Regions of magnetic support of a plasma around a black hole

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We establish necessary conditions for the trapping of charged particles by an electromagnetic field in the magnetosphere of a black hole. Three regions are defined according to the relative importance of electric and magnetic fields. Idealized models are used to illustrate the application of these criteria.

I. INTRODUCTION

The observations of binary x-ray sources have given clear evidence that plasma accreting from a normal star into a neutron star or a black hole gives rise to the emission of x rays of up to $10^5 L_\odot$.[1] The primary source of this radiated energy is the gravitational binding energy of the material accreting into the deep potential well of the gravitationally collapsed object.[2] Recent observations of x-ray bursts[3] also point to the possibility that processes involving black-hole magnetospheres occur in the cores of globular clusters.[2] Similar mechanisms have also been suggested as the origin of extended radio lobes in extragalactic radio sources.[2]

In order to reach deeper implications from these observations it is therefore relevant to obtain a theoretical understanding of the magnetosphere of a gravitationally collapsed object. Moreover, since the emission processes are expected to occur within a few radii of the surface of the black holes, it is likely that these observations, duly interpreted, will give information about the most novel and strong-field effects of general relativity.[2]

In this paper we introduce basic criteria for establishing the regions of magnetic support for a plasma accreting in the field of a black hole, and we apply these criteria to idealized models of astrophysical systems containing black holes.

II. THREE CONDITIONS FOR MAGNETIC TRAPPING

The motion of a test particle of mass m and charge q in an electromagnetic field $F_{\mu\nu}$ in a fixed curved background is determined by the well-known relativistic generalization of the Lorentz force law,[4,5]

$$m \frac{Du^\mu}{DS} = qF^{\mu\nu} u_\nu \,, \qquad \text{(1a)}$$

where u_ν is the four-velocity of the particle and D/DS denotes the covariant derivative taken along the trajectory of the test particle.

In a local Lorentz frame the electromagnetic tensor $F_{\mu\nu}$ is related to the components of the electric and magnetic fields[6,7]

$$E_{\hat{j}} = F_{\hat{j}\hat{0}} \text{ and } B_{\hat{j}} = \epsilon_{\hat{j}\hat{k}\hat{l}} F^{\hat{k}\hat{l}} \,, \qquad \text{(2a)}$$

where the careted indices indicate projection onto the orthonormal components of the tetrad defining the local inertial frame. In such a frame Eq. (1a) acquires the familiar form

$$m \frac{d}{dt} \left[\vec{v}/(1-v^2)^{1/2} \right] = q(\vec{E} + \vec{v} \times \vec{B}) \,. \qquad \text{(1b)}$$

Let us consider the possibility of trapping test charges in a magnetic field. The minimal local condition for magnetic dominance is

$$\frac{\vec{B}^2 - \vec{E}^2}{2} > 0 \,, \qquad \text{(3a)}$$

which may be expressed from Eq. (2) in covariant form as

$$F = \tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} > 0 \,. \qquad \text{(3b)}$$

This local criterion is a necessary condition only when the Lorentz force given in Eq. (1) dominates the motion of the charged particles. In view of the strength of the electromagnetic fields expected near a collapsed object[6] and the large charge-to-mass ratio of the particles under consideration, the direct effect of the gravitational field on the motion of the particles can be neglected expect very near the horizon of a black hole.

The condition of Eq. (3) does not take the relative direction of the electric and magnetic fields into account. A stronger requirement would be that the magnetic force can balance the electric force in direction as well as modulus. In a local Lorentz frame this is equivalent to the Lorentz force on the charge being orthogonal to the electric field:

$$\vec{E}\cdot(\vec{E}+\vec{v}\times\vec{B})=\vec{E}^2-\vec{v}\cdot(\vec{E}\times\vec{B})=0\,. \tag{4a}$$

Since the magnitude of the velocity is less than one, Eq. (4a) implies that

$$|\vec{E}\times\vec{B}|>\vec{E}^2\,. \tag{5}$$

The electric and magnetic fields can be expressed in terms of η^α, the four-velocity of an observer at rest in a local inertial frame, as follows:

$$E_\alpha=F_{\alpha\beta}\,\eta^\beta$$

and

$$B_\alpha=-\sqrt{-g}\;\epsilon_{\alpha\beta\gamma\delta}F^{\beta\gamma}\eta^\delta\,. \tag{2b}$$

Equation (4a) may then be rewritten in covariant form

$$E_\alpha F^{\alpha\beta}u_\beta=V^\beta u_\beta=0\,, \tag{4b}$$

where

$$V^\beta=E_\alpha F^{\alpha\beta}=F^{\alpha\beta}F_{\alpha\gamma}\,\eta^\gamma\,.$$

Equation (4b) requires that the four-velocity of the test charge, u_β, be orthogonal to the four-vector V^β. Since u_β is timelike, V^β must be spacelike or zero to satisfy condition (4b). If V^β is spacelike, then there exists a timelike four-velocity u_β orthogonal to V^β. Thus the regions in which the magnetic field can balance the electric field are determined by the condition

$$V_\gamma V^\gamma>0\,. \tag{4c}$$

The boundary of these regions is called the plasma horizon.

To study the flow of plasma in regions of magnetic dominance, we will use the guiding-center approximation. Consider the local Lorentz frame in which the electric and magnetic fields are parallel and have magnitudes E_0 and B_0.[8] In this frame the solution of Eq. (1b) consists of a helical motion about the common direction of $\vec{E}_0$ and $\vec{B}_0$. To characterize the global behavior of these particles we average out the gyration and keep only the acceleration along $\vec{E}_0$. Such a guiding-center approximation is valid only when the radius of gyration is small compared to the distance over which the magnitude of the magnetic field changes significantly.

The velocity of the boost from an arbitrary local Lorentz frame to the frame in which the electric and magnetic fields are parallel is

$$\vec{v}_D=(\vec{E}\times\vec{B})/(B^2+E_0^{\,2})\,. \tag{6}$$

The electric and magnetic fields in the arbitrary local Lorentz frame, Eq. (2b), are

$$\vec{E}=(\vec{E}_0-\vec{v}_D\times\vec{B}_0)(1-v_D^{\,2})^{-1/2}\,, \tag{7a}$$

$$\vec{B}=(\vec{B}_0+\vec{v}_D\times\vec{E}_0)(1-v_D^{\,2})^{-1/2}\,, \tag{7b}$$

The motion of the guiding center is composed of the drift velocity of Eq. (6) and an acceleration parallel to $\vec{E}_0$, which can be expressed in terms of these fields as

$$\frac{dx^\alpha}{d\lambda}=E_0E^\alpha+B_0B^\alpha\,, \tag{8}$$

where λ is an appropriately normalized arclength parameter, and

$$E_0^{\,2}=(F^2+G^2)^{1/2}-F\,,\qquad B_0^{\,2}=(F^2+G^2)^{1/2}+F\,, \tag{9}$$

and the invariants of the electromagnetic field are

$$F=\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}=\frac{B_0^{\,2}-E_0^{\,2}}{2}=\frac{\vec{B}^2-\vec{E}^2}{2}\,,$$

$$G=\tfrac{1}{4}F_{\mu\nu}{}^*F^{\mu\nu}=E_0B_0=\vec{E}\cdot\vec{B}\,.$$

The flow lines determined by Eq. (8) can be divided into two classes, those that intersect the horizon and those that do not. Plasma flowing along a line that intersects the horizon will either accrete to the black hole or be swept away from it, depending on its charge. Plasma flowing along a line that does not intersect the horizon can oscillate back and forth on the flow line. A magnetosphere can form only in this region. The flow line that divides these classes of flow lines from each other is of particular interest. It may have a cusp, unless the electric field is everywhere less than the magnetic field. In either case, the surface of revolution it generates is the boundary of the region in which a magnetosphere is possible.

III. BLACK HOLES IN EXTERNAL ELECTROMAGNETIC FIELDS

The three criteria developed in the preceding section will be applied to idealized models containing black holes. These general considerations clarify the basic features of the magnetosphere of a collapsed object embedded in a plasma and external electromagnetic fields.

Let us first consider a charged static black hole in an asymptotically uniform magnetic field. The exact solution of the Einstein-Maxwell equations for this system was derived by Ernst.[10] The linearization of that solution for a weak magnetic field near the black hole is

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\varphi^2 + 4BQr \sin^2\theta\, dt\, d\varphi\,, \quad (10)$$

and the nonzero tetrad components of the electromagnetic field are

$$E_{\hat{r}} = Q/r^2\,, \quad (11a)$$

$$B_{\hat{r}} = B\left(1 - \frac{3Q^2}{r^2}\right)\cos\theta\,, \quad (11b)$$

$$B_{\hat{\theta}} = -B\sin\theta\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{1/2}. \quad (11c)$$

The basis 1-forms of the tetrad to first order in B are

$$\omega^{\hat{t}} = \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{1/2} dt\,, \quad \omega^{\hat{\theta}} = r\,d\theta\,,$$

$$\omega^{\hat{r}} = \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1/2} dr\,,$$

$$\omega^{\hat{\varphi}} = r\sin\theta\left(d\varphi + \frac{2BQ}{r}\, dt\right),$$

and the four-velocity used in Eq. (2b) is $\eta^\alpha = N(1,0,0,-2BQ/r)$, where N is the appropriate constant of normalization. In this case the invariants of the electromagnetic field are

$$F = \frac{B^2}{2}\left[1 + \left(\frac{Q^2}{r^2} - \frac{2M}{r}\right)\sin^2\theta + \left(\frac{9Q^4}{r^4} - \frac{6Q^2}{r^2}\right)\cos^2\theta\right] - \frac{Q^2}{2r^4}\,, \quad (12a)$$

$$G = \frac{QB}{r^2}\left(1 - \frac{3Q^2}{r^2}\right)\cos\theta\,. \quad (12b)$$

The separation of the plasma horizon from the axis is

$$r\sin\theta = \frac{Q}{Br}\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1/2}, \quad (13)$$

and the flow lines are determined by

$$\frac{d(r\sin\theta)}{d\theta} = \frac{3Q^2}{r}\cos\theta - \frac{Q}{Br}\left[\frac{(F^2+G^2)^{1/2} - F}{(F^2+G^2)^{1/2} + F}\right]^{1/2}. \quad (14)$$

They are plotted along with the surface $F=0$ for a specific choice of parameters in Fig. 1.

Let us now consider a slightly charged rotating black hole in an asymptotically uniform weak magnetic field. Since there is no background electromagnetic field, the magnetic field and charge make no first-order corrections to the Kerr metric:

$$ds^2 = \Sigma\Delta^{-1}dr^2 + \Sigma d\theta^2 + \Sigma^{-1}\sin^2\theta[a\,dt - (r^2+a^2)d\varphi]^2 - \Sigma^{-1}\Delta(dt - a\sin^2\theta\, d\varphi)^2\,, \quad (15)$$

where $\Sigma = r^2 + a^2\cos^2\theta$ and $\Delta = r^2 - 2Mr + a^2$. The tetrad components of the electromagnetic field are the superposition of those of a Kerr-Newman hole and an asymptotically uniform magnetic field:

$$E_{\hat{r}} = F_{\hat{r}\hat{t}} = \Sigma^{-2}A^{-1/2}\{BaM[2r^2\sin^2\theta\,\Sigma - (r^2+a^2)(r^2 - a^2\cos^2\theta)(1+\cos^2\theta)] + Q(r^2+a^2)(r^2 - a^2\cos^2\theta)\}\,, \quad (16a)$$

$$E_{\hat{\theta}} = F_{\hat{\theta}\hat{t}} = \Sigma^{-2}A^{-1/2}\Delta^{1/2}2ra^2\sin\theta\cos\theta[BaM(1+\cos^2\theta) - Q]\,, \quad (16b)$$

$$B_{\hat{r}} = F_{\hat{\theta}\hat{\varphi}} = \Sigma^{-2}A^{-1/2}\cos\theta(B\{(r^2+a^2)\Sigma^2 - 2Mra^2[2r^2\cos^2\theta + a^2(1+\cos^4\theta)]\} + 2Qar(r^2+a^2))\,, \quad (16c)$$

$$B_{\hat{\theta}} = F_{\hat{\varphi}\hat{r}} = \Sigma^{-2}A^{-1/2}\Delta^{1/2}\sin\theta\{Qa(r^2 - a^2\cos^2\theta) - B[Ma^2(r^2 - a^2\cos^2\theta)(1+\cos^2\theta) + r\Sigma^2]\}\,, \quad (16d)$$

where $A = (r^2+a^2)^2 - \Delta a^2\sin^2\theta$. The basis 1-forms of the tetrad are

$$\omega^{\hat{t}} = (\Sigma\Delta/A)^{1/2}dt\,, \quad \omega^{\hat{\theta}} = \Sigma^{1/2}d\theta\,,$$

$$\omega^{\hat{r}} = (\Sigma/\Delta)^{1/2}dr\,,$$

$$\omega^{\hat{\varphi}} = (A/\Sigma)^{1/2}\sin\theta\, d\varphi - \frac{2Mra\sin\theta}{(\Sigma A)^{1/2}}\, dt\,,$$

and its four-velocity is $\eta^\alpha = N(1,0,0,2Mra/A)$, where N is the normalization factor.

Figures 2 and 3 show the electric and magnetic lines of force for specific systems with the dipole moment of the black hole parallel and antiparallel to the external magnetic field. The plasma horizons [Eq. (4)] are plotted for the tetrad defined in the preceding paragraph. Figures 4 and 5 show the corresponding flow lines. When the magnetic moment of the black hole is aligned with the external field, the magnetic field is weaker in the equatorial plane. Consequently there is less support for a charged particle in that region and both the plasma horizon and the cusped flow line are farther from the black hole than in the counter-aligned case.

In the limit that the external magnetic field vanishes, $B=0$, the flow lines are strictly radial. It is not possible for the magnetic field of an isolated black hole to support a magnetosphere against the electric field of the black hole.

As a final example, consider a slightly charged static black hole at the center of an oppositely charged ring of current. This idealized model is related to Wilson's[9] numerical analysis of the accretion of a magnetic plasma to a black hole. Neglecting second-order perturbations of the background, we have the Schwarzschild metric

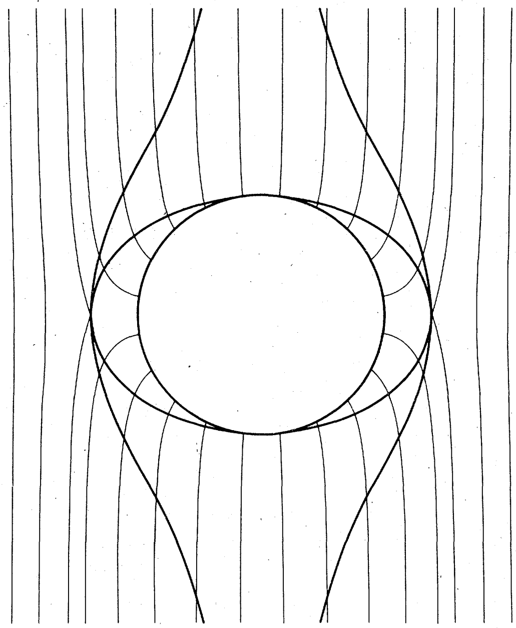

FIG. 1. The circle represents the event horizon of a Reissner-Nordström hole, whose charge is one-tenth of its mass. The ellipse defines the surface of revolution on which the asymptotically uniform magnetic field ($B=Q/M^2$) has the same magnitude as the monopole electric field of the black hole. The other pair of heavy lines are the plasma horizon [Eq. (4)] associated with the tetrad given after Eq. (11). The lighter lines are the flow lines defined in Eq. (14). Plasma spiraling around the flow lines outside the cusped flow line will oscillate through the equatorial plane. Plasma gyrating about flow lines inside the cusped line will be pulled from the magnetosphere by the electric field.

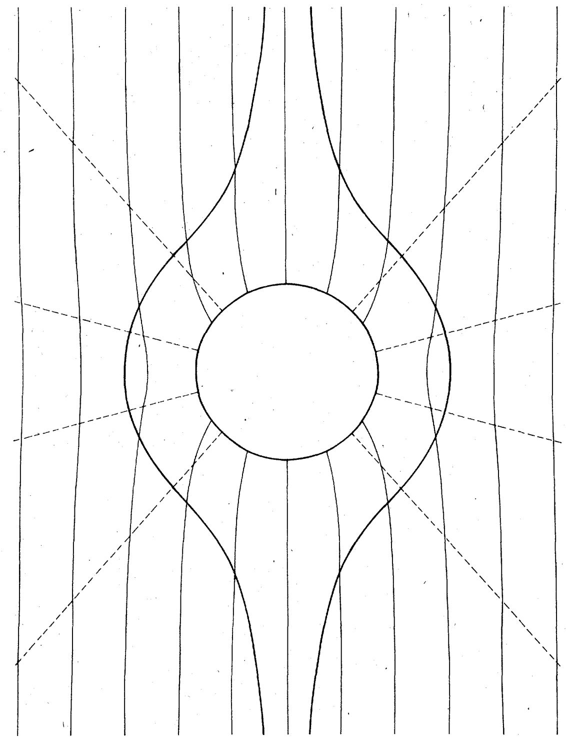

FIG. 2. Magnetic (light continuous) lines of force and electric (dashed) lines of force of a weakly charged black hole ($a/M=3/4$, $Q/M=10^{-2}$) embedded in an asymptotically uniform weak magnetic field ($B=Q/4M^2$) are plotted in Boyer-Lindquist coordinates. The magnetic moment of the black hole is aligned with the external magnetic field. The heavy lines represent the plasma horizon [Eq. (4)] for the tetrad of Eq. (16).

$$ds^2=-(1-2M/r)\,dt^2+(1-2M/r)^{-1}dr^2 + r^2(d\theta^2+\sin^2\theta\, d\varphi^2)\,. \tag{17}$$

The orthonormal tetrad of a static observer has the basis

$$\omega^{\hat t}=\left(1-\frac{2M}{r}\right)^{1/2}dt\,,\quad \omega^{\hat\theta}=r\,d\theta\,,$$

$$\omega^{\hat r}=\left(1-\frac{2M}{r}\right)^{-1/2}dr\,,\quad \omega^{\hat\varphi}=r\sin\theta\, d\varphi\,.$$

The weak-field solution for a ring of azimuthal current J^φ and radius a centered about an oppositely charged static black hole follows immediately from the solution of Linet.[11] The nonvanishing tetrad components are

$$F_{\hat r\hat t}=-\frac{\partial V}{\partial r}\,,\quad F_{\hat\theta\hat t}=-\frac{1}{r(1-2M/r)^{1/2}}\frac{\partial V}{\partial\theta}\,, \tag{18a}$$

where

$$V=\frac{Q}{r}\left(1-\frac{M}{a}\right)-\frac{Q}{2\pi a r}\int_0^{2\pi} d\varphi[(r-M)(a-M)-M^2\sin\theta\cos\varphi]$$
$$\times[(r-M)^2+(a-M)^2-M^2-2(r-M)(a-M)\sin\theta\cos\varphi+M^2\sin^2\theta\cos^2\varphi]^{-1/2}$$

and

$$F_{\hat\theta\hat\varphi}=\frac{\partial\chi}{\partial r}\,,\quad F_{\hat\varphi\hat r}=\frac{1}{r(1-2M/r)^{1/2}}\frac{\partial\chi}{\partial\theta}\,, \tag{18b}$$

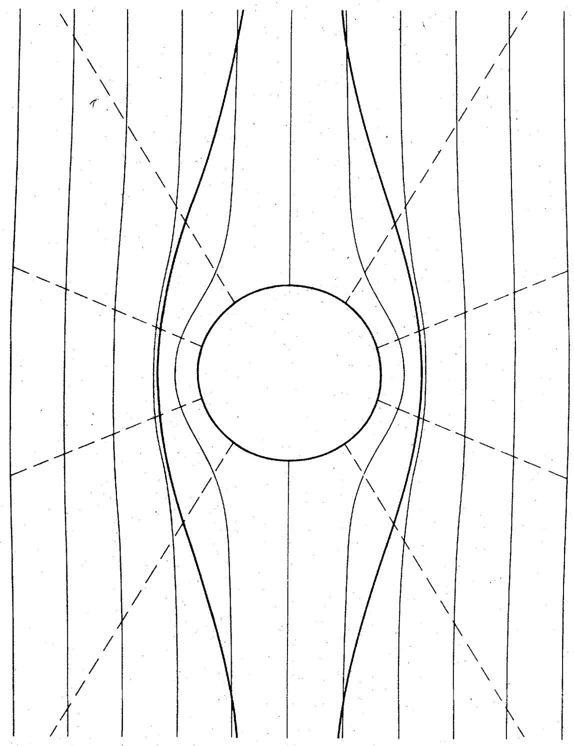

FIG. 3. This figure is identical to Fig. 2, except that the magnetic moment of the black hole is antiparallel to the external magnetic field.

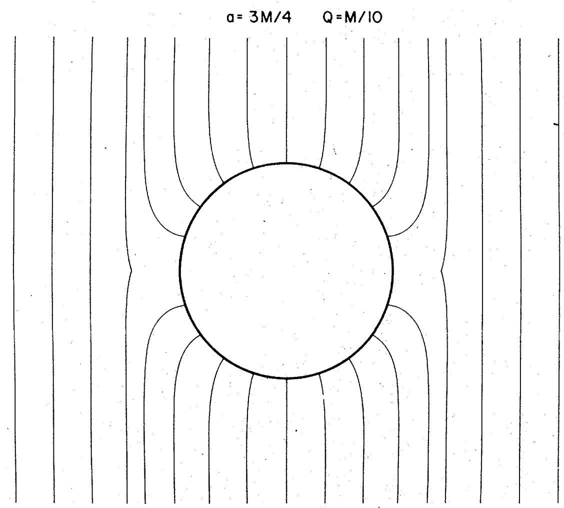

FIG. 5. This figure is identical to Fig. 4, except that the magnetic moment of the black hole is directly opposite to the external magnetic field.

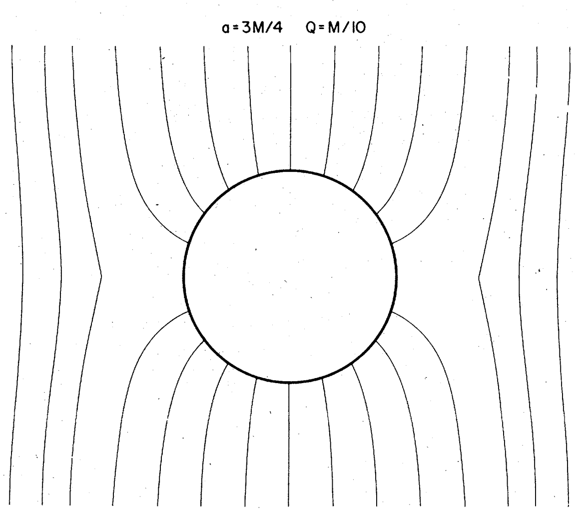

FIG. 4. The integration along the flow of the plasma was started at regular intervals along the circular event horizon. Integrating out in the equatorial plane produced the cusped line, which divides the lines that intersect the horizon from those that do not.

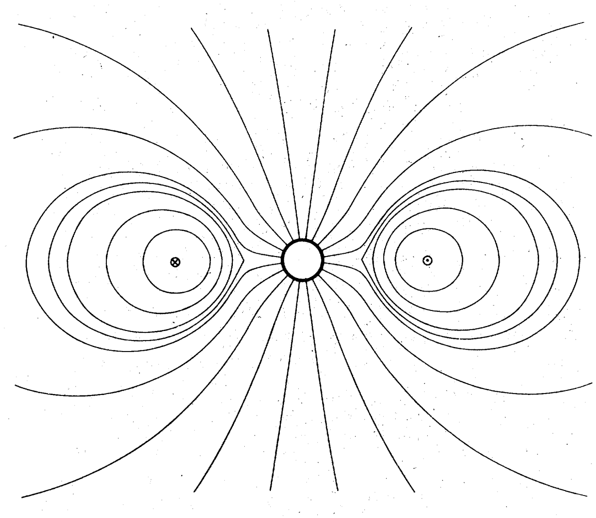

FIG. 6. The heavy circle represents the event horizon of a static black hole, whose charge is one-hundredth of its mass. A ring of current whose charge is opposite that of the black hole is shown at $r = 10M$ ($J^{\varphi}/J^0 = 2$). The cusped line encloses a toroidal region in which the lines of flow do not intersect the horizon.

where

$$\chi = J^{\varphi}(r-2M)\cos\theta \int_0^{2\pi}\int_a^{\infty} \rho d\rho\, d\varphi[(r-M)^2+(\rho-M)^2-M^2-2(r-M)(\rho-M)\sin\theta\cos\varphi+M^2\sin^2\theta\cos^2\varphi]^{-3/2}.$$

The flow of the plasma is determined through Eq. (14) by the invariants of the electromagnetic field. Figure 6 shows the flow lines of this system for a specific choice of parameters. The plasma horizon [Eq. (4)] and lines of force have been published elsewhere.[12] The region surrounded by the cusped flow line is qualitatively different from that of the systems with an asymptotically uniform magnetic field. Not only is the region of flow to the horizon infinite, but it contains the equatorial plane beyond the toroidal magnetosphere. The source of the magnetic field is, however, within the toroidal magnetosphere.

IV. CONCLUSIONS

When the possibility of strong electromagnetic fields around an accreting black hole was first proposed,[13] it was thought that the selective accretion of oppositely charged particles would neutralize the charge of the black hole on a short time scale. That argument was based on the dominance of the electric field and did not take into account the possibility of external magnetic fields.

Figures 1, 2, and 3 delineate the regions in which the electric field of a charged black hole can be balanced by asymptotically uniform weak magnetic fields. While these examples manifest the restrictions of the selective-accretion argument, they are only a first step toward demonstrating the possibility of trapping charged particles in the magnetosphere of a collapsed object.

Ruffini and Wilson[14] gave an explicit example of charge separation and strong electromagnetic fields near the surface of a collapsed object accreting plasma of infinite conductivity. The possibility of charge separation allows for the overall neutrality of the system, which screens it from selective accretion.

The flow lines derived in this paper provide a global criterion for the trapping of charged particles in the magnetosphere of a black hole. The idealized models considered here are indicative of the processes occurring in real astrophysical systems.

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[1]For a review of these topics see, e.g., *Neutron Stars, Black Holes and Binary X-Ray Sources*, edited by M. Gursky and R. Ruffini (Reidel, Dordrecht, 1975).
[2]For a review of these topics and their observational consequences see, e.g., *Physics and Astrophysics of Neutron Stars and Black Holes*, edited by R. Giacconi and R. Ruffini (North-Holland, Amsterdam, 1977).
[3]W. Lewin, *Physics and Astrophysics from Spacelab*, edited by P. Bernacca and R. Ruffini (Reidel, Dordrecht, 1977).
[4]L. Landau and E. Lifshitz, *Classical Theory of Fields*, 3rd edition (Pergamon, Oxford, 1971).
[5]Greek indices run from 0 to 3; Latin indices run from 1 to 3; $G=c=1$.
[6]See R. Ruffini in Ref. 2.
[7]D. Christodoulou and R. Ruffini, in *Black Holes*, edited by B. DeWitt and C. DeWitt (Gordon and Breach, New York, 1973).
[8]See Ref. 4.
[9]J. R. Wilson, in *Proceedings of the Marcel Grossman Meeting*, edited by R. Ruffini (North-Holland, Amsterdam, 1977); J. R. Wilson in Ref. 2.
[10]F. J. Ernst, J. Math. Phys. 17, 54 (1976).
[11]B. Linet, J. Phys. A 9, 1081 (1976).
[12]R. S. Hanni, in *Proceedings of the Marcel Grossman Meeting*, edited by R. Ruffini (North-Holland, Amsterdam, 1977).
[13]R. Ruffini, in *Black Holes*, edited by B. DeWitt and C. DeWitt (Gordon and Breach, New York, 1975).
[14]R. Ruffini and J. R. Wilson, Phys. Rev. D 12, 2959 (1975).