-particle propagation function is formed as an antisymmetric product of k two-point propagators, $\det_k S(x_m, y_n)$, corresponding to the Fermi statistics of spin-$\frac{1}{2}$ particles. On account of the factor Det γD, the quantum amplitude will vanish unless $k \geq N$ and all N zero-mode states are included in the k-particle propagation function. By virtue of the complete antisymmetry of the determinant (the Pauli principle), the remaining k-N particles in $\det_k S(x_m, y_n)$ cannot be in zero-mode states. Thus the propagation of these remaining particles, the particles with which we are concerned, is described by a propagator

$$S(x, y) = \sum_n{}' \frac{\psi_n(x)\psi_n^\dagger(y)}{\lambda_n}, \tag{2.18}$$

where the prime on the summation sign indicates that the zero modes (states with $\lambda_n = 0$) are to be deleted.

It follows from Eq. (2.18) that the spin-$\frac{1}{2}$ propagator obeys the Green's function equation

$$\gamma DS(x, y) = Q(x, y) , \tag{2.19}$$

where

$$Q(x, y) = \delta(x - y) - \sum_n \psi_n^{(0)}(x)\psi_n^{(0)\dagger}(y) , \tag{2.20}$$

with the summation running over all the zero-mode functions. The quantity $Q(x, y)$ represents the projection operator into the subspace of all nonzero modes, the complement of the zero-mode subspace. It also follows from Eq. (2.18) that the spin-$\frac{1}{2}$ propagator is orthogonal to all the zero-mode functions,

$$\int (d_E{}^4x)\psi_n^{(0)\dagger}(x)S(x, y) = 0 . \tag{2.21}$$

The Green's function equation (2.19) and the orthogonality constraint (2.21) serve to define the spin-$\frac{1}{2}$ propagation function $S(x, y)$.

A construction of this propagator is easily achieved with operator techniques. We write the function $S(x, y)$ as the matrix element of an operator S,

$$S(x, y) = \langle x|S|y\rangle . \tag{2.22}$$

Similarly, we write the corresponding spin-0 propagation function $\Delta(x, y)$, which is defined by

$$-D^2\Delta(x, y) = \delta(x - y) , \tag{2.23}$$

as the matrix element of an operator $1/-D^2$,

$$\Delta(x, y) = \left\langle x \left| \frac{1}{-D^2} \right| y \right\rangle . \tag{2.24}$$

We now assert that the operator expression of the spin-$\frac{1}{2}$ propagator is

$$S = \gamma D \frac{1}{-D^2}\left(\frac{1+\gamma_5}{2}\right) + \frac{1}{-D^2}\gamma D\left(\frac{1-\gamma_5}{2}\right) . \tag{2.25}$$

The proof of this assertion is quick. We multiply Eq. (2.25) by γD and use Eq. (2.12) to get

$$\gamma DS = Q , \tag{2.26}$$

where

$$Q = \frac{1+\gamma_5}{2} + \gamma D \frac{1}{-D^2}\gamma D\left(\frac{1-\gamma_5}{2}\right) . \tag{2.27}$$

Now Eq. (2.26) implies that Q contains no zero modes since these modes are annihilated by γD. On the other hand, using Eq. (2.12) again, we find that

$$\gamma DQ = \gamma D . \tag{2.28}$$

Hence Q is the operator which projects into the subspace of all the nonzero modes, and we have

$$Q(x, y) = \langle x|Q|y\rangle , \tag{2.29}$$

where $Q(x, y)$ is the function defined in Eq. (2.20). The matrix elements of Eq. (2.26) thus reproduce the Green's function equation (2.19), and therefore the matrix elements of Eq. (2.25) produce a spin-$\frac{1}{2}$ propagation function which obeys this Green's function equation. [Incidentally, it is a simple matter to use Eq. (2.12) to check directly that the definition (2.27) does obey the projection property $Q^2 = Q$.] It remains to be shown that the assertion (2.25) gives a propagator which is orthogonal to all the zero modes, i.e., a propagator which obeys the constraints (2.21). This, however, is immediate, for Eq. (2.12) implies that

$$QS = S , \tag{2.30}$$

which is tantamount to the constraints (2.21).

The matrix elements of Eq. (2.25) express the massless spin-$\frac{1}{2}$ propagation function in a self-dual but otherwise arbitrary non-Abelian gauge field in terms of the corresponding massless, spin-0 propagation function,

$$S(x, y) = \gamma D^{(x)} \Delta(x, y)\left(\frac{1+\gamma_5}{2}\right) + \Delta(x, y)\gamma \overleftarrow{D}^{(y)}\left(\frac{1-\gamma_5}{2}\right). \tag{2.31}$$

(Note that when the symbol D_μ acts to the left, $\overleftarrow{D}_\mu$, it involves the derivative ∂_μ with its sign reversed, $-\overleftarrow{\partial}_\mu$.) Equation (2.31) expresses the principal result of this section. The spin-$\frac{1}{2}$ propagator in an arbitrary anti-self-dual field is obtained from Eq. (2.31) by changing the sign of the γ_5 matrices and using the appropriate spin-0 propagator.

III. SPIN 1

The non-Abelian gauge field is governed by the action

$$g^2 W = -\int (d_E{}^4x)\left[\tfrac{1}{4}(F_{\mu\nu a})^2 + \frac{1}{2\xi}(D^{\rm cl}_{\mu ab}A_{\mu b})^2\right]. \tag{3.1}$$

With our normalization of the gauge field [cf. Eq. (2.9)] the coupling constant g appears as an overall factor. Here we have chosen a "background gauge" specified by the parameter ξ: The operator $D^{\rm cl}_{\mu ab}$ is the gauge-covariant derivative

$$D^{\rm cl}_{\mu ab} = \partial_\mu \delta_{ab} + f_{acb}A^{\rm cl}_{\mu c}, \tag{3.2}$$

where the vector potential $A^{\rm cl}_{\mu a}$ describes a classical solution of the non-Abelian field equations and is fixed in all variations of the action. We shall study the small fluctuations $\phi_{\mu a}$ of the gauge field about the classical solution,

$$A_{\mu a} = A^{\rm cl}_{\mu a} + \phi_{\mu a}. \tag{3.3}$$

Inserting this decomposition of the vector potential into the field-strength tensor [Eq. (2.9)] and extracting pieces quadratic in ϕ_μ from the resulting action [Eq. (3.1)] yields the small-fluctuation, vector-field action

$$g^2 W_2 = -\tfrac{1}{2}\int (d_E{}^4x)\phi_\mu[-D^2\delta_{\mu\nu} - 2F_{\mu\nu} + (1-1/\xi)D_\mu D_\nu]\phi_\nu. \tag{3.4}$$

Here and henceforth we omit the superscript cl on the classical fields $A^{\rm cl}_\mu$, $F^{\rm cl}_{\mu\nu}$, and on the corresponding gauge-covariant derivative $D^{\rm cl}_\mu$. We have also adopted a matrix notation with regard to group indices a, b, ..., defined

$$(F_{\mu\nu})_{ab} = f_{acb}F_{\mu\nu c}, \tag{3.5}$$

and used the commutation relation

$$[D_\mu, D_\nu] = F_{\mu\nu}, \tag{3.6}$$

which follows from the fact that the structure-constant matrices

$$(f_c)_{ab} = f_{acb} \tag{3.7}$$

obey the relation

$$[f_a, f_b] = f_{abc}f_c. \tag{3.8}$$

From the definition [Eq. (2.9)] of the field-strength tensor, it follows algebraically that the dual tensor ${}^dF_{\mu\nu,a}$ obeys

$$D_{\mu ab}\,{}^dF_{\mu\nu,b} = 0. \tag{3.9}$$

Hence any gauge field with a self-dual (or anti-self-dual) field-strength tensor ${}^dF_{\mu\nu,a} = \pm F_{\mu\nu,a}$ provides automatically a solution of the field equations

$$D_{\mu ab}F_{\mu\nu b} = 0. \tag{3.10}$$

Moreover, by a suitable gauge transformation, we may impose the background gauge condition

$$D_{\mu ab}A_{\mu b} = \partial_\mu A_{\mu a} = 0. \tag{3.11}$$

In general, a continuously connected family of self-dual (or anti-self-dual) fields exists, labeled by some set of continuously varying parameters. [For example, if $A_\mu(x)$ yields a self-dual field-strength tensor, then so does the translated field $A_\mu(x-z)$, where z_μ are four constant parameters.] Thus, given a self-dual (or anti-self-dual) field, we can take its derivative with respect to one of its parameters to get a small-fluctuation field which is also self-dual (or anti-self-dual). Moreover, an infinitesimal gauge transformation can be added to this small-fluctuation field to bring it into the background gauge. We find that any self-dual (or anti-self-dual) field will support some number of zero-mode fluctuation fields $\phi^{(s)}_{\mu,a}$, $s=1,2,\ldots,N$, satisfying

$$D_\mu \phi^{(s)}_\mu = 0 , \tag{3.12}$$

and the field equation which follows from the action (3.4),

$$(D^2\delta_{\mu\nu} + 2F_{\mu\nu})\phi^{(s)}_\nu = 0 . \tag{3.13}$$

In the background gauge, the fluctuation fields $\phi^{(s)}_\mu$ are square integrable,[12] and their contribution to the spin-1 propagator must be deleted just as the zero modes were deleted from the spin-$\frac{1}{2}$ propagator in the preceding section. Thus, according to the action (3.4), the spin-1 propagator $G_{\mu\nu}(x,y)$ in an arbitrary self-dual (or anti-self-dual) field obeys the Green's function equation

$$-[D^2\delta_{\mu\lambda} + 2F_{\mu\lambda} - (1-1/\xi)D_\mu D_\lambda]G_{\lambda\nu}(x,y) = Q_{\mu\nu}(x,y) , \tag{3.14}$$

where

$$Q_{\mu\nu}(x,y) = \delta_{\mu\nu}\delta(x-y) - \sum_{S=1}^{N} \phi^{(s)}_\mu(x)\phi^{(s)}_\nu(y) \tag{3.15}$$

projects onto the space of the nonzero-mode functions. (We should note that the zero modes play a different role here from that in the spin-$\frac{1}{2}$ case. The zero modes in the spin-1 field are accounted for by the introduction of appropriate collective coordinates.[13])

The vector propagation function in an arbitrary self-dual (or anti-self-dual) field can be constructed in terms of the corresponding scalar-field propagator using simple operator techniques akin to those employed in the preceding section for the spin-$\frac{1}{2}$ propagator. To proceed with this construction, we note that

$$p^{(\pm)}_{\mu\nu\lambda\kappa} = \tfrac{1}{4}(\delta_{\mu\lambda}\delta_{\nu\kappa} - \delta_{\mu\kappa}\delta_{\nu\lambda} \pm \epsilon_{\mu\nu\lambda\kappa}) \tag{3.16}$$

projects out the self-dual part ($p^{(+)}$) or anti-self-dual part ($p^{(-)}$) of an antisymmetrical tensor. We now define

$$q^{(\pm)}_{\mu\nu\lambda\kappa} = \delta_{\mu\nu}\delta_{\lambda\kappa} + 4p^{(\pm)}_{\mu\nu\lambda\kappa} \tag{3.17}$$

and, for an arbitrary operator X,

$$\{X\}^{(\pm)}_{\mu\nu} = q^{(\pm)}_{\mu\nu\lambda\kappa} D_\lambda X D_\kappa . \tag{3.18}$$

(Some motivation for the introduction of the bracket operation is presented in the Appendix.) This bracket operation has several useful properties when the field $F_{\mu\nu}$ is either self-dual [for the (+) brackets] or anti-self-dual [for the (−) brackets]. First, we note that

$$p^{(\pm)}_{\mu\nu\lambda\kappa} D_\lambda D_\kappa = p^{(\pm)}_{\mu\nu\lambda\kappa}\tfrac{1}{2}[D_\lambda, D_\kappa] = \tfrac{1}{2}F_{\mu\nu} . \tag{3.19}$$

Hence

$$\{1\}^{(\pm)}_{\mu\nu} = D^2\delta_{\mu\nu} + 2F_{\mu\nu} , \tag{3.20}$$

which expresses the field equation operator (with $\xi=1$) as the bracket operation on the identity [cf. Eq. (3.4)]. Second, we use the commutation relation of the gauge-covariant derivatives [Eq. (3.6)] and the self-duality property

$$\tfrac{1}{2}\epsilon_{\mu\nu\lambda\kappa}F_{\lambda\kappa} = \pm F_{\mu\nu} \tag{3.21}$$

to establish that

$$D_\mu\{X\}^{(\pm)}_{\mu\nu} = D^2 X D_\nu \tag{3.22a}$$

and

$$\{X\}^{(\pm)}_{\mu\nu} D_\nu = D_\mu X D^2 . \tag{3.22b}$$

Third, we make use of the algebraic relation proved in the Appendix:

$$q^{(\pm)}_{\mu\sigma\alpha\beta}q^{(\pm)}_{\sigma\nu\lambda\kappa} = \delta_{\beta\lambda}q^{(\pm)}_{\mu\nu\alpha\kappa} + r^{(\pm)}_{\mu\nu\alpha\kappa\beta\lambda} . \tag{3.23}$$

Here $r^{(\pm)}_{\mu\nu\alpha\kappa\beta\lambda}$ is a numerical tensor whose detailed structure need not concern us; we need only the fact that it is antisymmetrical in its last pair of indices, and that its duality character in this last index pair is reversed:

$$\tfrac{1}{2} r^{(\pm)}_{\mu\nu\alpha\kappa\rho\sigma}\epsilon_{\rho\sigma\beta\lambda} = \mp r^{(\pm)}_{\mu\nu\alpha\kappa\beta\lambda} . \tag{3.24}$$

Hence,

$$r^{(\pm)}_{\mu\nu\alpha\kappa\beta\lambda}D_\beta D_\lambda = r^{(\pm)}_{\mu\nu\alpha\kappa\beta\lambda}\tfrac{1}{2}F_{\beta\lambda} = 0 , \tag{3.25}$$

and we secure the bracket composition law

$$\{X\}^{(\pm)}_{\mu\sigma}\{Y\}^{(\pm)}_{\sigma\nu} = \{XD^2Y\}^{(\pm)}_{\mu\nu} . \tag{3.26}$$

Henceforth, so as to achieve a simpler notation, we shall consider only self-dual fields and delete the superscript (±). (The treatment of the anti-self-dual case is obvious.)

The construction of the spin-1 propagation function $G_{\mu\nu}(x,y)$ can now be quickly performed with the aid of these operator techniques. We assert that $G_{\mu\nu}(x,y)$ has the formal operator realization

$$G_{\mu\nu} = -\left\{\left(\frac{1}{D^2}\right)^2\right\}_{\mu\nu} + (1-\xi)D_\mu\left(\frac{1}{D^2}\right)^2 D_\nu . \tag{3.27}$$

(An alternative derivation of this result is sketched in the Appendix.) To prove this assertion, we first observe that

$$\begin{aligned}[-D^2\delta_{\mu\lambda} - 2F_{\mu\lambda} + (1-1/\xi)D_\mu D_\lambda]D_\lambda &= -\{1\}_{\mu\lambda}D_\lambda + (1-1/\xi)D_\mu D^2 \\ &= -(1/\xi)D_\mu D^2 , \end{aligned}\tag{3.28}$$

whence

$$[-D^2\delta_{\mu\lambda}-2F_{\mu\lambda}+(1-1/\xi)D_\mu D_\lambda]G_{\lambda\nu}$$
$$=\{1\}_{\mu\lambda}\left\{\left(\frac{1}{D^2}\right)^2\right\}_{\lambda\nu}=Q_{\mu\nu}\,, \quad (3.29)$$

where

$$Q_{\mu\nu}=\{1/D^2\}_{\mu\nu}\,. \quad (3.30)$$

The quantity in square brackets on the left-hand side of Eq. (3.29) is the field-equation operator which appears on the left-hand side of the Green's function, Eq. (3.14). Thus, we will have proved that the operator $G_{\mu\nu}$ given by Eq. (3.27) is a formal operator realization of the spin-1 propagation function $G_{\mu\nu}(x,y)$ if we prove that $Q_{\mu\nu}$ [cf. Eqs. (3.29) and (3.30)] is an operator realization of the projector $Q_{\mu\nu}(x,y)$ onto the nonzero modes, Eq. (3.15). The proof that $Q_{\mu\nu}$ is the correct projection operator proceeds exactly as in the spin-$\frac{1}{2}$ case discussed in the preceding section. Since Eq. (3.29) expresses $Q_{\mu\nu}$ as the field equation applied to some operator, $Q_{\mu\nu}$ itself can contain no zero modes. On the other hand, it follows from Eqs. (3.20), (3.26), and (3.30) that

$$(D^2\delta_{\mu\lambda}+2F_{\mu\lambda})Q_{\lambda\nu}=\{1\}_{\mu\lambda}\{1/D^2\}_{\lambda\nu}$$
$$=\{1\}_{\mu\nu}$$
$$=D^2\delta_{\mu\nu}+2F_{\mu\nu}\,, \quad (3.31)$$

which implies that $Q_{\mu\nu}$ contains all the nonzero modes. Hence $Q_{\mu\nu}$ is indeed the correct projection operator. [Incidentally, the fact that $Q_{\mu\nu}$ is a projection operator, $Q_{\mu\lambda}Q_{\lambda\nu}=Q_{\mu\nu}$, follows immediately from Eqs. (3.26) and (3.30)].

There are problems with the formal operator construction (3.27) for the spin-1 propagators. These problems arise from the convolution integral that defines the matrix element of the operator $(1/D^2)^2$,

$$\left\langle x\left|\left(\frac{1}{D^2}\right)^2\right|y\right\rangle=\int (d_E{}^4z)\Delta(x,z)\Delta(z,y)\,. \quad (3.32)$$

At large distances ($z^2\to\infty$) the spin-0 propagator $\Delta(x,z)$ behaves as $1/z^2$ and the integral in Eq. (3.32) diverges logarithmically. We shall discuss this difficulty in Sec. VI after we have derived explicit forms for the spin-0 propagators $\Delta(x,y)$ in specific pseudoparticle fields.

IV. ISOSPIN-$\frac{1}{2}$ SCALAR PROPAGATORS

Here we shall construct explicitly the massless, spin-0 propagation function with isospin $\frac{1}{2}$ in the self-dual (or anti-self-dual) SU(2) gauge field of n pseudoparticles. First we shall review some properties of the pseudoparticle fields. The vector potential of a single pseudoparticle is given by[1]

$$A_{\mu a}^{(\pm)}(x)=\frac{2\eta_{\mu\nu a}^{(\pm)}(x-z)_\nu}{(x-z)^2+\rho^2}\,, \quad (4.1)$$

where $a=1,2,3$ is the SU(2) group index and $\eta_{\mu\nu a}^{(\pm)}$ is antisymmetrical in $\mu\nu$,

$$\eta_{\mu\nu a}^{(\pm)}=-\eta_{\nu\mu a}^{(\pm)}\,, \quad (4.2a)$$

with

$$\eta_{kla}^{(\pm)}=\epsilon_{kla} \quad (4.2b)$$

and

$$\eta_{k4a}^{(\pm)}=\pm\delta_{ka}\,. \quad (4.2c)$$

The (+) superscript denotes a pseudoparticle solution which has a self-dual field-strength tensor while the (−) superscript denotes an antipseudoparticle solution which has an anti-self-dual field-strength tensor. (These self-duality properties will become evident in the following.) The four constants z_λ parametrize the position of the pseudoparticle and the fifth constant ρ parametrizes its size.

It is convenient to introduce the Hermitian 2×2 Pauli spin matrices τ_a which obey

$$\tau_a\tau_b=\delta_{ab}+i\epsilon_{abc}\tau_c\,, \quad (4.3)$$

and define the matrix field

$$A_\mu^{(\pm)}=A_{\mu a}^{(\pm)}\tfrac{1}{2}\tau_a\,. \quad (4.4)$$

It is also convenient to introduce the four-vector symbols

$$\tau_\mu=(\vec{\tau},i)\,,\quad \tau_\mu^\dagger=(\vec{\tau},-i)\,, \quad (4.5)$$

which have the useful properties

$$\tau_\mu^\dagger\tau_\nu=\delta_{\mu\nu}+i\eta_{\mu\nu a}^{(+)}\tau_a \quad (4.6a)$$

and

$$\tau_\mu\tau_\nu^\dagger=\delta_{\mu\nu}+i\eta_{\mu\nu a}^{(-)}\tau_a\,. \quad (4.6b)$$

The coordinate-dependent matrix

$$\Omega(x)=\frac{-i\tau_\mu x_\mu}{(x^2)^{1/2}} \quad (4.7)$$

is a unitary matrix

$$\Omega(x)^\dagger=\Omega^{-1}(x)\,, \quad (4.8)$$

which connects each space-time point with an element of the SU(2) gauge group in a particular way: As the coordinate x_μ ranges once over the S_3 hypersphere $x^2=$const, the matrix $\Omega(x)$ covers the SU(2) group space once; this mapping thus has winding number +1. It follows from Eqs. (4.6) that

$$\Omega(x)^{\mp 1}i\partial_\mu\Omega(x)^{\pm 1}=\eta_{\mu\nu a}^{(\pm)}\frac{x_\nu}{x^2}\tau_a\,. \quad (4.9)$$

Hence, the matrix form (4.4) of the pseudoparticle solution (4.1) can be expressed as

$$A_\mu^{(\pm)}(x)=\frac{(x-z)^2}{(x-z)^2+\rho^2}\,\Omega(x-z)^{\mp 1}\times i\partial_\mu\Omega(x-z)^{\pm 1}. \tag{4.10}$$

This form shows immediately that at large distances the vector potential approaches a pure gauge transformation,

$$A_\mu^{(\pm)}(x)\xrightarrow[x^2\to\infty]{}\Omega(x)^{\mp 1}i\partial_\mu\Omega(x)^{\pm 1}. \tag{4.11}$$

Thus, the pseudoparticle potential $A_\mu^{(+)}(x)$ provides a mapping with winding number +1, and the antipseudoparticle potential $A_\mu^{(-)}(x)$ provides a mapping with winding number −1.

In order to motivate the construction of the n-pseudoparticle solution, we first gauge transform the single pseudoparticle solution [Eq. (4.10)] so that it vanishes more rapidly at infinity,

$$\begin{aligned}\bar{A}_\mu^{(\pm)}(x)&=\Omega(x-z)^{\pm 1}[i\partial_\mu+A_\mu^{(\pm)}(x)]\Omega(x-z)^{\mp 1}\\&=\left[1-\frac{(x-z)^2}{(x-z)^2+\rho^2}\right]\Omega(x-z)^{\pm 1}i\partial_\mu\Omega(x-z)^{\mp 1}.\end{aligned} \tag{4.12}$$

We use Eqs. (4.4) and (4.9) to write this as

$$\begin{aligned}\bar{A}_{\mu a}^{(\pm)}(x)&=\frac{2\rho^2}{(x-z)^2+\rho^2}\eta_{\mu\nu a}^{(\mp)}\frac{(x-z)_\nu}{(x-z)^2}\\&=-\eta_{\mu\nu a}^{(\mp)}\partial_\nu\ln\left[1+\frac{\rho^2}{(x-z)^2}\right].\end{aligned} \tag{4.13}$$

The solution in this gauge is singular at $x_\mu=z_\mu$. This singularity arises because the gauge transformation matrix $\Omega(x-z)$ is singular at $x_\mu=z_\mu$ in the sense that its value at this point depends upon the direction in which the limit $x_\mu=z_\mu$ was approached. Nonetheless, since the singular solution [Eq. (4.13)] can be gauge transformed into a regular solution [Eq. (4.10)], the singular solution is an acceptable one. In view of the structure of Eq. (4.13), it is natural to try the form[9]

$$\bar{A}_{\mu a}^{(\pm)}(x)=-\eta_{\mu\nu a}^{(\mp)}\partial_\nu\ln\Pi(x), \tag{4.14}$$

for a general n-pseudoparticle solution. This will be a solution to the field equations if the field-strength tensor

$$\bar{F}_{\mu\nu a}^{(\pm)}=\partial_\mu\bar{A}_{\nu a}^{(\pm)}-\partial_\nu\bar{A}_{\mu a}^{(\pm)}+\epsilon_{abc}\bar{A}_{\mu b}^{(\pm)}\bar{A}_{\nu c}^{(\pm)} \tag{4.15}$$

is self-dual (or anti-self-dual), i.e.,

$$p_{\mu\nu\lambda\kappa}^{(\mp)}\bar{F}_{\lambda\kappa a}^{(\pm)}=0, \tag{4.16}$$

where the projection operators $p^{(\pm)}$ are defined in Eq. (3.16). The symbols $\eta_{\mu\nu a}^{(\pm)}$ define transformation matrices that take the three independent components of an antisymmetrical, self-dual (or anti-self-dual) tensor into a three-vector labeled by the index a. Thus they decompose the O(4) rotation group into its SU(2)⊗SU(2) subgroups. Indeed

$$p_{\mu\nu\lambda\kappa}^{(\pm)}=\tfrac{1}{4}\eta_{\mu\nu a}^{(\pm)}\eta_{\lambda\kappa a}^{(\pm)}, \tag{4.17}$$

and the conditions (4.16) are tantamount to the conditions

$$\eta_{\mu\nu b}^{(\mp)}\bar{F}_{\mu\nu a}^{(\pm)}=0. \tag{4.18}$$

The $\eta_{\mu\nu a}^{(\pm)}$ symbols play another role. They are isomorphic to the generators $i\tau_a$ of the SU(2) group since

$$\eta_{\mu\lambda a}^{(\pm)}\eta_{\nu\lambda b}^{(\pm)}=\delta_{ab}\delta_{\mu\nu}+\epsilon_{abc}\eta_{\mu\nu c}^{(\pm)}. \tag{4.19}$$

Inserting Eqs. (4.14) and (4.15) into the self-duality condition (4.18) and using Eq. (4.19), we find that it is satisfied if

$$\Pi(x)^{-1}\partial^2\Pi(x)=0. \tag{4.20}$$

If the function $\Pi(x)$ approaches unity at large distances, then the vector potential (4.14) will vanish at infinity. Hence, a general solution[9] of Eq. (4.20) is given by[10]

$$\Pi(x)=1+\sum_{s=1}^{n}\frac{\rho_s^2}{(x-z_s)^2}. \tag{4.21}$$

This describes an n-pseudoparticle configuration specified by $5n$ parameters: $4n$ position variables $z_{s\mu}$ and n sizes ρ_s. It is not the most general n-pseudoparticle solution, for the latter involves[8] $8n$ parameters, with $3n$ additional parameters specifying the gauge orientation of each pseudoparticle. [Specializing to $n=1$ we recover the solution (4.13) and hence, by the gauge transformation (4.12), verify that the vector potential (4.1) is the regular, single-pseudoparticle solution.]

The n-pseudoparticle solution (4.14) is singular at each of the pseudoparticle positions, i.e., as $x\to z_s$,

$$\bar{A}_{\mu a}^{(\pm)}(x)\to 2\eta_{\mu\nu a}^{(\mp)}\frac{(x-z_s)_\nu}{(x-z_s)^2}. \tag{4.22}$$

If we write the vector potential in a matrix notation and use Eq. (4.9), we see that this singularity has the structure of a pure gauge transformation,

$$\bar{A}_\mu^{(\pm)}(x)\xrightarrow[x\to z_s]{}\Omega(x-z_s)^{\pm 1}i\partial_\mu\Omega(x-z_s)^{\mp 1}. \tag{4.23}$$

Hence, these singularities of $\bar{A}_{\mu a}^{(\pm)}(x)$ can be removed by a gauge transformation[14]

$$\tilde{A}_\mu^{(\pm)}(x)=U^{(\pm)}(x)^{-1}[i\partial_\mu+\bar{A}_\mu^{(\pm)}(x)]U^{(\pm)}(x), \tag{4.24}$$

if a unitary matrix $U^{(\pm)}(x)$ can be found which obeys for all z_s, $s=1,\ldots,n$,

$$U^{(\pm)}(x)\xrightarrow[x\to z_s]{}\Omega(x-z_s)^{\pm 1}R_s^{(\pm)}(x), \tag{4.25}$$

with $R_s^{(\pm)}(x)$ a unitary matrix that is regular at x

$=z_s$. Such a matrix $U^{(\pm)}(x)$ obeying Eq. (4.25) can be constructed, and thus the divergences in $\bar{A}_\mu^{(\pm)}(x)$ can be removed by a gauge transformation. Indeed, for a single pseudoparticle, $n=\pm1$, we have

$$U^{(\pm)}(x)=\Omega(x-z_1)^{\pm1}, \tag{4.26a}$$

for two pseudoparticles, $n=\pm2$, we have

$$U^{(\pm)}(x)=\Omega(x-z_1)^{\pm1}\Omega(z_2-z_1)^{\mp1}\Omega(x-z_2)^{\pm1}, \tag{4.26b}$$

while for three pseudoparticles, $n=\pm3$, we have

$$\begin{aligned}U^{(\pm)}(x)=&\Omega(x-z_1)^{\pm1}\Omega(z_2-z_1)^{\mp1}\Omega(x-z_2)^{\pm1}\\&\times\Omega(z_3-z_2)^{\mp1}\Omega(z_2-z_1)^{\pm1}\Omega(z_3-z_1)^{\mp1}\\&\times\Omega(x-z_3)^{\pm1},\end{aligned} \tag{4.26c}$$

and so forth. Note that this construction shows that the $U^{(\pm)}(x)$ matrix for $\pm n$ pseudoparticles covers the SU(2) group $\pm n$ times as the space-time coordinate x ranges once over the surface of a large hypersphere S_3 that encloses all of the pseudoparticle positions z_s. Since as $x^2\to\infty$,

$$\tilde{A}_\mu^{(\pm)}(x)\to U^{(\pm)}(x)^{-1}i\partial_\mu U^{(\pm)}(x), \tag{4.27}$$

we find explicitly that the vector potential has a winding number $\pm n$. We should remark that although the gauge transformation (4.24) does remove all the divergent pieces of the vector potentail, there remain "singularities" in the transformed potential $\tilde{A}_\mu^{(\pm)}(x)$ involving ill-defined quantities of the form $(x-z)_\nu[(x-z_s)^2]^{-1/2}$. These weaker singularities, however, cause no trouble in that they do not give rise to singularities in the action.

With this lengthy introduction completed, we now proceed with the construction of the massless, spin-0, isospin-$\frac{1}{2}$ propagation function $\bar{\Delta}^{(\pm)}(x-y)$ in the presence of the pseudoparticle field $\bar{A}_\mu^{(\pm)}(x)$ [Eqs. (4.14) and (4.21)]. This propagator is defined by

$$-\bar{D}^{(\pm)2}\bar{\Delta}^{(\pm)}(x,y)=\delta(x-y), \tag{4.28}$$

where

$$\bar{D}_\mu^{(\pm)}=\partial_\mu-\tfrac{1}{2}i\tau_a\bar{A}_{\mu a}^{(\pm)}(x). \tag{4.29}$$

Utilizing Eqs. (4.3), (4.19). and (4.14), we find that

$$\bar{D}^{(\pm)2}=\partial^2-\tfrac{3}{4}(\partial_\mu\ln\Pi)^2+i\tau_a\eta_{\mu\nu a}^{(\mp)}(\partial_\nu\ln\Pi)\partial_\mu. \tag{4.30}$$

The propagator can be easily constructed because $\bar{D}^{(\pm)2}$ can be factored. We use Eq. (4.6a) or Eq. (4.6b) and Eq. (4.20) to secure

$$\bar{D}^{(\pm)2}=\Pi^{1/2}\tau\partial\Pi^{-1}\tau^\dagger\partial\Pi^{1/2} \tag{4.31a}$$

and

$$\bar{D}^{(-)2}=\Pi^{1/2}\tau^\dagger\partial\Pi^{-1}\tau\partial\Pi^{1/2}. \tag{4.31b}$$

(Here and henceforth we write scalar products such as $\tau_\mu\partial_\mu$ simply as $\tau\partial$.)

The propagation function must have the same short-distance singularity as the free propagation function,

$$\bar{\Delta}^{(\pm)}(x,y)\underset{x\to y}{\sim}\frac{1}{4\pi^2(x-y)^2}, \tag{4.32}$$

to produce the inhomogeneous term $\delta(x-y)$ in the Green's function equation (4.28). On the other hand, the appearance of the factors of $\Pi^{1/2}$ on the right-hand side of Eqs. (4.31) suggests that $\bar{\Delta}^{(\pm)}(x-y)$ should contain a factor of $\Pi(x)^{-1/2}\Pi(y)^{-1/2}$. Thus, we are led to write

$$\bar{\Delta}^{(\pm)}(x,y)=\Pi(x)^{-1/2}\frac{F^{(\pm)}(x,y)}{4\pi^2(x-y)^2}\Pi(y)^{-1/2}, \tag{4.33}$$

where the function $F^{(\pm)}(x,y)$ must obey the boundary condition

$$F^{(\pm)}(x,x)=\Pi(x). \tag{4.34}$$

Inserting the decomposition (4.33) into the Green's function equation (4.28) and using Eq. (4.31a) and the fact that

$$\Pi(x)^{-1/2}\delta(x-y)\Pi(y)^{1/2}=\delta(x-y), \tag{4.35}$$

we get

$$\tau\partial[\Pi(x)4\pi^2(x-y)^2]^{-1}\left[\tau^\dagger\partial F^{(+)}(x,y)-\frac{2\tau^\dagger(x-y)}{(x-y)^2}F^{(+)}(x,y)\right]=\delta(x-y). \tag{4.36}$$

Now according to Eq. (4.6b) we have

$$\tau\partial\tau^\dagger\partial=\partial^2, \tag{4.37}$$

and hence

$$\tau\partial\frac{\tau^\dagger(x-y)}{2\pi^2[(x-y)^2]^2}=\delta(x-y). \tag{4.38}$$

This allows us to write Eq. (4.36) as

$$\tau\partial[\Pi(x)(x-y)^2]^{-1}\left\{\tau^\dagger\partial F^{(+)}(x,y)-2\frac{\tau^\dagger(x-y)}{(x-y)^2}[F^{(+)}(x,y)-\Pi(x)]\right\}=0. \tag{4.39}$$

Since $\tau\partial$ has no zero eigenvalues [cf. Eq. (4.37)], we conclude that the quantity in curly brackets in Eq. (4.39) must vanish,

$$\tau^\dagger\partial F^{(+)}(x,y)-2\frac{\tau^\dagger(x-y)}{(x-y)^2}[F^{(+)}(x,y)-\Pi(x)]=0. \tag{4.40}$$

The same first-order differential equation holds for $F^{(-)}(x,y)$, but with $\tau_\nu^\dagger$ replaced by τ_ν.

We recall that

$$\Pi(x)=1+\sum_{s=1}^{n}\frac{\rho_s{}^2}{(x-z_s)^2}, \tag{4.21}$$

and assert that Eq. (4.39) is satisfied if $F^{(+)}(x,y)$ is given by

$$F^{(+)}(x,y)=1+\sum_{s=1}^{n}\rho_s^2\,\frac{\tau(x-z_s)}{(x-z_s)^2}\,\frac{\tau^\dagger(y-z_s)}{(y-z_s)^2}\,. \tag{4.41}$$

The proof rests on the repeated use of Eqs. (4.6). They imply that the boundary condition (4.34) is obeyed [as is necessary for any solution of Eq. (4.40)]. They also imply that

$$\tau^\dagger\partial F^{(+)}(x,y)=2\sum_{s=1}^{n}\frac{\rho_s^2}{(x-z_s)^2}\,\frac{\tau^\dagger(y-z_s)}{(y-z_s)^2}\,, \tag{4.42}$$

and the relations

$$\tau^\dagger(x-z_s)\tau(x-z_s)=(x-z_s)^2, \tag{4.43}$$

$$\tau^\dagger(y-z_s)\tau(x-z_s)\tau^\dagger(y-z_s)=-(y-z_s)^2\tau^\dagger(x-z_s)+2(x-z_s)(y-z_s)\tau^\dagger(y-z_s). \tag{4.44}$$

Hence, on writing

$$(x-y)=(x-z_s)-(y-z_s), \tag{4.45}$$

we get

$$2\,\frac{\tau^\dagger(x-y)}{(x-y)^2}[F^{(+)}(x,y)-\Pi(x)]=\frac{2}{(x-y)^2}\sum_{s=1}^{n}\rho_s^2\left\{\frac{\tau^\dagger(y-z_s)}{(y-z_s)^2}+\frac{\tau^\dagger(x-z_s)}{(x-z_s)^2}-2(x-z_s)(y-z_s)\,\frac{\tau^\dagger(y-z_s)}{(x-z_s)^2(y-z_s)^2}-\frac{\tau^\dagger[(x-z_s)-(y-z_s)]}{(x-z_s)^2}\right\}$$
$$=2\sum_{s=1}^{n}\frac{\rho_s^2}{(x-z_s)^2}\,\frac{\tau^\dagger(y-z_s)}{(y-z_s)^2}\,, \tag{4.46}$$

which, in view of Eq. (4.42), proves that the structure (4.41) does indeed satisfy the first-order differential equation (4.39). The antipseudoparticle solution is given by

$$F^{(-)}(x,y)=1+\sum_{s=1}^{n}\rho_s^2\,\frac{\tau^\dagger(x-z_s)}{(x-z_s)^2}\,\frac{\tau(y-z_s)}{(y-z_s)^2}\,. \tag{4.47}$$

Equations (4.33), (4.41), and (4.47) provide a simple algebraic expression for the massless, spin-0, isospin-$\frac{1}{2}$ propagators in the field of an arbitrary configuration of n pseudoparticles or antipseudoparticles.

We have calculated the propagators $\bar{\Delta}^{(\pm)}(x,y)$ in a singular vector potential $\bar{A}_\mu^{(\pm)}(x)$ [Eq. (4.14)]. A regular vector potential $\tilde{A}_\mu^{(\pm)}(x)$ can be obtained by the gauge transformation $U^{(\pm)}(x)$ [Eq. (4.24)]. The propagator $\tilde{\Delta}^{(\pm)}(x,y)$ in this (regular) vector potentail is given by a gauge rotation of $\bar{\Delta}^{(\pm)}(x,y)$:

$$\tilde{\Delta}^{(\pm)}(x,y)=U^{(\pm)}(x)^{-1}\bar{\Delta}^{(\mp)}(x,y)U^{(\pm)}(y). \tag{4.48}$$

For a single pseudoparticle (or antipseudoparticle) it is a simple matter to perform this transformation explicitly. Using Eq. (4.7), we may write

$$F^{(\pm)}(x,y)=1+\frac{\rho^2}{(x^2y^2)^{1/2}}\Omega(x)^{\pm1}\Omega(y)^{\mp1}, \tag{4.49}$$

where, for simplicity, we have located the pseudoparticle at the origin, $z_1=0$. Thus, using Eq. (4.26a), we get

$$\tilde{\Delta}^{(\pm)}(x,y)=\frac{\Omega(x)^{\mp1}\Omega(y)^{\pm1}+\rho^2(x^2y^2)^{-1/2}}{\Pi^{1/2}(x)\Pi^{1/2}(y)4\pi^2(x-y)^2}\,, \tag{4.50}$$

and, employing Eqs. (4.6),

$$\tilde{\Delta}^{(\pm)}(x,y)=\frac{\rho^2+xy+i\eta^{(\pm)}_{\mu\nu a}x_\mu y_\nu\tau_a}{(\rho^2+x^2)^{1/2}(\rho^2+y^2)^{1/2}4\pi^2(x-y)^2}\,. \tag{4.51}$$

This solution (4.51) was obtained previously[15] by exploiting the invariance of the single pseudoparticle field under an O(5) group of conformal coordinate transformations.

V. ISOSPIN-1 SCALAR PROPAGATORS

We turn now to the construction of the massless, spin-0, isospin-1 propagation function $\Delta_{ab}(x,y)$ in a general pseudoparticle field. This propagator obeys the Green's function equation

$$-D^2{}_{ac}\Delta_{cb}(x,y)=\delta_{ab}\delta(x-y), \tag{5.1}$$

where

$$D_{\mu ab}=\partial_\mu\delta_{ab}+\epsilon_{acb}A_{\mu c}(x). \tag{5.2}$$

In order to simplify the notation, we shall work out explicitly only the propagation function for the n-pseudoparticle field in the singular gauge, $\bar{A}_{\mu a}^{(+)}(x)$, and delete the various superscripts that refer to this field.

To proceed with the construction, we note that the isospin-$\frac{1}{2}$ propagator has the form

$$\Delta(x,y)=\frac{M(x,y)}{4\pi^2(x-y)^2}\,, \tag{5.3}$$

with

$$M(x,y)=\Pi^{-1/2}(x)F(x,y)\Pi^{-1/2}(y), \tag{5.4}$$

and $F(x,y)=F^{(\ )}(x,y)$ [Eq. (4.41)]. The matrix $M(x,y)$ is Hermitian in the sense that

$$M(x,y)^\dagger=M(y,x). \tag{5.5}$$

By inserting the form (5.3) into the Green's function equation (4.28) for the isospin-$\frac{1}{2}$ propagator, we find that

$$D^2M(x,y)=4\frac{(x-y)_\mu}{(x-y)^2}D_\mu M(x,y)\,, \tag{5.6}$$

where here the gauge-covariant derivative refers to isospin $\frac{1}{2}$, the derivative displayed in Eq. (4.29). The symmetry (5.5) implies that we also have

$$M(y,x)D^2=-4M(y,x)\overleftarrow{D}_\mu\frac{(x-y)_\mu}{(x-y)^2}\,, \tag{5.7}$$

where it should be remembered that the derivative $\overleftarrow{D}_\mu$ acting to the left involves $-\overleftarrow{\partial}_\mu$.

We combine two isospin-$\frac{1}{2}$ matrices $M(x,y)$ to form an isospin-1 matrix

$$W_{ab}(x,y)=\tfrac{1}{2}\operatorname{tr}\tau_aM(x,y)\tau_bM(y,x) \tag{5.8}$$

and express the isospin-1 propagator in terms of another matrix $C_{ab}(x,y)$:

$$\Delta_{ab}(x,y)=\frac{W_{ab}(x,y)}{4\pi^2(x,y)^2}+\frac{C_{ab}(x,y)}{4\pi^2\Pi(x)\Pi(y)}\,. \tag{5.9}$$

Since

$$W_{ab}(x,x)=\delta_{ab}, \tag{5.10}$$

the decomposition (5.9) will produce the inhomogeneous term $\delta_{ab}\delta(x-y)$ in the Green's function equation if $C_{ab}(x,y)$ is regular when $x\to y$. We insert the decomposition (5.9) into the Green's function equation (5.1); note that the isospin-1 covariant derivatives acting on $W_{ab}(x,y)$ are converted to isospin-$\frac{1}{2}$ covariant derivatives acting on $M(x,y)$ by the commutation relation

$$[\tau_a,\tau_b]=2i\epsilon_{abc}\tau_c, \tag{5.11}$$

and use Eqs. (5.6) and (5.7) to secure

$$D^2{}_{ac}\frac{C_{cb}(x,y)}{\Pi(x)\Pi(y)}=\frac{1}{(x-y)^2}\operatorname{tr}\tau_aD_\mu M(x,y)\tau_bM(y,x)\overleftarrow{D}_\mu\,. \tag{5.12}$$

The covariant derivatives on the left-hand side of this equation refer to isospin 1 [Eq. (5.2)], while those on the right-hand side refer to isospin $\frac{1}{2}$ [Eq. (4.29)]. Specializing to the particular pseudoparticle vector potential

$$A_{\mu a}(x)=-\eta^{(-)}_{\mu\nu a}\,\partial_\nu\ln\Pi(x), \tag{5.13}$$

writing $M(x,y)$ in terms of $F(x,y)$ [Eq. (5.4)], and commuting factors of $\Pi(x)$ with covariant derivatives, we obtain an explicit differential equation for the unknown function $C_{ab}(x,y)$:

$$\left\{\partial^2\delta_{ac}-2\left[\eta^{(-)}_{\mu\nu d}\epsilon_{adc}\frac{\partial_\nu\Pi(x)}{\Pi(x)}+\delta_{ac}\frac{\partial_\mu\Pi(x)}{\Pi(x)}\right]\partial_\mu\right\}C_{cb}(x,y)=K_{ab}(x,y)\,, \tag{5.14}$$

in which

$$K_{ab}(x,y)=\frac{1}{(x-y)^2}\operatorname{tr}\tau_a\left[\partial_\mu-\frac{1}{2}\frac{\tau\partial\Pi(x)}{\Pi(x)}\tau^\dagger_\mu\right]F(x,y)\tau_bF(y,x)\left[\overleftarrow{\partial}_\mu-\tau_\mu\frac{1}{2}\frac{\tau^\dagger\partial\Pi(x)}{\Pi(x)}\right]. \tag{5.15}$$

Here we have also used Eqs. (4.6) to introduce the matrices τ_μ and $\tau^\dagger_\mu$. We shall often make use of Eqs. (4.6) in the algebraic reductions that follow.

The quantity $K_{ab}(x,y)$ can be expressed in terms of fairly simple, explicit formulas. First, we note that since

$$\tau_\mu\tau^\dagger\partial\Pi\,\tau_a\tau\partial\Pi\,\tau^\dagger_\mu=0\,, \tag{5.16}$$

we can write

$$K_{ab}(x,y)=K^{(1)}_{ab}(x,y)+K^{(2)}_{ab}(x,y)\,, \tag{5.17}$$

with

$$K^{(1)}_{ab}(x,y)=\frac{1}{(x-y)^2}\operatorname{tr}\tau_a[\partial_\mu F(x,y)-\tfrac{1}{2}\tau\partial\Pi(x)\tau^\dagger_\mu]\tau_b[F(y,x)\overleftarrow{\partial}_\mu-\tfrac{1}{2}\tau_\mu\tau^\dagger\partial\Pi(x)] \tag{5.18}$$

and

$$K^{(2)}_{ab}(x,y)=\frac{-1}{2(x-y)^2}\frac{1}{\Pi(x)}\operatorname{tr}\{\tau_a\tau\partial\Pi(x)\tau^\dagger_\mu[F(x,y)-\Pi(x)]\tau_b[F(y,x)\overleftarrow{\partial}_\mu-\tfrac{1}{2}\tau_\mu\tau^\dagger\partial\Pi(x)]$$
$$+\tau_a[\partial_\mu F(x,y)-\tfrac{1}{2}\tau\partial\Pi(x)\tau^\dagger_\mu]\tau_b[F(y,x)-\Pi(x)]\tau_\mu\tau^\dagger\partial\Pi(x)\}\,. \tag{5.19}$$

In order to remove some of the notational clutter, we define

$$x_s = x - z_s\,, \quad y_s = y - z_s\,, \tag{5.20}$$

so that

$$\Pi(x) = 1 + \sum_s \frac{\rho_s^2}{x_s^2} \tag{5.21}$$

and

$$F(x,y) = 1 + \sum_s \rho_s^2 \frac{\tau x_s}{x_s^2}\,\frac{\tau^\dagger y_s}{y_s^2}\,. \tag{5.22}$$

Manipulations similar to those done before [cf. Eqs. (4.43)–(4.45)] give

$$\partial_\mu F(x,y) - \tfrac{1}{2}\tau\partial\Pi(x)\tau_\mu^\dagger = -\sum_s \frac{\rho_s^2}{(x_s^2)^2 y_s^2}\,\tau x_s \tau_\mu^\dagger \tau(x-y)\tau^\dagger y_s \tag{5.23a}$$

and

$$F(y,x)\overleftarrow{\partial}_\mu - \tfrac{1}{2}\tau_\mu\tau^\dagger\partial\Pi(x) = -\sum_t \frac{\rho_t^2}{(x_t^2)^2 y_t^2}\,\tau y_t \tau^\dagger(x-y)\tau_\mu\tau^\dagger x_t\,. \tag{5.23b}$$

Therefore, the quantity $K_{ab}^{(1)}(x,y)$ involves

$$\tau_\mu^\dagger\tau(x-y)\tau^\dagger y_s \tau_b \tau y_t \tau^\dagger(x-y)\tau_\mu = -4i\eta_{\mu\nu b}^{(-)} y_{s\mu} y_{t\nu}(x-y)^2\,, \tag{5.24}$$

and we get

$$K_{ab}^{(1)}(x,y) = 8\sum_{s,t} \frac{\rho_s^2}{(x_s^2)^2 y_s^2}\,\frac{\rho_t^2}{(x_t^2)^2 y_t^2} \times \eta_{\lambda\kappa a}^{(-)} x_{s\lambda} x_{t\kappa} \eta_{\mu\nu b}^{(-)} y_{s\mu} y_{t\nu}\,. \tag{5.25}$$

Using

$$F(x,y) - \Pi(x) = \sum_s \frac{\rho_s^2}{x_s^2 y_s^2}\,\tau(x-y)\tau^\dagger y_s\,, \tag{5.26a}$$

and

$$F(y,x) - \Pi(x) = \sum_t \frac{\rho_t^2}{x_t^2 y_t^2}\,\tau y_t \tau^\dagger(x-y)\,, \tag{5.26b}$$

together with Eqs. (5.23), we find that the quantity $K_{ab}^{(2)}(x,y)$ also involves the expression displayed in Eq. (5.24), and we get

$$K_{ab}^{(2)}(x,y) = \frac{-16}{\Pi(x)} \sum_{s,t,u} \frac{\rho_s^2}{x_s^2 y_s^2}\,\frac{\rho_t^2}{(x_t^2)^2 y_t^2}\,\frac{\rho_u^2}{(x_u^2)^2} \times \eta_{\lambda\kappa a}^{(-)} x_{u\lambda} x_{t\kappa} \eta_{\mu\nu b}^{(-)} y_{s\mu} y_{t\nu}\,. \tag{5.27}$$

The y coordinate enters into both $K_{ab}^{(1)}(x,y)$ and $K_{ab}^{(2)}(x,y)$ only through the functions

$$\Phi_{stb}^{(+)}(y) = \frac{\rho_s\rho_t}{y_s^2 y_t^2}\,\eta_{\mu\nu b}^{(-)} y_{s\mu} y_{t\nu}\,, \tag{5.28}$$

and in terms of these functions

$$K_{ab}^{(1)}(x,y) = 8\sum_{s,t} \frac{1}{x_s^2 x_t^2}\,\Phi_{sta}^{(+)}(x)\Phi_{stb}^{(+)}(y)\,, \tag{5.29}$$

$$K_{ab}^{(2)}(x,y) = -\frac{16}{\Pi(x)} \sum_{s,t,u} \frac{\rho_s\rho_u}{x_s^2 x_t^2 x_u^2}\,\Phi_{uta}^{(+)}(x)\Phi_{stb}^{(+)}(y)\,. \tag{5.30}$$

The propagation function is symmetrical under the interchange of its coordinates and isospin indices and this symmetry must be shared by the function $C_{ab}(x,y)$,

$$C_{ab}(x,y) = C_{ba}(y,x)\,. \tag{5.31}$$

The differential equation (5.14) for $C_{ab}(x,y)$ has a driving term $K_{ab}(x,y)$ which depends upon the y coordinate only through functions $\Phi_{stb}^{(+)}(y)$ which are the antisymmetrical in their indices s and t. Hence, by virtue of the symmetry (5.31) of $C_{ab}(x,y)$ it can only involve these functions,

$$C_{ab}(x,y) = \sum_{r,s,t,u} \Phi_{rsa}^{(+)}(x)\,c_{rs,tu}\,\Phi_{tub}^{(+)}(y)\,, \tag{5.32}$$

with coefficients $c_{rs,tu}$ that do not involve the coordinates x or y and have the symmetries

$$c_{rs,tu} = c_{tu,rs}\,, \tag{5.33}$$

$$c_{rs,tu} = -c_{sr,tu} = -c_{rs,ut}\,. \tag{5.34}$$

After a little calculation, we find that

$$\left\{\partial^2\delta_{ac} - 2\left[\eta_{\mu\nu d}^{(-)}\epsilon_{adc}\frac{\partial_\nu\Pi(x)}{\Pi(x)} + \frac{\partial_\mu\Pi(x)}{\Pi(x)}\,\delta_{ac}\right]\partial_\mu\right\}\Phi_{rsc}^{(+)}(x)$$
$$= -4\,\frac{(z_r - z_s)^2}{x_r^2 x_s^2}\,\Phi_{rsa}^{(+)}(x) + \frac{4}{\Pi(x)}\sum_t \frac{1}{x_r^2 x_s^2 x_t^2}\left\{\rho_t\rho_r[(z_r - z_s)^2 - x_r^2]\Phi_{tsa}^{(+)}(x) + \rho_s\rho_t[(z_r - z_s)^2 - x_s^2]\Phi_{rta}^{(+)}(x)\right\}. \tag{5.35}$$

We now insert the expansion (5.32) of $C_{ab}(x,y)$ in terms of the Φ functions into the differential equation (5.14) obeyed by $C_{ab}(x,y)$, use the explicit forms Eqs. (5.17), (5.29), and (5.30) for the driving term $K_{ab}(x,y)$ in this differential equation, and identify the coefficients of $\Phi_{rsa}^{(+)}(x)\Phi_{uvb}^{(+)}(y)$ to obtain a matrix equation for the constants $c_{rs,uv}$:

$$\Pi(x)(z_r - z_s)^2 c_{rs,uv} - \sum_p\left\{\frac{\rho_r\rho_p}{x_p^2}\,[(z_p - z_s)^2 - x_p^2]c_{ps,uv} + \frac{\rho_s\rho_p}{x_p^2}\,[(z_r - z_p)^2 - x_p^2]c_{rp,uv}\right\}$$
$$= \Pi(x)(\delta_{ru}\delta_{sv} - \delta_{rv}\delta_{su}) - \left(\frac{\rho_r\rho_u}{x_u^2}\,\delta_{sv} - \frac{\rho_s\rho_u}{x_u^2}\,\delta_{rv} - \frac{\rho_r\rho_v}{x_v^2}\,\delta_{su} + \frac{\rho_s\rho_v}{x_v^2}\,\delta_{ru}\right). \tag{5.36}$$

Although this equation involves the coordinate x explicitly, it is solved by constant coefficients $c_{rs,uv}$. The solution is of the form

$$c_{rs,uv}=\frac{\delta_{ru}\delta_{sv}-\delta_{rv}\delta_{su}}{(z_r-z_s)^2} -\frac{1}{(z_r-z_s)^2}(\rho_r\rho_u f_{sv}-\rho_r\rho_v f_{su}-\rho_s\rho_u f_{rv}-+\rho_s\rho_v f_{rv})\frac{1}{(z_u-z_v)^2}\,. \tag{5.37}$$

On substituting this structure into Eq. (5.36), we find that all the x-dependent terms cancel independent of the values of the constants f_{sv} and that these constants are determined by

$$\sum_t g_{st}f_{tv}=\delta_{sv}\,, \tag{5.38}$$

in which

$$g_{st}=\left[1+\sum_{r\neq s}\frac{\rho_r^{\,2}}{(z_r-z_s)^2}\right]\delta_{st} -\frac{\rho_s\rho_t}{(z_s-z_t)^2}(1-\delta_{st}). \tag{5.39}$$

Thus, the constants f_{st} are the elements of the matrix which is the inverse of the matrix g_{st}. Since the latter is symmetrical, so is f_{st}. This symmetry ensures that the structure (5.37) obeys the necessary symmetry conditions on $c_{rs,uv}$ given in Eqs. (5.33) and (5.34).

We have now completed our construction of the isospin-1 propagation function. It is given by Eq. (5.9) with the functions $C_{ab}(x,y)$ determined by Eqs. (5.32), (5.28), and (5.37)–(5.39). Our development has been for the case of an n-pseudoparticle field, but the result for n antipseudoparticles is obvious: One need only replace the function $F=F^{(+)}$ by $F^{(-)}$ and the functions $\Phi^{(+)}$ by $\Phi^{(-)}$, where $\Phi^{(-)}$ is constructed with $\eta^{(+)}$ symbols rather than with the $\eta^{(-)}$ symbols that appear in $\Phi^{(+)}$ [Eq. (5.28)].

The result for a single pseudoparticle (or antipseudoparticle) field is particularly simple. In this case the function $C_{ab}(x,y)$ vanishes and the isospin-1 propagation function is determined entirely in terms of the isospin-0 function. Our result agrees, when properly gauge transformed, with that obtained from an O(5) group theory construction.[15] The result for two pseudoparticles is also quite simple. In this case the function $C_{ab}(x,y)$ no longer vanishes but is determined by the single constant

$$c_{12,12}=[(z_1-z_2)^2+\rho_1^{\,2}+\rho_2^{\,2}]^{-1}. \tag{5.40}$$

VI. DISCUSSION

We have shown that the propagation functions for massless spin-$\frac{1}{2}$ and spin-1 particles moving in a self-dual but otherwise arbitrary non-Abelian gauge field are determined by the corresponding spin-0 propagators. The spin-0 propagators were constructed for particles of isospin $\frac{1}{2}$ and 1 moving in the self-dual field of n pseudoparticles. These scalar propagators have a remarkably simple algebraic form. This simplicity suggests the existence of some deeper underlying principles which, however, we have not yet been able to fathom.

The field of a single pseudoparticle is invariant under an O(5) group of conformal coordinate transformations.[16] The corresponding massless, spin-0 propagators which we have constructed for isospin-$\frac{1}{2}$ and isospin-1 particles are covariant under this O(5) group and, in fact, coincide with the functions obtained directly from an O(5) group theory analysis.[15] The situation is different for massless particles with spin $\frac{1}{2}$ or 1. The propagation functions for these particles do not transform covariantly under the O(5) group. This lack of covariance results because the propagators obey Green's function equations [Eqs. (2.19) and (2.20) and Eqs. (3.14) and (3.15)] from which the zero-mode components of the inhomogeneous δ function have been subtracted. The products of zero-mode functions which thus appear on the right-hand side of these equations have the same conformal weight as does the propagator (-3 for spin $\frac{1}{2}$ and -2 for spin 1). This weight differs from the conformal weight of the δ function (-4). Hence the inhomogeneous, driving terms in the spin-$\frac{1}{2}$ and spin-1 Green's function equations do not have a single, pure conformal weight. These equations—and hence the propagators they define—are thus not covariant under the transformations of the O(5) conformal group. Accordingly, the propagators that we have constructed for spin $\frac{1}{2}$ and spin 1 do not agree with those obtained from an O(5) conformal group analysis.[16]

The Green's function equations [Eqs. (2.19) and (2.20) for spin $\frac{1}{2}$ and Eqs. (3.14) and (3.15) for spin 1] dictate the asymptotic behavior of the propagators S and $G_{\mu\nu}$. For large x, the dominant zero-mode functions in Eqs. (2.20) and (3.15) fall as x^{-3}. The term $\delta(x-y)$ in Eqs. (2.19) and (3.15) is irrelevant in this limit, and S and $G_{\mu\nu}$ must therefore fall as x^{-2} and x^{-1}, respectively. This behavior should be contrasted with the large-x behavior of the free propagators for spin-$\frac{1}{2}$ and spin-

1 particles, namely x^{-3} and x^{-2}, respectively. This feature of our results is gauge independent and may be of considerable importance for physical processes occurring in pseudoparticle fields.

Note that this large-distance behavior conflicts with that derived from an O(5) conformal group analysis. The O(5) results are physically relevant only if they are supplemented by a corresponding O(5)-covariant treatment of the various collective coordinates which enter any pseudoparticle calculation. An O(5)-covariant approach to the collective coordinates has not been formulated and may in fact be an impossibility. It may also be true that such a formulation is possible and that it resolves the discrepancies between O(5)-covariant propagators and those derived in the present paper.

There are, as noted in Sec. III, some problems with the spin-1 propagation function that we have constructed. Now that we have found an explicit expression for the scalar, isospin-1 propagator,

$$\Delta_{ab}(x, y) = \frac{W_{ab}(x, y)}{4\pi^2(x-y)^2} + \frac{C_{ab}(x, y)}{4\pi^2\Pi(x)\Pi(y)}, \tag{5.9}$$

we can discuss these problems more fully. The matrix $C_{ab}(x, y)$ is a bilinear combination [Eq. (5.32)] of functions $\Phi^{(+)}_{rsa}(x)\,\Phi^{(+)}_{tub}(y)$ [Eq. (5.28)] which vanish sufficiently rapidly at infinity so as to be square integrable:

$$\Phi^{(+)}_{rsa}(x) \underset{x^2\to\infty}{\sim} O(x^{-3}). \tag{6.1}$$

The matrix $W_{ab}(x, y)$, on the other hand, does not vanish when one of its coordinates becomes large:

$$W_{ab}(x, y) \underset{x^2\to\infty}{\longrightarrow} \delta_{ab}\Pi(y)^{-1} \tag{6.2}$$

[see Eqs. (5.8) and (5.4)]. Thus, in the formal operator construction of the spin-1 propagator

$$G^{(\pm)}_{\mu\nu} = -\left\{\left(\frac{1}{D^2}\right)^2\right\}^{(\pm)}_{\mu\nu} + (1-\xi)D_\mu\left(\frac{1}{D^2}\right)^2 D_\nu, \tag{3.27}$$

the convolution integral which defines $(1/D^2)^2$ is divergent. In a coordinate representation we have

$$\left\langle x\left|\left(\frac{1}{D^2}\right)^2_{ab}\right|y\right\rangle = \int (d_E{}^4z)\,\Delta_{ac}(x, z)\,\Delta_{cb}(z, y), \tag{6.3}$$

and the z integration diverges logarithmically at large z, giving a contribution

$$\left\langle x\left|\left(\frac{1}{D^2}\right)^2_{ab}\right|y\right\rangle_{\rm div} = \frac{\ln\infty}{8\pi^2}\,\delta_{ab}\Pi(x)^{-1}\Pi(y)^{-1}. \tag{6.4}$$

The bracket operation appearing in the formal operator expression for the propagator is defined by

$$\{X\}^{(\pm)}_{\mu\nu} = q^{(\pm)}_{\mu\nu\lambda\kappa}D_\lambda X D_\kappa, \tag{3.18}$$

where, according to Eqs. (3.17) and (3.16),

$$q^{(\pm)}_{\mu\nu\lambda\kappa} = \delta_{\mu\nu}\delta_{\lambda\kappa} + \delta_{\mu\lambda}\delta_{\nu\kappa} - \delta_{\mu\kappa}\delta_{\nu\lambda} \pm \epsilon_{\mu\nu\lambda\kappa}. \tag{6.5}$$

The character of the divergent piece in the spin-1 propagator is revealed when we rewrite $q^{(\pm)}_{\mu\nu\lambda\kappa}$ in terms of an appropriate combination of the symbols $\eta^{(\mp)}$. To this end, we recall that these symbols project out the self-dual (or anti-self-dual) part of an antisymmetrical tensor [Eq. (4.17)], giving

$$\eta^{(\pm)}_{\alpha\beta a}\eta^{(\pm)}_{\gamma\delta a} = \delta_{\alpha\gamma}\delta_{\beta\delta} - \delta_{\alpha\delta}\delta_{\beta\gamma} \pm \epsilon_{\alpha\beta\gamma\delta}. \tag{6.6}$$

Thus we have

$$q^{(\pm)}_{\mu\nu\lambda\kappa} = \delta_{\mu\lambda}\delta_{\nu\kappa} + \eta^{(\mp)}_{\mu\lambda a}\eta^{(\mp)}_{\nu\kappa a}, \tag{6.7}$$

and we can write the divergent piece of the spin-1 propagator as

$$G^{(\pm)}_{\mu\nu ab}(x, y)_{\rm div} = -[\eta^{\mp}_{\mu\lambda d}D_{\lambda ac}\Pi(x)^{-1}\eta^{(\mp)}_{\nu\kappa d}\Pi(y)^{-1}\overleftarrow{D}_{\kappa cb} + \xi D_{\mu ac}\Pi(x)^{-1}\Pi(y)^{-1}\overleftarrow{D}_{\nu cb}]\,\frac{\ln\infty}{8\pi^2}. \tag{6.8}$$

The functions

$$\phi^{(\pm)c}_{\mu a}(x) = D_{\mu ac}\Pi(x)^{-1} = -\Pi(x)^{-2}[\partial_\mu\Pi(x)\delta_{ac} + \eta^{(\mp)}_{\mu\nu d}\partial_\nu\Pi(x)\epsilon_{adc}] \tag{6.9}$$

are three ($c = 1, 2, 3$) zero-mode functions of the small-fluctuation, vector field. They correspond to an overall, global gauge rotation of the small-fluctuation vector field.[17] In general,[12] a zero-mode function $\phi^{(\pm)}_{\mu a}(x)$ obeys the background gauge constraint

$$D_{\mu ab}\phi^{(\pm)}_{\mu b} = 0 \tag{6.10}$$

and produces a small-fluctuation field-strength tensor

$$f^{(\pm)}_{\mu\nu a}(x) = D_{\mu ab}\phi^{(\pm)}_{\nu b}(x) - D_{\nu ab}\phi^{(\pm)}_{\mu b}(x), \tag{6.11}$$

which is self-dual (or anti-self-dual), i.e.,

$$\eta^{(\mp)}_{\mu\nu c}D_{\mu ab}\phi^{(\pm)}_{\nu b}(x) = 0, \tag{6.12}$$

for $c = 1, 2, 3$. It is a simple matter to show that the gauge-rotation functions (6.9) do obey Eqs. (6.10) and (6.12) and thus prove that they are indeed zero-mode functions. Since the η symbols obey the Pauli-matrix algebra

$$\eta^{(\pm)}_{\mu\lambda a}\eta^{(\pm)}_{\nu\lambda b} = \delta_{\mu\nu}\delta_{ab} + \epsilon_{abc}\eta^{(\pm)}_{\mu\nu c}, \tag{4.19}$$

we see that if $\phi^{(\pm)}_{\nu a}(x)$ is a zero-mode solution of Eqs. (6.10) and (6.12), then so are the three functions $\overline{\phi}^{(\pm)d}_{\mu a}(x)$ ($d = 1, 2, 3$) defined by

$$\overline{\phi}^{(\pm)d}_{\mu a}(x) = \eta^{(\mp)}_{\mu\nu d}\,\phi^{(\pm)}_{\nu a}. \tag{6.13}$$

Thus the other functions

$$\eta^{(\mp)}_{\mu\lambda d} D_{\lambda ac}\, \Pi(x)^{-1} = \eta^{(\mp)}_{\mu\lambda d}\phi^{(\pm)c}_{\lambda a}(x) \tag{6.14}$$

appearing in the divergent piece (6.8) of the propagator are also zero-mode functions. Now, using the explicit form for the gauge-rotation mode displayed in Eq. (6.9), we find that

$$\eta^{(\mp)}_{\mu\lambda d}\,\phi^{(\pm)c}_{\lambda a}(x) = -\epsilon_{dce}\,\phi^{(\pm)e}_{\mu a}(x) + \delta_{dc}\,\psi^{(\pm)}_{\mu a}(x)\,, \tag{6.15}$$

where

$$\psi^{(\pm)}_{\mu a}(x) = -\Pi(x)^{-2}\eta^{(\mp)}_{\mu\nu a}\partial_\nu \Pi(x)\,. \tag{6.16}$$

Thus there is a fourth zero-mode function $\psi^{(\pm)}_{\mu a}(x)$ attached to the three gauge-rotation, zero-mode functions $\phi^{(\pm)}_{\mu a}(x)$. Recalling that

$$A^{(\pm)}_{\mu a}(x) = -\eta^{(\mp)}_{\mu\nu a}\partial_\nu \ln\Pi(x)\,, \tag{4.14}$$

with

$$\Pi(x) = 1 + \sum_{s=1}^{n} \frac{\rho_s{}^2}{(x-z_s)^2}\,, \tag{4.21}$$

we see that

$$\psi^{(\pm)}_{\mu a}(x) = \sum_{s=1}^{n} \rho_s{}^2 \frac{\partial}{\partial \rho_s{}^2} A^{(\pm)}_{\mu a}(x)\,, \tag{6.17}$$

which identifies $\psi^{(\pm)}_{\mu a}(x)$ as a zero mode corresponding to an overall scale change.[17]

Collecting these results, we can rewrite the divergent piece (6.8) of the propagator as

$$G^{(\pm)}_{\mu\nu ab}(x,y)_{\text{div}} = [3\psi^{(\pm)}_{\mu a}(x)\psi^{(\pm)}_{\nu b}(y) + (2+\xi)\,\phi^{(\pm)c}_{\mu a}(x)\,\phi^{(\pm)c}_{\nu b}(y)]\,\frac{\ln\infty}{8\pi^2}\,. \tag{6.18}$$

Since zero-mode functions can be subtracted from the vector propagator without any change in the Green's function equation, this propagator can be rendered finite by simply deleting its infinite terms. There is, however, no unique way in which to perform this subtraction. The propagator behaves asymptotically as $O(x^{-1})$ as dictated by the asymptotic behavior[11] $[O(x^{-3})]$ of the product of vector zero-mode functions which appear on the right-hand side of the Green's function equation. Thus, one cannot impose the constraint that the propagator be orthogonal to the zero-mode functions; the inner product integral diverges logarithmically. We believe that this ambiguity may be resolved by an appropriate redefinition of the collective coordinates associated with the zero modes, but of this we are not yet sure.

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APPENDIX

The construction of the spin-1 propagator in Sec. III can be motivated as follows. We decompose the propagator into a sum of various mode functions $\chi_{\mu a}(x)$. We set the gauge-fixing parameter $\xi = 1$ so that, in view of Eq. (3.14), the mode functions obey

$$-[D^2\delta_{\mu\lambda} + 2F_{\mu\lambda}]\chi_\lambda = \kappa^2\chi_\mu\,. \tag{A1}$$

Making use of the commutator of the gauge-covariant derivatives, Eq. (3.6), we find that

$$\chi^0_\mu = D_\mu\phi \tag{A2}$$

satisfies the vector mode equation (A1) if ϕ is a scalar mode function,

$$-D^2\phi = \kappa^2\phi\,. \tag{A3}$$

As we have remarked in the text, the $\eta^{(\pm)}$ symbols defined in Eq. (4.2) obey the algebra of the SU(2) generators $i\tau_a$,

$$\eta^{(\pm)}_{\mu\lambda a}\eta^{(\pm)}_{\nu\lambda b} = \delta_{\mu\nu}\delta_{ab} + \epsilon_{abc}\eta^{(\pm)}_{\mu\nu c}\,. \tag{4.19}$$

Moreover, the two $\eta^{(\pm)}$ symbols correspond to the two SU(2) spins in the decomposition O(4) = SU(2) ⊗ SU(2), and they commute,

$$\eta^{(\pm)}_{\mu\lambda a}\eta^{(\mp)}_{\lambda\nu b} = \eta^{(\mp)}_{\mu\lambda b}\eta^{(\pm)}_{\lambda\nu a}\,. \tag{A4}$$

Now if the field-strength tensor $F_{\mu\nu}$ is self-dual (or anti-self-dual), we can write [cf. Eq. (4.17)]

$$F^{(\pm)}_{\mu\nu} = \eta^{(\pm)}_{\mu\nu a}F_a \tag{A5}$$

and thus conclude that $\eta^{(\mp)}$ commutes with $F^{(\pm)}$,

$$F^{(\pm)}_{\mu\lambda}\eta^{(\mp)}_{\lambda\nu a} = \eta^{(\mp)}_{\mu\lambda a}F^{(\pm)}_{\lambda\nu}\,. \tag{A6}$$

In this case, $\eta^{(\mp)}$ commutes with the operator on the left-hand side of the vector mode equation (A1), so that if χ_μ is a solution of this equation, then so also are the three functions $\eta^{(\mp)}_{\mu\nu a}\chi_\nu$. In particular, starting from the scalar mode function ϕ, we can construct three more vector mode functions

$$\chi^a_\mu = \eta^{(\mp)}_{\mu\nu a}D_\nu\phi\,. \tag{A7}$$

It follows from Eq. (4.19) that all four vector mode functions χ^a_μ, $a = 0, 1, 2, 3$, are orthogonal to one another and from Eq. (A3) that their normalization differs from that of the scalar mode function ϕ by a factor of the eigenvalue κ^2,

$$\int (d_E{}^4x)\,\chi^a_\mu\chi^b_\mu = \delta_{ab}\kappa^2\int (d_E{}^4x)\phi^2\,. \tag{A8}$$

Thus the set of all χ_μ^a is a complete set of orthogonal vector mode functions, and with the scalar mode functions normalized to unity, the vector propagator can be written as

$$G_{\mu\nu}^{(\pm)}(x,y)=\sum_\chi \frac{\chi_\mu^a(x)\chi_\nu^a(y)}{\kappa^4}$$
$$=-(\delta_{\mu\lambda}\delta_{\nu\kappa}+\eta_{\mu\lambda a}^{(\mp)}\eta_{\nu\kappa a}^{(\mp)})D_\lambda\sum_\phi\frac{\varphi(x)\phi(y)}{\kappa^4}\overleftarrow{D}_\kappa\,. \tag{A9}$$

The tensor quantity in parentheses here is precisely the quantity $q_{\mu\nu\lambda\kappa}^{(\pm)}$ [cf. Eq. (6.7)]. The scalar mode sum in Eq. (A9) represents the operator $(1/D^2)^2$, and Eqs. (A9) and (3.27) are thus equivalent.

Finally, let us prove, as was asserted in Sec. III, that

$$q_{\mu\sigma\alpha\beta}^{(\pm)}q_{\sigma\nu\lambda\kappa}^{(\pm)}=\delta_{\beta\lambda}q_{\mu\nu\alpha\kappa}^{(\pm)}+r_{\mu\nu\alpha\kappa\beta\lambda}^{(\pm)}\,, \tag{3.23}$$

where $r_{\mu\nu\alpha\kappa\beta\lambda}^{(\pm)}$ is antisymmetrical in its last index pair with a reversed duality character,

$$\tfrac{1}{2}r_{\mu\nu\alpha\kappa\rho\sigma}^{(\pm)}\epsilon_{\rho\sigma\beta\lambda}=\mp r_{\mu\nu\alpha\kappa\beta\lambda}^{(\pm)}\,. \tag{3.24}$$

The proof is brief if the form (6.7) is employed, for the substitution of Eq. (6.7) in Eq. (3.23) and the use of Eq. (4.19) give

$$r_{\mu\nu\alpha\kappa\beta\lambda}^{(\pm)}=(\delta_{\mu\alpha}\eta_{\nu\kappa c}^{(\mp)}-\delta_{\nu\kappa}\eta_{\mu\alpha c}^{(\mp)}+\eta_{\mu\alpha a}^{(\mp)}\eta_{\nu\kappa b}^{(\mp)}\epsilon_{abc})\eta_{\beta\lambda c}^{(\mp)}\,. \tag{A10}$$

which is manifestly antisymmetric in $\beta\lambda$ with the opposite duality character (3.24).

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[17]These results differ slightly from the speculation offered in our letter, Ref. 7.