Simple physical interpretation of a neutrino radiation solution in general relativity

T. M. Davis
Department of Physics, North Georgia College, Dahlonega, Ga. 30533

J. R. Ray
Department of Physics and Astronomy, Clemson University, Clemson, S.C. 29631
(Received 5 December 1977)

We present a detailed study of an exact neutrino radiation solution that we have previously derived. We show that, after a coordinate transformation, the solution has a simple physical interpretation. The solution represents neutrino radiation propagating on, but not otherwise influencing, a static gravitational field.

I. INTRODUCTION

In this paper we continue our study of neutrinos in general relativity. Exact solutions to the Einstein-Dirac equations are important in pointing out explicit properties of the theory.

Previously, we have studied neutrinos in spacetimes with plane symmetry.[1] We were able to solve the Einstein-Dirac equations exactly for this case. At that time we did not present a discussion of the properties of the solution so obtained but only studied the special "ghost neutrino" solution.[2] Here we return to this exact solution and prove it can be written in a remarkably simple form that has a clear and interesting physical interpretation. The form of the solution is for a neutrino wave propagating on a curved static background gravitational field, but having the amazing property of leaving the curved background the same as when the wave is not present. It came as a surprise to us that our general solution could be written in such a physically transparent form. In Sec. II we review the solution from Ref. 1, while in Sec. III we derive the new form for the solution. Section IV considers the physical interpretation of the solution by studying geodesic deviation and transfer of energy. Finally, in Sec. V we give our conclusions along with suggestions for further work.

II. ORIGINAL SOLUTION

We refer to Ref. 1 for our notation. The general form for the plane-symmetric metric is

$$ds^2 = e^{2u}(dx^2 - dt^2) + e^{2v}(dy^2 + dz^2), \tag{2.1}$$

where u and v are functions of (x, t). The x axis is the symmetry axis. The exact neutrino solution to the Einstein-Dirac equations found in Ref. 1 is

$$ds^2 = \frac{4}{k^2}\dot{\alpha}\dot{\beta}(\alpha+\beta)^{-1/2}e^{C}(dx^2 - dt^2) + (\alpha+\beta)(dy^2+dz^2), \tag{2.2}$$

where k is a constant; α is an arbitrary function of $(x+t)$; β is an arbitrary function of $(x-t)$, and C is an arbitrary function of $(x+t)$. The function C is related to the neutrino wave function. The dots over α and β in (2.2) mean differentiation with respect to their arguments. For this solution the neutrinos are propagating down the $(-x)$ axis. There is also a different solution of the same form as (2.2) having C as a function of $x-t$ for neutrinos propagating down the $(+x)$ axis. We shall discuss only (2.2).

The other quantities of interest such as the neutrino wave function, conserved current, and energy-momentum tensor can be found in Ref. 1. We shall not repeat these here.

As shown in Ref. 1 the Taub vacuum plane-symmetry solution is obtained from (2.2) by setting $C=0$ (no real neutrinos) and performing the coordinate transformation

$$x' + t' = \frac{2}{k}(\alpha - \tfrac{1}{2}), \tag{2.3a}$$

$$x' - t' = \frac{2}{k}(\beta - \tfrac{1}{2}), \tag{2.3b}$$

which yields, after dropping the primes,

$$ds^2 = (kx+1)^{-1/2}(dx^2 - dt^2) + (kx+1)(dy^2+dz^2). \tag{2.4}$$

Equation (2.4) is the Taub vacuum gravitational field.

III. NEW FORM FOR THE SOLUTION

If we apply the coordinate transformation (2.3) to (2.2) then we obtain, after dropping the primes,

$$ds^2 = (kx+1)^{-1/2}e^{C(x+t)}(dx^2 - dt^2) \quad (kx+1)(dy^2+dz^2). \tag{3.1}$$

Here we see that the neutrinos show up only in the

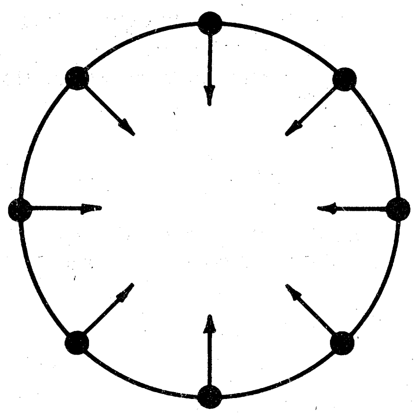

FIG. 1. The motion of a group of test particles, initially at rest, after the neutrinos pass through them. The motion of the test particles is due to the gravitational field carried by the neutrinos. The neutrinos are traveling into the paper.

e^C term in the metric. If we "turn the neutrinos off" then the metric (3.1) changes in a smooth way into the vacuum gravitational field (2.4). Thus we can interpret (3.1) as a neutrino wave propagating on the static vacuum solution. The gravitational effects of the neutrinos are given by the e^C term in the metric. It should be noted that the e^C term in the metric represents the gravitational field being carried along by the neutrinos. This propagating gravitational field can easily be distinguished from pure gravitational radiation. The easiest way to see this is to show that the gravitational field associated with (3.1) does not have the shearing property of gravitational radiation (see Fig 1).

Equation (3.1) describes the gravitational effects of the neutrino radiation. The neutrino energy-momentum tensor is

$$T_{11} = T_{44} = T_{14} = \frac{c^4 e^{-C} k\dot{C}}{8\pi\kappa(kx+1)^{1/2}}, \tag{3.2}$$

where κ is the Newtonian gravitational constant. In order to have positive energy ($T_{44}>0$), we require $k\dot{C}>0$. This then implies an energy flow (T^{14}) down the $(-x)$ axis. Equation (3.2) is written in the orthonormal frame for the metric (3.1). The gravitational energy associated with the neutrino radiation is obtained by calculating the pseudotensor for the metric (3.1). We present this in the next section. For a detailed discussion of the physical interpretation of classical neutrino solutions we refer the reader to Ref. (3).

IV. PHYSICAL INTERPRETATION

We first study geodesic deviation for the metric (3.1). Since we are interested in the effects of the neutrinos we shall subtract out the static-background field effects, that is, we shall calculate quantities

$$Q(\nu) = Q(C \neq 0) - Q(C=0), \tag{4.1}$$

where Q is some physical quantity, and C is the neutrino C in (3.1).

The equation for geodesic deviation is

$$\ddot{\eta}^i = R^i{}_{jkl} U^j U^k \eta^l, \tag{4.2}$$

where η^i is the connecting vector, U^i the four-velocity, and $R^i{}_{jkl}$ the Riemann tensor. Taking the particles to be initially at rest, the four-velocity has only a time component U^4. If we calculate $\ddot{\eta}^i$ due only to the neutrinos via (4.1), we find that

$$\ddot{\eta}^1(\nu) = 0, \tag{4.3a}$$

$$\ddot{\eta}^\alpha(\nu) = -(U^4)^2 \frac{\dot{C}k}{4}(kx+1)^{-1}\eta^\alpha, \quad \alpha = 2,3. \tag{4.3b}$$

Hence, the effect of the gravitational field carried by the neutrino is transverse, and the particles are affected in the same way as if a scalar gravitational wave had passed through them (see Fig. 1).

As another physical quantity associated with (3.1) we study the gravitational energy associated with the neutrino beam. Calculating the Landau-Lifshitz pseudotensor, we obtain, for the energy flux t^{14},

$$t^{14} = \frac{-k\dot{C}}{2}(kx+1)^{1/2} e^C, \tag{4.4}$$

which shows that there is gravitational energy associated with the propagating neutrino beam. Of course, we can also deduce this from the geodesic deviation results. The motion of the particles in Fig. 1 is associated with the effects of the gravitational energy carried by the neutrinos since we have subtracted out the background gravitational effects.

V. CONCLUSIONS

It is sometimes easier to find solutions in general relativity than to give a physical interpretation of the solutions. Here we have presented a case where the opposite statement is true. The neutrino radiation gravitational field in (3.1) has a very simple physical interpretation as we have shown in the last section. The most interesting thing about the solution is the way the neutrinos ride on top of the background field without changing it. This is the only case where this occurs, as far as we are aware. It is possible that this solution would have other uses in studies where it is important to separate a static gravitational field from the dynamical variables in the solution. Perhaps this solution would be useful to those studying quantum field theory on curved backgrounds. The neutrino field could be quantized while leaving the background field fixed. Other possible applications might be in studying perturbation solutions on curved backgrounds. It would also be interesting to have the general solution for a static background with neutrinos, of which the present solution is an example.

---

[1]T. M. Davis and J. R. Ray, J. Math. Phys. 16, 75 (1975).

[2]T. M. Davis and J. R. Ray, Phys. Rev. D 9, 331 (1974).

[3]J. R. Ray, Lett. Nuovo Cimento (to be published).