Coulomb gauge description of large Yang-Mills fields

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Recent observations on ambiguities of the Coulomb gauge in Yang-Mills theories are clarified. We point out that discontinuities of the transverse potentials are necessary in order to accommodate arbitrary Pontryagin index. A general argument is given along formal lines, and a numerical example is presented for an O(3)-symmetric gauge-field configuration. The one-pseudoparticle solution is transformed into the Coulomb gauge, and the mechanism for the emergence of an angle characterizing the vacuum is indicated.

I. INTRODUCTION

Attention has recently been drawn to the interesting circumstance that the Coulomb gauge description of Yang-Mills fields is ambiguous when the magnitude of the fields becomes large: Namely any one of several transverse and gauge-equivalent potentials A_a^μ represents the same physical field configuration.[1] A signal for this multivaluedness is the appearance of zero eigenvalues in the spectrum of the inverse ghost propagator for definite values of the potentials; consequently the Coulomb interaction Hamiltonian is not defined for such A_a^μ.

The purpose of this communication is to clarify the types of ambiguities that occur in the Coulomb gauge. We shall consider an SU(2) gauge theory and restrict attention to field configurations that have explicit O(3) symmetry. We shall then show that ambiguities of the gauge potential are a necessity if one wants to accommodate field configurations with nonzero Pontryagin index. Moreover, the time evolution of the potential must be discontinuous, involving transitions between gauge equivalent A_a^μ. The multiple vacuums[2] of the quantum theory emerge then from a requirement on the values that the wave functional takes for gauge-equivalent transverse potentials.

The analysis will proceed in Sec. II along a formal line, making use only of general properties of the theory. In Sec. III we shall then illustrate the results with an example: We transform from the $A_a^0=0$ gauge to the Coulomb gauge an especially simple gauge-field configuration with arbitrary Pontryagin index. Finally the original solution of Belavin, Polyakov, Schwartz, and Tyupkin[3] is presented in the Coulomb gauge, by numerical integration of the appropriate equations.

Our considerations are easily generalized to groups other than SU(2). The restriction to O(3) symmetry is motivated by the desire to render the relevant equations tractable. We do not expect, however, that our results are a consequence of the special geometry. It would be extremely pathological if the Coulomb gauge description of spherically symmetric, gauge-invariant field configurations necessitated nonsymmetric gauge potentials.

II. CLASSICAL VACUUM AND PONTRYAGIN INDEX

A potential with explicit O(3) symmetry has the following form (we use an anti-Hermitian matrix representation in the space of infinitesimal group generators, with the coupling constant scaled out; σ_i are Pauli matrices):

$$A_i = i\hat{r}_i\sigma_j\hat{r}_j\,\frac{a}{r} + i\epsilon_{ijk}\sigma_j\hat{r}_k\,\frac{b}{r} + i(\sigma_i - \hat{r}_i\sigma_j\hat{r}_j)\,\frac{c}{r}\,, \qquad A_0 = i\sigma_i\hat{r}_i\,\frac{d}{r}\,. \tag{2.1}$$

The functions a, b, c, and d depend on r and t; it will be frequently convenient to use $s=\ln r$ as the independent variable. The Coulomb gauge transversality condition is

$$\vec{\nabla}\cdot\vec{A}=0\,, \tag{2.2}$$

which implies

$$\dot{a}+a=2c\,. \tag{2.3}$$

(The overdot will indicate differentiation with respect to s.)

Let us consider the effect of a radially symmetric gauge transformation

$$U=e^{i\alpha\vec{\sigma}\cdot\hat{r}}\,, \tag{2.4}$$

$$\begin{aligned} &A_\mu \to \tilde{A}_\mu = U^{-1}A_\mu U + U^{-1}\partial_\mu U\,,\\ &a\to\tilde{a}=a+\dot{\alpha}\,,\\ &b-\tfrac{1}{2}\to\tilde{b}-\tfrac{1}{2}=(b-\tfrac{1}{2})\cos2\alpha + c\sin2\alpha\,,\\ &c\to\tilde{c}=c\cos2\alpha-(b-\tfrac{1}{2})\sin2\alpha\,,\\ &d\to\tilde{d}=d+\frac{\partial\alpha}{\partial t}\,. \end{aligned} \tag{2.5}$$

We define the classical vacuum as a zero-energy field configuration, thus F^{0i} and F^{ij} vanish and A_μ is a pure gauge:

$$\begin{aligned} A_\mu &= U^{-1}\partial_\mu U, \\ a &= \mathring{\alpha}, \\ b &= \frac{1-\cos 2\alpha}{2}, \\ c &= \frac{\sin 2\alpha}{2}, \\ d &= \frac{\partial\alpha}{\partial t}. \end{aligned} \tag{2.6}$$

Insertion of Eq. (2.6) into (2.3) gives the equation for a transverse gauge description of a [O(3) symmetric] classical vacuum[1,4]

$$\ddot{\alpha} + \mathring{\alpha} - \sin 2\alpha = 0. \tag{2.7}$$

Evidently (2.7) can be thought of as the equation of motion for a damped pendulum with angle α, moving with friction in a periodic potential $V(\alpha) = -\sin^2\alpha$.

We now discuss various aspects of the solution to (2.7). Near $\alpha = 0$ (or $\alpha = n\pi$ with integer n) the equation may be linearized and the solution is

$$\alpha = c_1 e^{s} + c_2 e^{-2s} + (n\pi). \tag{2.8}$$

Linearization is possible also near $\alpha = \pi/2$ (or $\alpha = \pi/2 + n\pi$) where one finds an oscillatory behavior

$$\alpha = c e^{-s/2}\cos(\omega s + \varphi) + \pi/2 + (n\pi), \quad \omega^2 = \tfrac{7}{4}. \tag{2.9}$$

Assuming that $\alpha = 0$ for $r = 0$ ($s = -\infty$), so that the gauge potential is regular at the origin, one finds, with Gribov,[1] that the only solutions to Eq. (2.7) are $\alpha = 0$ and

$$\alpha = \epsilon f(s - s_0), \quad \epsilon = \pm 1. \tag{2.10}$$

The function $f(s)$, illustrated in Fig. 1, describes a motion where α starts at 0 (with $\mathring{\alpha} = 0$) —this is a maximum point of the potential—and rolls down the potential valley approaching $\pi/2$—a minimum point of the potential. These nonvanishing solutions for α give origin to nonvanishing transverse potentials $\vec{A}$, behaving as $i(\hat{r}\times\vec{\sigma})/r$ for large r.

It is evident that, just as in electrodynamics, the transversality condition does not fix the gauge completely; an additional regularity condition is required, and in the following we adopt the requirement

$$\lim_{r\to\infty} r\vec{A} = 0. \tag{2.11}$$

Nevertheless, in the Appendix we shall discuss the features of the evolution of the potentials when

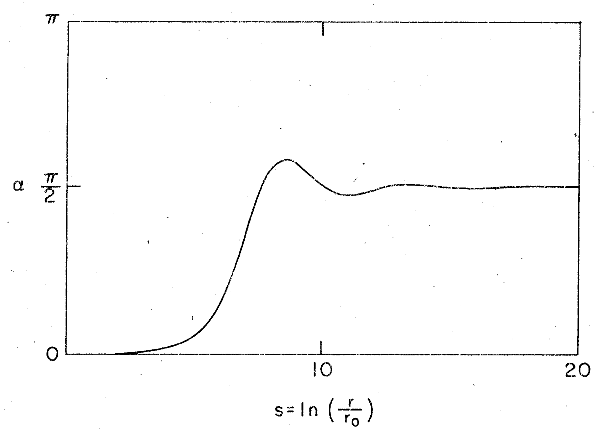

FIG. 1. The angle α that parametrizes the gauge transformation leading to a pure gauge transverse potential as a function of $\ln(r/r_0)$.

one allows long-range, $O(1/r)$, tails in $\vec{A}$.

There are various reasons why we should exclude the $O(1/r)$ asymptotic behavior when fixing the gauge. Without this condition, the specification of the vacuum would not be stable against continuous deformation. Also, in the corresponding quantum theory there appear to be no (tunneling) transitions between the trivial vacuum $\vec{A} = 0$ and those with $\lim_{r\to\infty} r\vec{A} \neq 0$.[5]

Adopting (2.11), we conclude that in the Coulomb gauge the classical vacuum is uniquely $\vec{A} = 0$. This is also true of the quantum theory, but there is no contradiction with previous analyses in the temporal gauge $A^0 = 0$.[2] Let us recall that while we found a family of vacuums parametrized by an angle θ, there are no transitions between different θ, and different inequivalent theories are built on each separate vacuum. In the Coulomb gauge we have only one vacuum; nevertheless, as we shall see presently, an angle θ emerges, which characterizes different theories, when wave functionals are constructed.

Let us compare in detail the evolution in a temporal gauge and in a Coulomb gauge. In the former, the classical vacuum can be described by nonvanishing potentials, still well behaved at infinity. If we set

$$A_i = e^{-i\beta(r)\vec{\sigma}\cdot\hat{r}}\,\nabla_i\, e^{i\beta(r)\vec{\sigma}\cdot\hat{r}}, \quad \beta(0) = 0, \tag{2.12}$$

(2.11) will be satisfied if $\lim_{r\to\infty}\beta = m\pi$, integer m, where m is the winding number of the gauge function. Consider now an evolution in the temporal gauge with the potential approaching vacuum configurations with different winding numbers at $t\to\pm\infty$. In particular, let us assume that $\vec{A}(\vec{r},t)$ tends to 0 at $t = -\infty$, while at $t = \infty$ it becomes the pure gauge Eq. (2.12) with $m \neq 0$. (Notice that any continuous field evolution can be brought into the temporal gauge with a continuous gauge transformation, so that starting in the temporal gauge is not a restriction.) Suppose that we want to describe the same evolution in the Coulomb gauge. We must perform a time-dependent gauge transformation from $A_i(\vec{r},t)$ to $\tilde{A}_i(\vec{r},t)$ with an amplitude $\alpha(r,t)$, such that the coefficient functions $\tilde{a}$ and $\tilde{c}$ obey the transversality condition (2.3). From (2.5) we deduce the equation for α,

$$\ddot{\alpha}+\dot{\alpha}-\sin 2\alpha=-\dot{a}-a-2b\sin 2\alpha+2c\cos 2\alpha\,, \tag{2.13}$$

where a, b, and c are the coefficients in the expansion of $\vec{A}(\vec{r},t)$ in the temporal gauge. An energy-conserving evolution in the temporal gauge cannot change the asymptotic behavior of $\vec{A}(\vec{r},t)$; hence $\lim_{r\to\infty} r\vec{A}(\vec{r},t)=0$ at all times,[5] which implies that a, b, and c vanish as s tends to ∞.

The solution to (2.13) as $t\to-\infty$ is clearly $\alpha=0$ because $\lim_{t\to-\infty}\vec{A}(\vec{r},t)=0$, which is already transverse, and the transverse vacuum is unique. However, for $t\to\infty$, we must have $\lim_{t\to\infty}\alpha=-\beta$ because the potential in the temporal gauge is the pure gauge (2.12) and the effect of the further gauge transformation, parametrized by α, must be to return $\vec{A}$ to zero again because the transverse vacuum is unique. This shows that α cannot be continuous in t, when the winding number is nonvanishing: α tends to zero as s tends to infinity when t is large and negative; conversely α tends to $-m\pi$ for large positive t. We conclude that the Coulomb gauge description involves a discontinuity at some time t_0 of the type

$$\begin{aligned}&\lim_{\epsilon\to 0}\tilde{A}_i(\vec{r},t_0+\epsilon)=A_i^{(1)}(\vec{r})\,,\\ &\lim_{\epsilon\to 0}\tilde{A}_i(\vec{r},t_0-\epsilon)=A_i^{(2)}(\vec{r}),\quad \epsilon>0\end{aligned} \tag{2.14}$$

where $\vec{A}^{(1)}$ and $\vec{A}^{(2)}$ are transverse potentials that are gauge equivalent to the same potential $\vec{A}(\vec{r},t_0)$ in the temporal gauge. Therefore, the Coulomb gauge description of a guage field with nonvanishing Pontryagin index cannot be single-valued.

This discussion may be reformulated by examining the expression for the Pontryagin index

$$\begin{aligned}&q=-\frac{1}{16\pi^2}\int d\vec{r}\,dt\,\mathrm{Tr}{}^*F^{\mu\nu}F_{\mu\nu},\quad {}^*F^{\mu\nu}=\tfrac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}\\ &F^{\mu\nu}=\partial^\mu A^\nu-\partial^\nu A^\mu+[A^\mu,A^\nu]\,.\end{aligned} \tag{2.15}$$

Let us assume that the potential is continuous and transverse. Since the integrand may be written as a total divergence, we may use Gauss's theorem to cast the formula for q into the following form:

$$q=-\frac{1}{16\pi^2}\left(\lim_{t\to\infty}\int d\vec{r}X^0-\lim_{t\to-\infty}\int d\vec{r}X^0+\int dt\lim_{r\to\infty}\int d\Omega\, r^2\hat{r}\cdot\vec{X}\right), \tag{2.16a}$$

$$\begin{aligned}&X^\mu=4\epsilon^{\mu\alpha\beta\gamma}\,\mathrm{Tr}(\tfrac{1}{2}A_\alpha\partial_\beta A_\gamma+\tfrac{1}{3}A_\alpha A_\beta A_\gamma)\,,\\ &\partial_\mu X^\mu=\mathrm{Tr}\,{}^*F^{\mu\nu}F_{\mu\nu}\,.\end{aligned} \tag{2.16b}$$

By hypothesis, the configuration becomes the vacuum as $t\to\pm\infty$; hence the first two contributions to (2.16a) vanish, since the unique transverse vacuum is zero. Moreover, if the potentials vanish faster than $1/r$ as $r\to\infty$, the last term in (2.16a) is also zero. We see then that a gauge potential with nonvanishing Pontryagin index cannot undergo continuous time evolution in the Coulomb gauge.

When the potentials are discontinuous, as in (2.14), the right-hand side of (2.16a) acquires an additional contribution:

$$q=-\frac{1}{16\pi^2}\int d\vec{r}\ \text{discontinuity}\ (X^0)\,. \tag{2.16c}$$

The potentials $A_\mu^{(1)}$ and $A_\mu^{(2)}$ in (2.14) are related by a gauge transformation

$$A_\mu^{(2)}=U^{-1}A_\mu^{(1)}U+U^{-1}\partial_\mu U\,. \tag{2.17a}$$

With some algebra one finds

$$\begin{aligned}X_\mu^{(2)}=X_\mu^{(1)}&-\tfrac{2}{3}\mathrm{Tr}\,\epsilon_{\mu\alpha\beta\gamma}(U^{-1}\partial^\alpha U)(U^{-1}\partial^\beta U)(U^{-1}\partial^\gamma U)\\ &+2\,\mathrm{Tr}\epsilon_{\mu\alpha\beta\gamma}\partial^\alpha(\partial^\beta U)U^{-1}A^\gamma\,.\end{aligned} \tag{2.17b}$$

From (2.16c) we see that Pontryagin index is given precisely by the winding number of the gauge transformation which relates the two different transverse descriptions of the same field configuration.

One may wonder what necessitates a discontinuous transition to an equivalent Coulomb gauge in a field configuration which evolves according to the Yang-Mills equations of motion. Recall that a time evolution is established by defining a Hamiltonian formalism. In a Yang-Mills theory the Hamiltonian contains the term $\frac{1}{2}\mathrm{Tr}\rho\mathcal{G}\rho$ where ρ is the charge density and $\mathcal{G}$ is a Coulomb Green's function:

$$\mathcal{G}^{-1}=d\,\frac{1}{\nabla^2}\,d,\quad d=\vec{\nabla}\cdot\mathfrak{D}(A)\,. \tag{2.18}$$

Here $\vec{\mathfrak{D}}(A)$ is the gauge-covariant gradient with transverse potential $\vec{A}$. As noted by Gribov,[1] $\mathcal{G}^{-1}$ develops a vanishing eigenvalue for those values of $\vec{A}$ which begin to allow more than one gauge-equivalent transverse representation. For completeness of this presentation, we repeat the argument. Let us consider a nonunique transverse $\vec{A}$ which is infinitesimally close to its gauge-equivalent partner. In that case, there exists an infinitesimal gauge transformation $U = I + i\Theta$ with the property that both $\vec{A}$ and $\vec{A} + \delta\vec{A} = \vec{A} + i\vec{\mathfrak{D}}(A)\Theta$ are in the transverse gauge. Consequently Θ satisfies the equation

$$\vec{\nabla} \cdot \vec{\mathfrak{D}}(A)\Theta = 0. \qquad (2.19)$$

For this $\vec{A}$, Det d vanishes, and $\mathcal{G}^{-1}$ becomes ill defined when two gauge-equivalent transverse potentials differ infinitesimally. But this is precisely the point where the number of gauge-equivalent gauge potentials changes and we see that the Hamiltonian procedure fails to produce a continuous time evolution when the transverse potentials reach definite magnitudes. The evolution of the system may be followed only by altering discontinuously the gauge representation.

We may now identify the mechanism which produces multiple quantum theories parametrized by an angle θ. When a quantum state is described by a wave functional $\Psi(\vec{A})$, with $\vec{A}$ transverse, sufficiently small $\vec{A}$ cause no difficulties. However, when $\vec{A}$ reaches a definite magnitude, gauge-equivalent $\vec{A}$'s describe the same physics. The wave functional for two such potentials must be equal in modulus, but a difference in phase can be accommodated. Choosing a definite value for this phase introduces the angle θ, and apparently different choices of θ define different theories.

The same ambiguity that is indicated by the null eigenvalue of Eq. (2.19) also affects the functional integral formalism and the Fadeev-Popov gauge-fixing procedure. The null eigenvalues cause Det d to vanish and the functional integral has to be treated with care.

III. EXPLICIT CALCULATION

We have proposed that the transversality condition of the Coulomb gauge be supplemented with the requirement that the transverse potentials decrease faster than $1/r$ at large r. This then led to the conclusion that sufficiently large potentials evolve discontinuously in time with sudden transitions between gauge-equivalent transverse configurations. The question still remains whether it is in fact possible to find such gauge potentials, i.e., whether the Eq. (2.13) has solutions with $\lim_{s\to\infty} \alpha(s) = n\pi$. We answer this question affirmatively by a detailed solution of (2.13) for various examples. All the features which we found in the general analysis of Sec. II will be explicitly exhibited.

Consider first the case where the original nontransverse potential is a pure gauge, parametrized by β. Of course (2.13) then reduces to the "free" equation (2.7), but now for $\alpha + \beta$. The only acceptable solution is $\alpha = -\beta$, and if β varies from 0 to $m\pi$ as s spans the real axis, then α moves in the opposite direction, from 0 to $-m\pi$. When we consider the parameters a, b, c of an arbitrary nontransverse potential, the analysis of (2.13) will not be so trivially straightforward. But one expects that if the original potential, as a function of time, interpolates between zero and a pure gauge with winding number m, then the effect of the right-hand side of (2.13), which may be thought of as a kind of "driving" term, is to force α for large s progressively closer to $-m\pi$, as time evolves.

This is illustrated by the following example. We take $\beta(s)$ to equal zero for $s \leq \bar{s}$, to pass smoothly to $m\pi$ between $-\bar{s}$ and $\bar{s}$, and to remain equal to $m\pi$ for $\bar{s} \leq s$. For the nontransverse potential in the temporal guage we take

$$A_i(\vec{r}, t) = \eta(t)\, e^{-i\beta\vec{\sigma}\cdot\hat{r}}\, \nabla_i\, e^{i\beta\vec{\sigma}\cdot\hat{r}}, \qquad (3.1)$$

where the function $\eta(t)$ interpolates monotonically between $\eta(-\infty) = 0$ and $\eta(\infty) = 1$. This field configuration has Pontryagin index m. From (2.6), we find that for $\bar{s} \to 0$, a approaches $m\pi\eta(t)\delta(s)$, whereas b and c vanish. So in the limit where the variation in β occurs at a point, the equation for α becomes

$$\ddot{\alpha} + \dot{\alpha} - \sin 2\alpha = -m\pi\eta(t)\left(\frac{d}{ds} + 1\right)\delta(s). \qquad (3.2)$$

This equation is solved by a function α which obeys the "free" equation (2.7) for $s < 0$ and for $s > 0$, while at the origin there is a discontinuity in the function, but not in its derivative:

$$\lim_{\bar{s}\to 0^+} (\alpha|_{s=\bar{s}} - \alpha|_{s=-\bar{s}}) = -m\pi\eta, \qquad (3.3a)$$

$$\lim_{\bar{s}\to 0^+} (\dot{\alpha}|_{s=\bar{s}} - \dot{\alpha}|_{s=-\bar{s}}) = 0. \qquad (3.3b)$$

Since for $s \to -\infty$, we want α to vanish, the solution for $s < 0$ is (2.10)

$$\alpha = \epsilon f(s - s_0), \quad \epsilon = \pm 1. \qquad (3.4)$$

We investigate in detail the solution for α at positive s; we impose the condition

$$\lim_{s\to\infty} \alpha = n\pi, \quad \text{integer } n, \qquad (3.5)$$

which excludes a long-range $O(1/r)$ tail in the transverse potential. (In the Appendix this restriction is removed.)

In order for α to approach and settle at a point of unstable equilibrium $\alpha = n\pi$ at $s = \infty$, one must carefully adjust the parameter s_0 of the motion for $s < 0$. An understanding of the behavior of α is obtained as follows. The condition (3.5) requires that after the jump at $s = 0$, α takes a value $n\pi + \Delta\alpha$ with velocity $\dot{\alpha}|_{s=0}$ such as to generate a motion directed toward the peak at $n\pi$: $\text{sgn}(\dot{\alpha}|_{s=0}) = -\text{sgn}(\Delta\alpha)$. Moreover, $|\Delta\alpha|$ must be less than π; otherwise the velocity is not sufficient to overcome the friction. Finally $\dot{\alpha}|_{s=0}$ and $\Delta\alpha$ must be in a definite functional relation, namely the functional form that is obtained by integrating the free equation from $s=\infty$, with $\lim_{s\to\infty}\alpha = n\pi$ and $\lim_{s\to\infty}\dot{\alpha}=0$, backward in s to $s=0$. For large s, where α and $\dot{\alpha}$ are small, the relation between velocity and position is obtained from (2.8) with $c_1=0$:

$$\dot{\alpha} + 2\alpha = 2n\pi \text{ (large } s). \tag{3.6}$$

Since both $\alpha - n\pi$ and $\dot{\alpha}$ must remain small—the allowed range of $\Delta\alpha$ is limited—and $\dot{\alpha}|_{s=0}$ always differs in sign from $\Delta\alpha$, we may extend (3.6) to all $s \geq 0$ and use it as an approximation to the desired relation between position and velocity after the discontinuity. Replacing α in (3.6) by $\epsilon f(-s_0) - m\pi\eta = n\pi + \Delta\alpha$, and $\dot{\alpha}$ by $\epsilon\dot{f}(-s_0)$ we find

$$\epsilon\dot{f}(-s_0) + 2[\epsilon f(-s_0) - m\pi\eta] = 2n\pi. \tag{3.7a}$$

Therefore the approximate equation which determines that value for s_0 which permits an unstable equilibrium to be the end point of motion is

$$F(s_0) \equiv \tfrac{1}{2}\epsilon\dot{f}(-s_0) + \epsilon f(-s_0) - n\pi = m\pi\eta. \tag{3.7b}$$

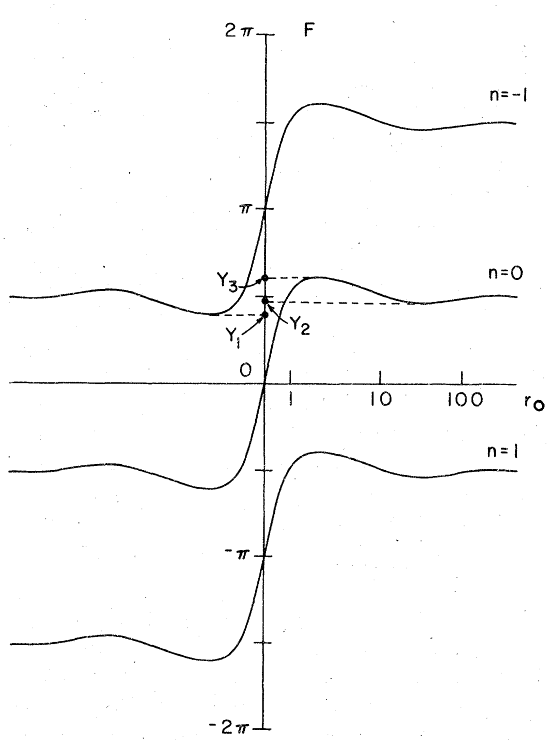

FIG. 2. A few branches ($n=0, \pm 1$) of the function $F(r_0)$ defined in Eq. (3.7b).

In Fig. 2 we plot three branches ($n=0, \pm1$) of F as a function of $r_0 = e^{-s_0}$, using the negative axis to plot the values of F for negative ϵ. It is apparent that for small values of η (i.e., small fields in the temporal gauge) there is a unique gauge transformation that brings the field configuration into a unique Coulomb gauge. As $m\pi\eta$ becomes larger, however, a multivaluedness sets in. For a critical value y_1 of $m\pi\eta$ a second solution appears, and for a range $y_1 < m\pi\eta < y_2$ (see Fig. 2) one has three solutions for s_0. Of the corresponding gauge transformations that bring the field into the Coulomb gauge, one has winding number zero (the one along the branch through the origin), the other two have winding number -1. As the field increases in magnitude, for some range of values of the fields it is still possible to follow the evolution without discontinuities in the transverse potential (i.e., in s_0), although the multivaluedness increases as $m\pi\eta$ approaches $\pi/2$. Eventually, for $m\pi\eta > y_3 = \pi - y_1$ (see Fig. 2) one must leave the original branch, and adopt a gauge-equivalent description, obtained by a gauge transformation of nontrivial winding number. Notice that for $m\pi\eta = y_3$, there are two possible choices of s_0 on the original branch, infinitesimally close to each other. Consequently there exists an infinitesimal gauge transformation that connects the two transverse gauges at the point, and the Coulomb propagator becomes ill defined, as explained in Sec. II. We thus see that all the features we have discussed are explicitly realized in the simple model.

Our final calculation is exhibited in Fig. 3 where we plot the values of $V(r) = 2 - 4b(r)$ for the pseudoparticle solution[3] of Belavin, Polyakov, Schwartz, and Tyupkin, transformed into the Coulomb gauge. A numerical integration of Eq. (2.13) was performed to find the function $\alpha(s)$ that transforms the pseudoparticle solution of Ref. 3, centered at $x^\mu = 0$ with scale Λ, to the Coulomb gauge, and Eq. (2.5) was used to obtain the parameters, a, b, and c of the pseudoparticle in the Coulomb gauge. The initial condition in the integration of Eq. (2.13) (i.e., the value of $\partial\alpha/\partial r$ for $r=0$) was set so as to fulfill Eq. (2.11). For a range of values of t, from $-\infty$ to a definite $-t_0$ ($t_0 > 0$) and from t_0 to ∞, the requirement that the transverse field vanishes faster than $1/r$ as $r\to\infty$ [Eq. (2.11)] fixes the initial condition uniquely. In particular, the transverse field $\vec{A}(r,t)$ vanishes for $t = \pm\infty$. We followed the time evolution of the field starting from $t=-\infty$. When more than one initial value of $\partial\alpha/\partial r$ allowed Eq. (2.11) to be satisfied, we resolved the ambiguity demanding continuity in time. We thus found that we could integrate the equations up to a maximum time $t = t_0 \approx 0.18\Lambda$. But

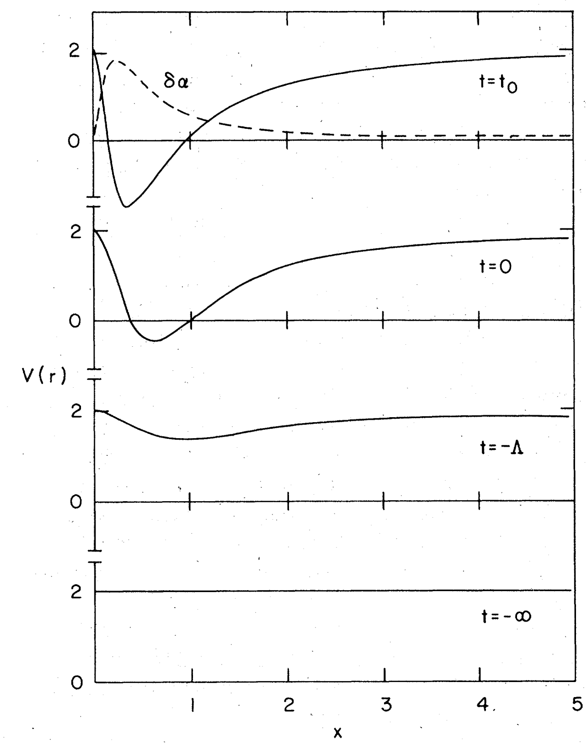

FIG. 3. The parameter $V=2-4b$ of the pseudoparticle in the Coulomb gauge for different values of the time variable and the solution of the small-deformation equation (dashed line) for $t=t_0$, displayed as functions of $x=r/(t^2+\Lambda^2)^{1/2}$.

for $t>t_0$ we could not satisfy Eq. (2.11) still maintaining continuity in $\partial\alpha/\partial r\,|_{r=0}(t)$. To follow the evolution of the pseudoparticle field up to $t=+\infty$ it is necessary to go discontinuously, at some time t_1, $-t_0 \le t_1 \le t_0$, to a gauge-equivalent description. The required gauge transformation has winding number -1 and the Pontryagin index gets its correct value through the contribution in Eq. (2.16c), as we anticipated in our general discussion of the evolution of the transverse fields.

We chose to plot the combination $V(r)=2-4b(r)$ because it appears as a potential in the equation for infinitesimal gauge transformations within the transverse gauge

$$\delta\ddot{\alpha}+\delta\dot{\alpha}-(2-4b)\delta\alpha=0\,. \tag{3.8}$$

[Equation (3.8) is obtained from Eq. (2.13) assuming that α is infinitesimal and that the field is already transverse so that Eq. (2.3) is satisfied.] As explained before, the appearance of a normalizable solution to Eq. (3.8) is a signal that one cannot follow the evolution on a given branch. We integrated Eq. (3.8) numerically from $r=0$, $\delta\alpha(0)=0$, with the potential $V(r)$ obtained for $t=t_0$. The result is the dashed curve in Fig. 3. It is nice to see that $\delta\alpha(r)$ does approach zero for large r, indicating the presence of a normalizable solution.

IV. CONCLUSION

We have confirmed the observations of Ref. 1 that the Coulomb gauge is ambiguous, even when the transversality condition is supplemented by the requirement $\lim_{r\to\infty} r\vec{A}=0$. Discontinuities related to such ambiguity appear in the time evolution of large transverse fields. Moreover, in the quantum theory, the ambiguities and discontinuities allow for the introduction of an angle which further specifies the theory. It should be clear that the notion of "large" fields entered our considerations since we studied configurations with nonvanishing Pontryagin index. The effects are consequently nonperturbative; when the gauge coupling constant e is reintroduced, then the amplitudes with which we are dealing are $O(1/e)$ and become infinite as $e\to 0$. Large fields also have consequences in the perturbative calculation of the invariant charge which determines the static potential evaluated to one-loop order in the Coulomb gauge.[1,6] These effects manifest themselves in a strengthening of the interaction already at moderate values of the coupling constant.

While all these properties of the Coulomb gauge are certainly peculiar, they do not, as far as we can see, expose any novel dynamical aspects of the gauge theory. It seems that all the pathologies are present merely to reproduce features that are easily seen in other gauges such as the temporal gauge: nontrivial Pontryagin index, θ multiplicity. In particular, it is an open question whether these phenomena, or further effects related to the $O(1/r)$ transverse gauge potentials, which we have excluded, have any bearing on confinement.[7]

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APPENDIX

While in the body of the paper we have excluded transverse potenentials with a long-range $O(1/r)$ tail, in this appendix we discuss properties of field evolution in the Coulomb gauge without imposing the supplementary condition $\lim_{r\to\infty} r\vec{A}=0$. Now there are nontrivial vacuums, given by pure gauges with α as in (2.10). It is easy to check that the winding number $-(1/16\pi^2)\int d\vec{r}X^0$ equals $\epsilon/2$. Therefore it is possible to obtain a Pontryagin index q of modulus not larger than one as a difference of two surface contributions at $t=\pm\infty$. In particular the pseudoparticle solution can be cast into the Coulomb gauge without any discontinuities in the fields.[8] However, it is also apparent that Pontryagin indices larger than one in modulus cannot be accommodated without some form of singularity. We shall show that additional contributions to q come from the surface integral at spatial infinity, the last term in (2.16a).

The angle α, appearing in the transformation from the temporal gauge to the Coulomb gauge, must still satisfy (2.7) for large s at all t, since the driving terms on the right-hand side of (2.13) vanish as $s\to\infty$. When allowing a long-range $O(1/r)$ tail, one permits α to approach any of the values $\pi/2+n\pi$ for $s\to\infty$.

The way in which contributions to the surface integral at spatial infinity emerge is that there is a discontinuity in $\lim_{s\to\infty}\alpha$ as a function of t. Typically, we will have $\lim_{s\to\infty}\alpha(s,t)=\pi/2+n\pi$ for $t<t_0$ and $\lim_{s\to\infty}\alpha(s,t)=\pi/2+(n\pm1)\pi$ for $t>t_0$. As a function of s with large s, $\alpha(s,t)$ gets closer and closer to $\pi/2+(n\pm\frac{1}{2})\pi$ for t approaching t_0, before falling into a potential valley. For t very close to t_0 (but $t<t_0$) α lingers at large s near $\pi/2+(n\pm\frac{1}{2})\pi$, then falls back to $\pi/2+n\pi$; as t becomes larger than t_0, α passes (for large s, and very slowly) the critical point of unstable equilibrium, and falls into the next potential valley. It appears therefore that discontinuities at finite r, t may be avoided in transverse gauge potentials with long-range $O(1/r)$ tails. They reappear at infinite r.

The crucial difference between the $\lim_{r\to\infty}r\vec{A}=0$ situation and the present case, $\lim_{r\to\infty}r\vec{A}\neq0$, is that with the former boundary condition, α must approach a point of unstable equilibrium at $s=\infty$. As we have seen, this fixes α as a function of s for all s with no free parameters. So for α to approach two different points of unstable equilibrium, it must be described by altogether different functions, and the time evolution is discontinuous. If instead, as in the present case, α is allowed to approach a point of stable equilibrium at $s\to\infty$, the parameter s_0 is left free in the determination of $\alpha(s)$ [see Eq. (2.10)]. By varying this parameter, one may go from a function α with limiting value $\pi/2+n\pi$, to another with limiting value $\pi/2+(n\pm1)\pi$, continuously at all s, albeit in a nonuniform way.

The discontinuity in t at infinite r produces a $\delta(t-t_0)$ singularity in the time component of the Coulomb gauge potential at $r=\infty$ [see Eq. (2.6)]. This, combined with the $1/r$ tail of the spatial components, contributes unit magnitude to q through the surface integral at spatial infinity in (2.16a). By allowing for several such discontinuities an arbitrary Pontryagin index may be regained.

These general considerations are well illustrated by calculation on the model of Sec. III. Allowing for a long-range $O(1/r)$ tail, Eq. (3.2) is solved straightforwardly. For $s<0$, $\alpha=\epsilon f(s-s_0)$, where s_0 is arbitrary. At $s=0$, α jumps to $\epsilon f(-s_0)-m\pi\eta$, keeping $\dot{\alpha}$ unchanged. For $s>0$, α executes damped oscillations and settles finally in one of the valleys $\pi/2+n\pi$, with n determined by the magnitude of the position and velocity at $s=0^+$. For fixed s, α is a continuous function of t (as long as ϵ, s_0, and η are continuous), whereas $\lim_{s\to\infty}\alpha$ develops discontinuities of magnitude equal to π, confirming the behavior described above.

Note added in proof. Further investigations of these and related questions have been made recently. I. Singer, U. C. Berkeley report (unpublished), has found that the problems exhibited here occur whenever one tries to fix a gauge condition completely on a compactified space-time. M. Ademollo, E. Napolitano, and S. Sciuto, CERN report (unpublished), have extended portions of our analysis to $O(3)$ noninvariant configurations, with results similar to ours.

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