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Unitarity, Ward identities, and new quantization rules of supergravity

George Sterman, Paul K. Townsend, and Peter van Nieuwenhuizen
Institute for Theoretical Physics, State University of New York at Stony Brook, Stony Brook, New York 11794
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Using diagrammatic techniques, we prove that supergravity is unitary to all orders in perturbation theory, subject to a generalized Becchi-Rouet-Stora (BRS) invariance of the effective action. In particular we show that new nonphysical modes in the propagator, not canceled by ghosts, decouple from the S matrix through appropriate Ward identities. We establish the generalized BRS invariance to order κ^3 and give a general recipe for extension to higher order. This new invariance requires new quantization rules for supergravity: A quartic supersymmetry ghost coupling must be added to the action. The same term has been found previously by Fradkin and Vasiliev from a Hamiltonian formulation of supergravity.

I. INTRODUCTION

The renormalizability and unitarity of a quantum field theory can be analyzed independently from each other. Most work on quantum supergravity has dealt with renormalizability, and spectacular improvements over quantized Einstein gravity have been found.[1] In this article we turn to the equally important question of unitarity.

At first sight there is no problem. One merely has to choose a gauge in which ghosts do not propagate (a so-called unitary gauge). In this gauge, unitarity is manifest and unitarity in any other gauge follows by invoking gauge invariance of the S matrix. For all other gauge theories this is true,[2] but not for supergravity. There are two surprises.

The following simple question illustrates that in supergravity unitarity has unusual features.[3] How does supergravity reduce the eight degrees of freedom of the spin-$\frac{3}{2}$ fermion to its two physical modes? For ordinary gravity the answer is easy[4]: Eight of the ten degrees of freedom in the graviton propagator (corresponding to the ten fields $g_{\mu\nu}$) are eliminated by the complex gravitational vector ghost field. But in supergravity there is a complex spin-$\frac{1}{2}$ ghost field[5,6]; hence, it is surprising that only four of the eight degrees of freedom of the spin-$\frac{3}{2}$ fermion can be eliminated by ghosts. (Fermions have half as many degrees of freedom as components because the other half are canonical momenta.) Where do the remaining two modes go?

We will show that, as a consequence of Ward identities, these two unphysical modes decouple from the theory. This is superficially similar to the decoupling of longitudinal and timelike polarizations in QED, but on closer inspection the similarity disappears. The unphysical polarizations in QED are really canceled by ghosts which just happen to decouple by themselves in a flat space, but as soon as we consider the interaction with gravity, the QED ghosts no longer decouple, and are needed for unitarity. Here, however, the extra two unphysical polarizations have no corresponding ghosts, and the mechanism by which they decouple is unique to supergravity.

The second surprise is that the functional procedure[7] for obtaining Ward identities, which are necessary both for unitarity and gauge invariance of the S matrix, cannot be used. This is due to the well-known nonclosure of the supergravity gauge algebra off the mass shell.[8] The derivation in Ref. 7 relies on closure, since the group property of gauge transformations is needed to prove the invariance of the product of functional measure and Faddeev-Popov determinant under the well-known class of restricted gauge transformations. Because of this, we use a different method, due to Becchi, Rouet, and Stora (BRS),[9] in order to prove the Ward identities we need for unitarity. These authors discovered that the effective quantum action containing gauge-fixing and ghost Lagrangians possesses a new invariance with a constant anticommuting parameter, and that this invariance enables one to obtain Ward identities and renormalizability proofs[10] in a very compact way.

One year ago two of us studied BRS transformations for supergravity[11] and found that the effective action was only invariant under these transformations when the classical equations of motion were used. We now understand this as a consequence of the nonclosure of the gauge algebra and can explain the many puzzling regularities then found from the fact that BRS invariance is an expression of gauge invariance *and* the group composition rule. We generalize the BRS method to include the case when the gauge transformations on the field do not form a group, as in supergravity, and we find that new quartic couplings of the ghosts are necessary for BRS invariance. Since the Jacobian of the BRS transformations is not unity, we also require an explicit factor of g^{-2} in the functional integral.

Our conclusion is that new quantization rules for supergravity are needed in order to ensure unitarity of the S matrix. They are summarized by the generating functional,

$$Z=\int (d\phi)g^{-2}\exp\left\{i\int d^4x\left[\mathcal{L}_{\rm inv}+\mathcal{L}_{\rm fix}+\mathcal{L}_{\rm FP}-\tfrac{5}{8}\kappa^2(B\gamma^\nu\hat{B})(\hat{C}\gamma_\nu C)+O(\kappa^3)\right]\right\}, \tag{1.1}$$

where C (B) is the ghost (antighost) of local supersymmetry and a caret denotes Majorana conjugation. This result was first obtained by Fradkin and Vasiliev[12] in an important article using canonical quantization in the Hamiltonian formalism. It is encouraging that our method based on covariant perturbation theory leads to new quantization rules which completely agree with those obtained in the Hamiltonian method because the methods are so very different. The appearance of quartic ghost couplings means that the ghost Lagrangian can no longer be written as a determinant, so that the BRS method is essential to our work.

Also, Batalin and Vilkovisky[13] have concluded that in theories with fermionic gauge invariances new quartic ghost couplings are needed. These authors have also studied BRS invariance in the Hamiltonian formalism but, according to Fradkin and Vasiliev,[12] their results are not sufficiently general to apply to supergravity.

The organization of this article is as follows. In Sec. II we discuss degrees of freedom in classical supergravity. In Sec. III we discuss degrees of freedom in the spin-$\frac{3}{2}$ fermion propagator. We find the propagator in the "unitary" gauge and show the existence of the extra unphysical modes both in the renormalizable and the "unitary" propagator. In Sec. IV we derive Ward identities assuming BRS invariance. For unitarity we need Ward identities for cut diagrams[14] and these are derived as well. In Sec. V we come to unitarity. Using the Ward identities we prove unitarity to all orders in perturbation theory following 't Hooft and Veltman's diagrammatic methods for Yang-Mills theory[15] and exhibit the cancellation mechanism by ghosts and the decoupling mechanism for the unphysical modes that are not canceled by ghosts. Finally we come to BRS invariance in Sec. VI. We find that the failure of "naive" BRS invariance leads to a violation of unitarity which necessitates new covariant quantization rules with quartic ghost couplings. The new effective action possesses a generalized BRS invariance which we exhibit to order κ^3 and we give a recipe for extension to higher orders. We stress that our proof of unitarity only needs the BRS transformations to order κ.

Before proceeding we present our conventions and notations for the effective supergravity action in the renormalizable gauge. The classical action is

$$\mathcal{L}_{\rm inv}=-\frac{1}{2\kappa^2}eR-\tfrac{1}{2}\epsilon^{\lambda\rho\mu\nu}\bar{\psi}_\lambda\gamma_5\gamma_\rho D_\mu\psi_\nu \tag{1.2}$$

and is invariant under $\delta\psi_\mu=(2/\kappa)D_\mu\epsilon$, $\delta e_{a\mu}=\kappa\bar{\epsilon}\gamma_a\psi_\mu$, where

$$D_\mu=\partial_\mu+\tfrac{1}{2}\omega_{\mu ab}(e,\psi)\sigma^{ab},$$
$$\omega_{\mu ab}(e,\psi)=\omega_{\mu ab}(e)+\frac{\kappa^2}{4}(\bar{\psi}_a\gamma_\mu\psi_b+\bar{\psi}_\mu\gamma_a\psi_b-\bar{\psi}_\mu\gamma_b\psi_a). \tag{1.3}$$

The effective quantum action is

$$\mathcal{L}_{\rm eff}=\mathcal{L}_{\rm inv}-\tfrac{1}{4}(\partial_\mu\tilde{h}^{\mu\nu})^2-\frac{\alpha}{\kappa^2}(h_{a\mu}-h_{\mu a})^2+\tfrac{1}{4}\bar{\psi}\cdot\bar{\gamma}\not{\partial}\bar{\gamma}\cdot\psi+\bar{\mathcal{C}}M\mathcal{C}, \tag{1.4}$$

where

$$\bar{\gamma}_\mu=\delta^a_\mu\gamma_a,\quad \tilde{h}^{\mu\nu}=\tfrac{1}{2}(h_{\mu\nu}+h_{\nu\mu}-\delta_{\mu\nu}h^\alpha_\alpha),\quad h_{\mu\nu}=h_{a\nu}\delta^a_\mu,$$

and $h_{a\mu}$ is the gravitational quantum field obtained by expanding $e_{a\mu}$ as

$$e_{a\mu}=\delta_{a\mu}+\tfrac{1}{2}\kappa h_{a\mu}-\tfrac{1}{32}\kappa^2(h_{a\lambda}+h_{\lambda a})(h_\mu{}^\lambda+h^\lambda{}_\mu)+\cdots, \tag{1.5}$$

which corresponds to $g_{\mu\nu}=\delta_{\mu\nu}+\kappa h_{\mu\nu}$ when $h_{a\mu}$ is symmetric. If $e_{a\mu}$ is split into symmetric and antisymmetric parts, one sees that our choice of gauge-fixing terms coincides with the conventional choice up to terms linear in the fields, so that the propagators are the conventional ones.

M is the Faddeev-Popov ghost matrix for the ghosts $\mathcal{C}=\{C^\nu, C_{ab}, C\}$ and antighosts $\bar{\mathcal{C}}=\{C^{\nu}{}^*, C^*_{ab}, B\}$ of general coordinate, local Lorentz symmetry, and local supersymmetry, respectively.

In (1.4) α is an arbitrary constant and we will find it convenient to allow $\alpha\to\infty$, which gives an unweighted gauge with $e_{a\mu}=e_{\mu a}$. This freezes the antisymmetric part of $e_{a\mu}$ out of the theory, but the ghosts C_{ab} and C^*_{ab} remain. Since they do not propagate they can be eliminated by solving the C^*_{ab} field equation, which is

$$C_{ab}=\tfrac{1}{2}(C_{b,a}-C_{a,b})+O(\kappa)\,, \tag{1.6}$$

where $C_b=C^\nu\delta_{\nu b}$ and the $O(\kappa)$ terms are also of higher order in fields. C^*_{ab} now drops from the action since it only appears in $\overline{\mathcal{C}}M\mathcal{C}$ as a Lagrange multiplier. In this gauge the Faddeev-Popov ghost Lagrangian is simply

$$C^{\nu*}\Box C_\nu+2B\bar\gamma^\mu D_\mu C+\kappa C^{\nu*}\partial^\mu(C^\lambda\tilde h_{\mu\nu,\lambda}+C^\lambda_{,\mu}h_{\nu\lambda}+C^\lambda_{,\nu}h_{\mu\lambda}-\delta_{\mu\nu}C^\lambda_{,\lambda}-\bar\psi_\mu\gamma_\nu C-\bar\psi_\nu\gamma_\mu C+\bar\psi\cdot\gamma C\delta_{\mu\nu})$$
$$+\kappa B\bar\gamma^\mu(-C^\lambda\psi_{\mu,\lambda}-C^\lambda_{,\mu}\psi_\lambda-\tfrac{1}{2}\sigma^{ab}C_{ab}\psi_\mu)+O(\kappa^2)\,. \tag{1.7}$$

The new quartic ghost couplings have not yet been included in (1.4). Note that the dimensions of the ghosts have been chosen to be

$$[C^\nu]=[C^{\nu*}]=1,\quad [C^*_{ab}]=[C_{ab}]=2,\quad [B]=[C]=\tfrac{3}{2}, \tag{1.8}$$

so that the kinetic ghost Lagrangian contains no factors of κ. Our other conventions are

$$\gamma_\mu{}^2=\gamma_5{}^2=1,\quad \mu=1,2,3,4\quad \epsilon^{1234}=\epsilon_{1234}=+1\,,$$
$$\gamma_5=\gamma_1\gamma_2\gamma_3\gamma_4,\quad \sigma_{\mu\nu}=\tfrac{1}{4}(\gamma_\mu\gamma_\nu-\gamma_\nu\gamma_\mu)\,, \tag{1.9}$$
$$\delta_{\mu\nu}=(+,+,+,+),\quad a\cdot b=\vec a\cdot\vec b+a_4b_4,$$

and in Sec. III we use the Dirac matrix conventions

$$\gamma_k=\begin{pmatrix}0 & -i\sigma_k\\ i\sigma_k & 0\end{pmatrix},\quad \gamma_4=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}. \tag{1.10}$$

II. DEGREES OF FREEDOM IN THE CLASSICAL THEORY

Classically there are two degrees of freedom in the real massless spin-$\frac{3}{2}$ field ψ_μ. The counting proceeds as follows. The classical equation of motion

$$\epsilon^{\mu\nu\rho\sigma}\gamma_5\gamma_\nu\partial_\rho\psi_\sigma=0 \tag{2.1}$$

is seen to be equivalent (after multiplication by γ_μ) to the following two equations:

$$\not\partial\psi_\mu=\partial_\mu\gamma\cdot\psi\,, \tag{2.2}$$
$$\partial\cdot\psi=\not\partial\gamma\cdot\psi\,. \tag{2.3}$$

Multiplying the first of these by $\not\partial$ one finds for a free massless field ψ_μ with $p^2=0$ that $\not\partial\gamma\cdot\psi=0$, and inserting this result into the last equation, one obtains the *gauge-independent equation* $\partial\cdot\psi=0$. Thus ψ_μ can be represented in momentum space as

$$\psi_\mu=\epsilon^+_\mu\lambda+\epsilon^-_\mu\chi+p_\mu\phi\,, \tag{2.4}$$

where λ, χ, and ϕ are spinors and $\epsilon^\pm_\mu$ the helicity ±1 polarization tensors of a photon.

Since the free field theory is invariant under the linearized gauge transformations

$$\psi_\mu\to\psi_\mu+p_\mu\eta\,, \tag{2.5}$$

one can choose the gauge $\gamma\cdot\psi=0$ by fixing $\eta=-\not p^{-1}\gamma\cdot\psi$.

As usual, *the gauge "shoots twice."* After one has used it in the inhomogeneous equation $\not p\eta=-\gamma\cdot\psi$, one can still use it once more in the homogeneous form, $\not p\eta=0$, in order not to undo the condition $\gamma\cdot\psi=0$. From $\gamma\cdot\psi=0$, Eq. (2.2) yields $\not p\psi_\mu=0$; hence $\not p\lambda=\not p\chi=\not p\phi=0$ in Eq. (2.4) since $\epsilon^\pm_\mu$ and p_μ are independent vectors. Thus, choosing $\eta=-\phi$ in Eq. (2.5), one can gauge away the longitudinal mode $p_\mu\phi$ in ψ_μ, leaving

$$\psi_\mu=\alpha\epsilon^+_\mu u^++\beta\epsilon^-_\mu u^++\gamma\epsilon^+_\mu u^-+\delta\epsilon^-_\mu u^-\,, \tag{2.6}$$

where u^+ and u^- are solutions of the Dirac equation with helicities $\pm\frac{1}{2}$. However, as the reader may verify explicitly, $\gamma_\mu(\epsilon^\pm_\mu u^\pm)=0$, $\gamma_\mu(\epsilon^\pm_\mu u^\mp)=\sqrt{2}iu^\pm$ so that there are only two degrees of freedom in ψ_μ. These results are a detailed version of the treatment in Ref. 16. (Reality requires $\alpha=\delta^*$.)

In quantum field theory, the in- and out-fields correspond to the free fields above, and although free fields change under gauge transformations, their changes are eliminated if they are coupled to a gauge-invariant S matrix. The conservation of the S matrix in the world indices of ψ_μ is proved from the Ward identities which we derive in this article.

The physical modes of the classical theory in the gauge $\gamma\cdot\psi=0$ propagate according to the propagator[5] [see also (2.6)]

$$p^2P^{\text{phys},3/2}_{\mu\nu}=[(\epsilon^+_\mu u^+)(\epsilon^{+*}_\nu\bar u^+)+(\epsilon^-_\mu u^-)(\epsilon^{-*}_\nu\bar u^-)]E$$
$$=\tfrac{1}{2}\bar\delta_{\nu\alpha}(\gamma_\alpha\not p\gamma_\beta)\bar\delta_{\beta\mu}\,, \tag{2.7}$$

where

$$\bar\delta_{\mu\nu}=\epsilon^+_\mu\epsilon^{+*}_\nu+\epsilon^-_\mu\epsilon^{-*}_\nu$$
$$=\delta_{\mu\nu}-(p_\mu\bar p_\nu+p_\nu\bar p_\mu)(p\cdot\bar p)^{-1} \tag{2.8}$$

and $\bar p_\mu=(\vec p,-p_4)$. The simple form in the second line of (2.7) is new.

In Yang-Mills theory the physical propagator corresponds also to a sum over the two physical helicities ± 1 and is given by

$$p^2 P^{\text{phys},1}_{\mu\nu} = -i\bar{\delta}_{\mu\nu}\,, \tag{2.9}$$

while in Einstein gravity the physical propagator is a sum over the physical helicities ± 2 and is given by

$$p^2 P^{\text{phys},2}_{\mu\nu,\rho\sigma} = -2i(\bar{\delta}_{\mu\rho}\bar{\delta}_{\nu\sigma} + \bar{\delta}_{\mu\sigma}\bar{\delta}_{\nu\rho} - \bar{\delta}_{\mu\nu}\bar{\delta}_{\rho\sigma})\,. \tag{2.10}$$

The renormalizable propagators for spin 2, $\frac{3}{2}$, and 1 are uniformly obtained from the three formulas above by the substitution $\bar{\delta}_{\mu\nu} \to \delta_{\mu\nu}$. These renormalizable propagators are not obtained in the unweighted gauges $\gamma\cdot\psi = 0$, etc. but by adding the squares of these expressions to the action with particular coefficients, as indicated in Eq. (1.4).

III. DEGREES OF FREEDOM IN THE QUANTUM PROPAGATOR

The quantum propagators are obtained by adding gauge-fixing terms to the classical action which contain terms quadratic in the classical fields in such a way that the kinetic matrix becomes regular. We will consider two choices for the spin-$\frac{3}{2}$ field, which we call the renormalizable gauge and the unitary gauge,

$$\mathcal{L}^{\text{ren}}_{\text{fix}} = \tfrac{1}{4}\bar{\psi}\cdot\gamma\not{\partial}\gamma\cdot\psi \quad \mathcal{L}^{\text{un}}_{\text{fix}} = \tfrac{1}{4}\bar{\psi}\cdot\hat{\gamma}\not{\partial}\hat{\gamma}\cdot\psi\,. \tag{3.1}$$

Here $\hat{\gamma}_\mu = \gamma_\mu(1-\delta_{\mu 4})$ and $\gamma\cdot\psi = \gamma_\mu\psi_\mu$. Since we consider terms quadratic in fields only, we need not specify whether γ_μ is field dependent or not. The coefficient $\frac{1}{4}$ in the renormalizable gauge-fixing term is carefully chosen[6]; a different value gives a rather complicated propagator. The choice of the unitary gauge-fixing term is strongly suggested by the unitary gauges in Yang-Mills and Einstein theory, where one replaces four-dimensional derivatives by three-dimensional ones,

$$-\tfrac{1}{2}(\partial_\mu W^a_\mu)^2 \to -\tfrac{1}{2}(\partial_i W^a_i)^2, \tag{3.2}$$

and similarly for gravity.[2] We do not consider the alternative $\partial\cdot\psi$ instead of $\gamma\cdot\psi$, since this gauge has the drawback of leading to extra singularities in the propagator, proportional to p^{-4}.[5]

The set of renormalizable gauges we use as a model below are the gauges used in explicit calculations.[1] For spin 2, $\frac{3}{2}$ and 1 we use

$$\mathcal{L}^{\text{ren}}_{\text{fix}} = -\tfrac{1}{4}(\partial_\mu \tilde{h}^{\mu\nu})^2 + \tfrac{1}{4}\bar{\psi}\cdot\gamma\not{\partial}\gamma\cdot\psi - \tfrac{1}{2}(\partial_\mu A_\mu)^2 \tag{3.3}$$

and with the invariant actions in second-order form for real spin 2, $\frac{3}{2}$, 1, $\frac{1}{2}$, and 0 fields normalized as

$$\mathcal{L}^{\text{inv}} = -(2\kappa^2)^{-1} e R(e) - \tfrac{1}{2}\epsilon^{\mu\nu\rho\sigma}\bar{\psi}_\mu\gamma_5\gamma_\nu D_\rho\psi_\sigma - \tfrac{1}{4}eF_{\mu\nu}F^{\mu\nu} - \tfrac{1}{2}e\bar{\lambda}\not{D}\lambda - \tfrac{1}{2}e(\partial_\mu\phi)(\partial^\mu\phi)\,, \tag{3.4}$$

the propagators are given by

$$P^{(2)}_{\mu\nu,\rho\sigma} = 2(\delta_{\mu\rho}\delta_{\nu\sigma} + \delta_{\mu\sigma}\delta_{\nu\rho} - \delta_{\mu\nu}\delta_{\rho\sigma})(-ip^{-2})\,, \tag{3.5}$$

$$P^{(1)}_{\mu\nu} = \delta_{\mu\nu}(-ip^{-2}), \quad P^{(0)} = (-ip^{-2})\,, \tag{3.6}$$

$$P^{(3/2)}_{\mu\nu} = \tfrac{1}{2}(\gamma_\nu\not{p}\gamma_\mu)p^{-2}, \quad P^{(1/2)} = -\not{p}\,p^{-2}\,. \tag{3.7}$$

Note the reverse order of the μ,ν indices in $P^{(3/2)}$. Other choices, for example $\mathcal{L}^{\text{ren}}_{\text{fix}} = (\partial_\mu g_{\mu\nu})^2$ or $(\partial_\mu e_{a\mu})^2$ where $e_{a\mu}$ is the vierbein field, yield cumbersome propagators. Since we work in the second-order formalism,[8] $\omega_{\mu ab}$ is not an independent field,[17] while the local Lorentz gauge is fixed by the unweighted gauge $-(\alpha/\kappa^2)(h_{a\mu} - h_{\mu a})^2$ with ultimately $\alpha\to\infty$. Thus $e_{a\mu}$ is symmetric. We can decompose the renormalizable propagator $\frac{1}{2}\gamma_\nu\not{p}\gamma_\mu p^{-2}$ on-shell into eight channels:

$$\begin{aligned} p^2 P^{(3/2)}_{\mu\nu} = &-i[(\epsilon^+_\mu u^+)(\epsilon^{+*}_\nu \bar{u}^+) + (\epsilon^-_\mu u^-)(\epsilon^{-*}_\nu\bar{u}^-)]E \\ &+\frac{i}{\sqrt{2}}[(\epsilon^+_\mu w^-)(\bar{p}^*_\nu\bar{u}^+) + (p_\mu u^-)(\epsilon^{-*}_\nu\bar{w}^+) \\ &\qquad -(\epsilon^-_\mu w^+)(\bar{p}^*_\nu\bar{u}^-) - (p_\mu u^+)(\epsilon^{+*}_\nu\bar{w}^-)] \\ &-\frac{i}{2E}[(p_\mu w^+)(\bar{p}^*_\nu\bar{w}^+) + (p_\mu w^-)(\bar{p}^*_\nu\bar{w}^-)]\,, \end{aligned} \tag{3.8}$$

where $E>0$ is the energy and where the four linearly independent spinors which span spinor space are given by

$$\begin{aligned} &u^+ = (1,0,1,0), \quad u^- = (0,1,0,-1), \quad w^\pm = \gamma_4 u^\pm\,, \\ &w^+ = (1,0,-1,0), \quad w^- = (0,1,0,1), \quad \not{p}w^\pm = 0\,. \end{aligned} \tag{3.9}$$

This result was obtained by substituting the decomposition of $\delta_{\mu\nu}$ in Eq. (2.8) into the following form of the renormalizable propagator:

$$P^{3/2}_{\mu\nu} = \tfrac{1}{2}\delta_{\nu\alpha}(\gamma_\alpha\not{p}\gamma_\beta)\delta_{\beta\mu}\,, \tag{3.10}$$

making the following *on-shell* expansions

$$\begin{aligned} &\not{p} = (u^+\bar{u}^+ + u^-\bar{u}^-)(iE)\,, \\ &\bar{\not{p}} = (w^+\bar{w}^+ + w^-\bar{w}^-)(-iE)\,, \\ &\epsilon^\pm = (\mp w^\pm\bar{u}^\mp \pm u^\pm\bar{w}^\mp)(i/\sqrt{2})\,, \end{aligned} \tag{3.11}$$

and using the orthogonality of the u's and w's.

This form of $P^{(3/2)}_{\mu\nu}$ was to be expected. Since the square of $p^2P^{(3/2)}_{\mu\nu}$ as a 16×16 matrix is proportional to $\delta_{\mu\nu}p^2$, its rank is on-shell at most eight, similar to the fact that the rank of $\not{p}$ is on-shell at most 2. This may be shown by first bringing $p^2P_{\mu\nu}$ into Jordan canonical form. It is in fact *equal* to 8, as one sees from Eq. (3.8). Since $P^{(3/2)}_{\mu\nu}$ conserves helicity, it is block-diagonal in the 2, 6, 6, 2 dimensional subspaces with helicities $\lambda = \frac{3}{2}$, $\frac{1}{2}$, $-\frac{1}{2}$, and $-\frac{3}{2}$, respectively. It is in fact diagonal in the $\lambda = \pm\frac{3}{2}$ subspaces, and has rank 3 in each of the $\lambda = \pm\frac{1}{2}$ spaces (since there is a regular 3×3 submatrix in each). The problem now is to show that the four modes bilinear in ϵ and p are eliminated by ghosts, whereas the two modes bilinear in p and $\bar{p}^*$ decouple separately.

Turning to the unitary propagator, the problem is to invert the spin-$\frac{3}{2}$ field equation $(\delta/\delta\bar{\psi}_\mu)[\mathcal{L}^{\text{inv}} + \mathcal{L}(\text{fix, un})]$. From the explicit form of the corresponding result in Einstein gravity,[2] one does not expect a simple result. We solved this problem by expanding the most general form for the propagator in 20 covariants O_i which depended upon p_μ, $\bar{p}_\mu$, $\delta_{\mu\nu}$, and $\epsilon_{\mu\nu\rho\sigma}$. By means of a numerical FORTRAN program and inverting one 16×16 and one 20×20 matrix, most of the coefficients of O_i were found in analytical form, while the algebraic program SCHOONSCHIP yielded the remaining coefficients in analytical form. The result is as follows:

$$\bar{P}_{\alpha\sigma} = p^{-2}\left[-\tfrac{1}{2}\not{p}\delta_{\alpha\sigma} + \left(\frac{l^2+t^2}{8l^2t^2}\right)\not{p}\,p_\alpha\bar{p}_\sigma + \left(\frac{3l^2-t^2}{8l^2t^2}\right)\not{p}\,\bar{p}_\alpha p_\sigma - \left(\frac{l^2+t^2}{8l^2t^2}\right)\epsilon_{\alpha\sigma\rho\tau}p_\rho\bar{p}_\tau\not{p}\gamma_5 + \left(\frac{1}{2t^2}+\frac{3}{2l^2}\right)\bar{\not{p}}p_\alpha p_\sigma\right]$$
$$+\left(\frac{l^2+t^2-\not{p}\bar{\not{p}}}{8l^2t^2}\right)(\gamma_\alpha p_\sigma - p_\alpha\gamma_\sigma - \gamma_\alpha\bar{p}_\sigma + \bar{p}_\alpha\gamma_\sigma) - \frac{\not{p}p_\alpha p_\sigma}{2l^2t^2}$$
$$-\left(\frac{1}{8l^2t^2}\right)\bar{\not{p}}p_\alpha\bar{p}_\sigma - \left(\frac{3}{8l^2t^2}\right)\bar{\not{p}}\bar{p}_\alpha p_\sigma + \left(\frac{1}{8l^2t^2}\right)\epsilon_{\alpha\sigma\rho\tau}p_\rho\bar{p}_\tau\bar{\not{p}}\gamma_5\,, \quad 4l^2=(p+\bar{p})^2,\quad 4t^2=-(p-\bar{p})^2\,. \tag{3.12}$$

Only one thing is of importance for our work. On-shell ($p^2=0$, $l^2=t^2$), the residue of the pole is given by

$$(p^2\bar{P}_{\alpha\sigma})\big|_{p^2=0} = \tfrac{1}{2}\bar{\delta}_{\sigma\sigma'}\gamma_{\sigma'}\not{p}\gamma_{\alpha'}\bar{\delta}_{\alpha'\alpha} + 4(p\cdot\bar{p})^{-1}\bar{\not{p}}p_\alpha p_\sigma\,. \tag{3.13}$$

Thus, in addition to the two physical modes, a third scalar mode, proportional to $p_\alpha p_\sigma$, is propagated, and it is to be shown that this mode decouples from the S matrix by means of appropriate Ward identities.

IV. WARD IDENTITIES

There are two ingredients in the proof of the necessary Ward identities. The first is the BRS transformation, or rather its generalization, which leaves invariant the full effective action. This will be discussed further in Sec. VI. Here we need only that the BRS transformation exists and the explicit form of its terms linear in the fields. We further assume that the linear parts of the full BRS rules are just those given by the "naive" prescription.[11] This will be justified in Sec. VI. In an unweighted gauge, where $e_{a\mu}$ is symmetric and C_{ab}, C^*_{ab} have been eliminated, these linear BRS transformations are

$$\begin{aligned}
&\delta^L_{\text{BRS}}h_{\mu\nu} = \kappa(C_{\mu,\nu}+C_{\nu,\mu})\Lambda\,, \quad \chi_2^\nu = -\tfrac{1}{2}(\partial_\mu\tilde{h}^{\mu\nu})\,,\\
&\delta^L_{\text{BRS}}\psi_\mu = 2\kappa C_{,\mu}\Lambda\,, \quad \chi_{3/2} = +\tfrac{1}{2}\bar{\psi}\cdot\bar{\gamma}\overleftarrow{\not{\partial}}\,,\\
&\delta^L_{\text{BRS}}\bar{C}^{\nu *} = \kappa\chi_2^\nu\Lambda\,, \quad \delta^L_{\text{BRS}}C^\nu = 0\,,\\
&\delta^L_{\text{BRS}}B = \kappa\chi_{3/2}\Lambda\,, \quad \delta^L_{\text{BRS}}C = 0\,,
\end{aligned} \tag{4.1}$$

where we define the χ's for the renormalizable gauge. Λ is an anticommuting constant. Notice that the ghost (as opposed to antighost) transformation has no linear part.

The second ingredient is the lowest-order form of a theorem due to Lam.[18,19] If, in a theory with a set of fields ϕ_i and Lagrangian $\mathcal{L}(\phi)$, the fields undergo a canonical transformation

$$\phi_i \to \phi_i + f_i(\phi)\,, \tag{4.2}$$

then the following Green's function identity holds for any set of function F_i of the fields:

$$\langle F_1(\phi_1),\ldots,F_n(\phi_n)\rangle^{\mathcal{L}(\phi)} = \langle F_1(\phi_1+f_1(\phi)),\ldots,F_n(\phi_n+f_n(\phi))\rangle^{\mathcal{L}(\phi+f(\phi))}\,. \tag{4.3}$$

Here the superscripts indicate the Lagrangian used to calculate the Green's function in perturbation theory.

Let $f_i(\phi)=\epsilon g_i(\phi)$. Then (4.3) is very easy to prove to first order in ϵ. We give this brief argument because we will want to generalize (4.3) slightly. In general, $\mathcal{L}$ can be split into kinetic, interaction, and source parts:

$$\mathcal{L}(\phi) = \tfrac{1}{2}\phi_i M_{ij}\phi_j + V(\phi) + J_i\phi_i\,. \tag{4.4}$$

Under the canonical transformation this becomes

$$\mathcal{L}(\phi+f(\phi)) = \tfrac{1}{2}\phi_i M_{ij}\phi_j + \epsilon\phi_i M_{ij}g_j(\phi) + V(\phi+\epsilon g(\phi)) + J_i(\phi_i+\epsilon g_i(\phi))\,. \tag{4.5}$$

Consider now an arbitrary graph linear in ϵ. In any such graph there is a single vertex, ϵu, to which some (possibly multilinear) function of the fields g_i attaches, as shown in Fig. 1(a). Suppose the vertex u is from $V(\phi+\epsilon g(\phi))$. Then there is another graph, Fig. 1(b), where u is replaced by v, the vertex in the untransformed $V(\phi)$ which differs from u only in that $g_i(\phi)$ is replaced by ϕ_i, while ϕ_i is connected to $g_i(\phi)$ at the vertex $\epsilon\phi_i M_{ij} g_j(\phi)$ generated from the kinetic term. The two contributions cancel. In just the same way, graphs with $g_i(\phi)$ attached to a source J_i are canceled, and (4.3) is verified to first order in ϵ.

It is now easy to derive an on-shell Ward identity. Consider a Green's function with a single external antighost and a number of external gauge fields, all on-shell.[19] Such an object vanishes identically for the supersymmetry Lagrangian, but we will be interested in its first-order variation under the BRS transformation whose linear terms are given in (4.1). Because of (4.3), this first-order variation also vanishes, and, because the supersymmetry Lagrangian is (by assumption) invariant under the transformation, the variation comes only from the variation of external fields.

The resulting on-shell Ward identity can be written as

$$\langle(\delta^L_{\rm BRS}\,\overline{\mathbb{C}}(z))\phi_1(x_1)\cdots\phi_n(x_n)\rangle = -\sum_{i=1}^{n}\langle\overline{\mathbb{C}}(z)\phi_1(x_1)\cdots(\delta^L_{\rm BRS}\,\phi_i(x_i))\cdots\phi_n(x_n)\rangle\,, \tag{4.6}$$

where we only retain single-particle contributions to each external line, so that all variations multilinear in the fields drop out. In (4.6) $\overline{\mathbb{C}}$ is one of $C^{*\nu}$ and B, so that $\delta^{(L)}_{\rm BRS}\,\overline{\mathbb{C}}(z)$ is a gauge-fixing term.

With $\overline{\mathbb{C}}=B(z)$, $\delta^{(L)}_{\rm BRS}B(z)=\frac{1}{2}\bar{\psi}(z)\cdot\bar{\gamma}\overleftarrow{\not{\partial}}\Lambda$ in the renormalizable gauge. This means that associated with this external line there is a single-particle pole

$$\frac{-i}{2}\left(\frac{1}{2}\frac{\gamma_\mu\not{p}\gamma_\nu}{p^2}\right)\gamma_\mu\not{p}=2p_\nu(2\not{p})^{-1} \tag{4.7}$$

in the momentum-space version of the left-hand side of (4.6). The term $\delta^{(L)}_{\rm BRS}B(z)=\chi_{3/2}(z)\Lambda$ may thus be taken as representing a longitudinally polarized spin-$\frac{3}{2}$ line, whose propagator $-\frac{1}{2}\gamma_\mu\not{p}\gamma_\nu(p^2)^{-1}$ has been replaced by twice the supersymmetry ghost propagator.

When $\overline{\mathbb{C}}(z)=C^{\nu*}(z)$, the analog of (4.7) is found from $\delta^L_{\rm BRS}C^{\nu*}$. In momentum space there will thus be a factor

$$\frac{p_\mu}{p^2}\,(\delta_{\mu\mu'}\delta_{\nu\nu'}-\tfrac{1}{2}\delta_{\mu\nu}\delta_{\mu'\nu'})(\delta_{\mu\rho}\delta_{\nu\sigma}+\delta_{\mu\sigma}\delta_{\nu\rho}-\delta_{\mu\nu}\delta_{\rho\sigma}) =(p_\rho\delta_{\nu\sigma}+p_\sigma\delta_{\nu\rho})(p^2)^{-1} \tag{4.8}$$

for this external line of the Green's function. In the unweighted gauge $\alpha\to\infty$ we have chosen, the antisymmetric part of $h_{\mu\nu}$ vanishes, and (4.8) may be replaced by

$$2p_\rho\delta_{\nu\sigma}(p^2)^{-1}\,. \tag{4.9}$$

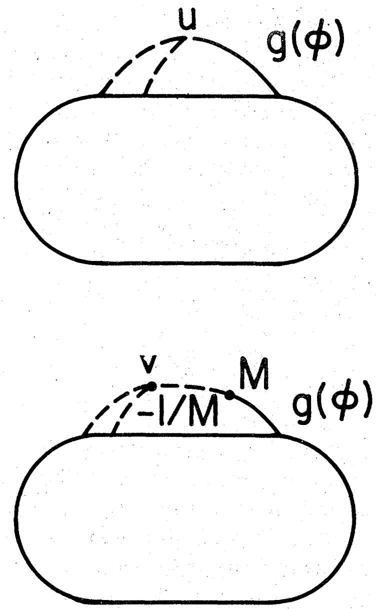

FIG. 1. Scheme of cancellation of graphs which are first order in a canonical transformation. Vertex u in (a) differs from v in (b) only in that $g(\phi)$ has been substituted for ϕ. Dashed lines represent ϕ, solid lines $g(\phi)$.

As with the supersymmetry ghost, the term $\delta^{(L)}_{\rm BRS}C^{\nu*}(z)$ can therefore be represented by a longitudinally polarized graviton, whose propagator has been replaced by twice the gravitational ghost propagator.

According to (4.1), variations of the gauge field are proportional to derivatives of their respective ghost fields. This fact, along with (4.7) and (4.9), enables us to represent the on-shell Ward identity (4.6) graphically as in Fig. 2.

Notice that on the right-hand side of Fig. 2, the continuous ghost line may change back and forth from a general coordinate to a supersymmetry ghost as it interacts. In particular, it can enter the blob as one, and leave as the other.

Figure 2 expresses current conservation in supergravity. It shows that a single longitudinally polarized gauge field decouples from a Green's function whose other external lines attach to gauge-invariant sources ($p_\mu J_\mu=0$ or $p_\mu J_{\mu\nu}=0$).

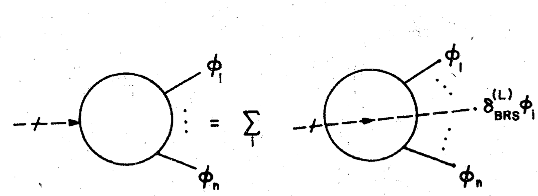

FIG. 2. On-shell Ward identity for Green's functions. The slash on the incoming ghost line indicates that its propagator has been amputated.

This result can be generalized to an arbitrary number of longitudinally polarized gauge fields as follows. Consider a Green's function with a single external antighost $\overline{\mathfrak{C}}(z)$, a number of external gauge fields ϕ_i coupled to gauge-invariant sources J_i, and a set of on-shell external gauge-fixing terms χ_j,

$$\langle\overline{\mathfrak{C}}(z)\phi_1(x_1)\cdots\phi_m(x_m)\chi_1(y_1)\cdots\chi_n(y_n)\rangle \times J_1(x_1)\cdots J_m(x_m) \quad (4.10)$$

As we have seen, each field χ_j represents a longitudinally polarized gauge field.

The vanishing of the first-order BRS variation of (4.10) gives the on-shell Ward identity

$$\left[\langle\chi_\alpha(z)\phi_1(x_1)\cdots\chi_n(y_n)\rangle+\sum_i\langle\overline{\mathfrak{C}}(z)\cdots\delta^L_{\mathrm{BRS}}\phi_i(x_i)\cdots\rangle+\sum_j\langle\overline{\mathfrak{C}}(z)\cdots\delta^L_{\mathrm{BRS}}\chi_j(x_j)\cdots\rangle\right]J_1(x_1)\cdots J_m(x_m)=0\,. \quad (4.11)$$

All terms in (4.11) except the first vanish on-shell. Those from the variation of the fields vanish by (4.1) and the gauge invariance of the sources. Those from the variation of the gauge-fixing terms vanish because

$$\delta^L_{\mathrm{BRS}}\chi_j(y_j)\propto M_j\overline{\mathfrak{C}}_j(y_j)\,, \quad (4.12)$$

where $\overline{\mathfrak{C}}_j$ is the ghost field corresponding to the gauge-fixing term χ_j, and M_j is the kinetic part of its Lagrangian. (We should point out that the recent work mentioned in the Introduction does not require a redefinition of ghost kinetic terms.) In momentum space, the M_j cancels the one-particle pole of the ghost, and all these terms vanish on-shell (i.e., after multiplying by on-shell inverse propagators). The graphical representation of the resulting Ward identity is shown in Fig. 3. Here and in the following, external gauge lines attached to gauge-invariant sources will be suppressed in figures.

In the unitary gauge $\hat{\gamma}\cdot\psi=0$, $\nabla_i\tilde{h}_{i\nu}=0$, very similar identities can be derived, using the same methods. Consider the BRS variation of the same Green's function (4.10), but now in the noncovariant gauge. The result is again (4.11), and again the terms with variations of fields and gauge-fixing terms vanish on-shell.

In place of (4.7) we now find a single-particle pole

$$-\frac{i}{2}\not{p}\,\hat{\gamma}_\mu\left[\tfrac{1}{2}(\bar{\delta}_{\nu\nu'}\gamma_{\nu'}\not{p}\,\gamma_{\mu'}\bar{\delta}_{\mu'\mu})+4(p\cdot\bar{p})^{-1}p_\mu p_\nu\right]p^{-2} \quad (4.13)$$

external to the graph for each field $\chi_{3/2}$ in (4.10). A straightforward computation using the explicit form of $\bar{\delta}$ [see Eq. (2.8)] shows that the first term in (4.13) has no pole; the entire single-particle pole comes from the unphysical residue $(4/p\cdot\bar{p})p_\mu p_\nu$, and is given by

$$-2i(\not{p}^{-1}p\cdot\hat{\gamma}\,p_\nu)(p\cdot\bar{p})^{-1}\,. \quad (4.14)$$

Except for $p=0$, $p\cdot\hat{\gamma}$ is a nonsingular matrix with an inverse, so that the Ward identity Fig. 3 also holds in the unitary gauge. As we will see, this identity alone is sufficient for unitarity in this gauge.

To complete the proof of unitarity in the renormalizable gauge, we also require a slightly different class of on-shell Ward identities. Consider again a Green's function with a single external antighost and N external gauge fields, all on-shell. Suppose that this Green's function is cut, so that the intermediate state has P particles and so that M external lines are on one side of the cut, and N-M on the other. Such an object will be represented as in Fig. 4, and will be denoted by $H_{P,M,N-M}$. Cut lines may be either gauge fields or ghosts.

We now observe that to first order in ϵ, $H_{P,M,N-M}$ obeys (4.3) under any canonical trans-

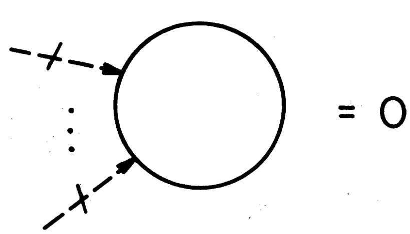

FIG. 3. On-shell Ward identity expressing the decoupling of longitudinally polarized gauge lines.

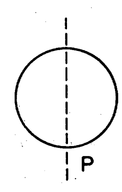

FIG. 4. Generic representation of a cut Green's function. The cut passes through P lines.

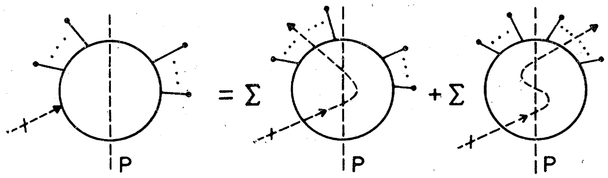

FIG. 5. On-shell Ward identity for cut Green's functions.

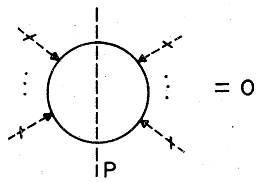

FIG. 6. Decoupling of longitudinally polarized gauge lines for cut Green's functions.

formation. This is shown in just the same way as for the Green's functions. Starting with any set of cut graphs with transformed external fields [the right-hand side of (4.3)], we find that all first-order graphs cancel pairwise.

Notice that because the BRS transformation includes linear terms, identical denominators can occur. Because of the presence of the kinetic term at the first-order two-point vertex, however, there is still only a single pole between the pair of lines, and there is no ambiguity in how to cut such a structure. Thus, the contributions of Figs. 1(a) and 1(b) cancel, whether the explicitly shown lines are cut or not.

We can now simply write down the on-shell Ward identity for the quantities $H_{P,N,N-M}$ by analogy to the Green's function case. It is shown in Fig. 5. Notice that there is no restriction on how many times the continuous ghost line on the right-hand side of Fig. 5 passes through the cut; all possibilities will occur in the sum over diagrams. As in the uncut case, if all gauge lines couple to gauge-invariant sources, the right-hand side of the Ward identity vanishes, and Fig. 5 expresses current conservation for the cut diagrams.

The same reasoning enables one to derive Ward identities of the type shown in Fig. 6, where multiple longitudinally polarized gauge lines attach to a cut diagram.

V. UNITARITY

In our proof of unitarity we will follow closely the method described by 't Hooft for the Feynman gauge in Yang-Mills theory. We discuss first the renormalizable gauge, and then turn briefly to the unitary gauge.

The goal of our argument is to show that a sum over intermediate states gives the same answer whether performed using only physical states (helicity $\pm\frac{3}{2}$ and ± 2), or the full set of "gauge" states, both ghost particles and gauge particles. This condition is represented in Fig. 7, where (following 't Hooft, Ref. 14) the shaded cut refers to physical lines

$$2\pi i(\tfrac{1}{2}\bar{\delta}_{\nu\nu'}\gamma_{\nu'}\not{k}\gamma_{\mu'}\bar{\delta}_{\mu\mu'})\theta(k_0)\delta(k^2) \tag{5.1}$$

for each cut spin $\frac{3}{2}$, and

$$2\pi i[-2i(\bar{\delta}_{\mu\sigma}\bar{\delta}_{\nu\rho}+\bar{\delta}_{\nu\sigma}\bar{\delta}_{\mu\rho}-\bar{\delta}_{\mu\nu}\bar{\delta}_{\rho\sigma})]\theta(k_0)\delta(k^2) \tag{5.2}$$

for each cut graviton. The dashed cut refers to the gauge states. In the renormalizable gauge these are

$$\begin{aligned}&2\pi i(\tfrac{1}{2}\gamma_\nu\not{k}\gamma_\mu)\theta(k_0)\delta(k^2)\,,\\&2\pi i[-2i(\delta_{\mu\rho}\delta_{\nu\sigma}+\delta_{\mu\sigma}\delta_{\nu\rho}-\delta_{\mu\nu}\delta_{\rho\sigma})]\theta(k_0)\delta(k^2)\end{aligned} \tag{5.3}$$

for cut gauge lines and

$$\begin{aligned}&2\pi i(-\tfrac{1}{2}\not{k})\theta(k_0)\delta(k^2)\,,\\&2\pi i(-i\delta_{\mu\nu})\theta(k_0)\delta(k^2)\end{aligned} \tag{5.4}$$

for cut ghost lines.

We start out by verifying Fig. 7 for $M=2$, and then extend the result to arbitrary M by iteration. In each case we concentrate on the spin-$\frac{3}{2}$ particle, and we will see how the problem of the "extra" degrees of freedom is resolved.

The residue of the physical spin-$\frac{3}{2}$ pole can be expanded as follows:

$$\tfrac{1}{2}\bar{\delta}_{\nu\nu'}\gamma_{\nu'}\not{k}\gamma_{\mu'}\bar{\delta}_{\mu'\mu}=\tfrac{1}{2}\left(\gamma_\nu\not{k}\gamma_\mu-\frac{k_\nu\bar{\not{k}}\not{k}\gamma_\mu}{k\cdot\bar{k}}-\frac{\gamma_\nu\not{k}\bar{\not{k}}k_\mu}{k\cdot\bar{k}}+\frac{k_\nu\bar{\not{k}}\not{k}\bar{\not{k}}k_\mu}{(k\cdot\bar{k})^2}\right). \tag{5.5}$$

Graphical notation for these terms is given in Fig. 8.

Figure 9 gives the proof of unitarity for $M=2$. Line (b) is a definition. Line (c) follows from cur-

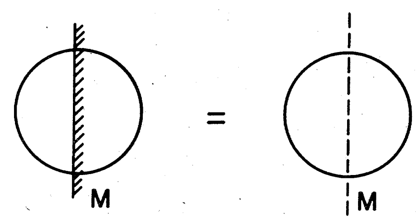

FIG. 7. The condition for unitarity. The shaded cut indicates physical states only. The dashed cut indicates gauge states.

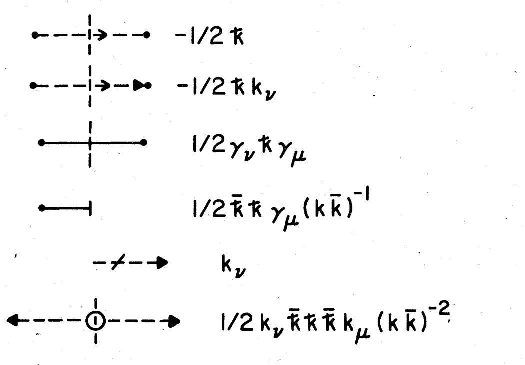

FIG. 8. Notation for cut lines. A factor $2\pi i\theta(k_0)\delta(k^2)$ is understood in each case.

rent conservation, Fig. 3, since the other external lines of both blobs to which the top line attaches are physical. (d) is simply an expansion of (c), using (5.5) and the notation of Fig. 8. To derive (e) we apply the Ward identity Fig. 2 on the second, third, and fourth terms. In each case a ghost line is produced on one of the blobs, ending in a longitudinally polarized external gauge line which acts on the other.

In the second term of (e) the lower residue is of the form

$$-2\pi i\,\frac{\bar{\not p}\not p\,\gamma_\mu}{(p\cdot\bar p)}=2\pi i\,\tfrac{1}{2}(\gamma_\alpha\not p\,\gamma_\mu)\left(\frac{-\bar p_\alpha}{p\cdot\bar p}\right). \qquad (5.6)$$

This is of the form of the residue of a gauge propagator, $\frac{1}{2}\gamma_\alpha\not p\,\gamma_\mu(p^2)^{-1}$, dotted into a "source," $J_\alpha=-\bar p_\alpha/p\cdot\bar p$. We can therefore apply Fig. 2 to the left-hand blob of the second term in (e). The gauge residue $(2\pi i)(\frac{1}{2}\gamma_\alpha\not p\,\gamma_\mu)$ is replaced by a ghost residue $(2\pi i)(-\frac{1}{2}\not p)$ times a momentum factor p_α. Since $p_\alpha J_\alpha=-1$, all extra factors except an overall minus sign cancel, and we get the second term of (f), in which the two cut lines are part of a ghost loop whose relative minus sign is explicit. Precisely analogous reasoning leads to the third term in (f), which is the same, except that the sense of the ghost loop is reversed. The last term in (e) vanishes by Fig. 3. Line (g) follows by definition, and Fig. 7 is verified for $M=2$. It may be worth noting that the upper line in Fig. 9 may have been either spin $\frac{3}{2}$ or 2.

This argument shows what becomes of the eight degrees of freedom found for the spin-$\frac{3}{2}$ propagator in Sec. III. Two are present in the physical propagator, and as expected an additional four are canceled by a complex spin-$\frac{1}{2}$ ghost. The remaining two, associated with the fourth term in (d), are seen to vanish independently by current conservation.

So far we have only used the Ward identities of Figs. 2 and 3. Going to $M>2$, however, requires

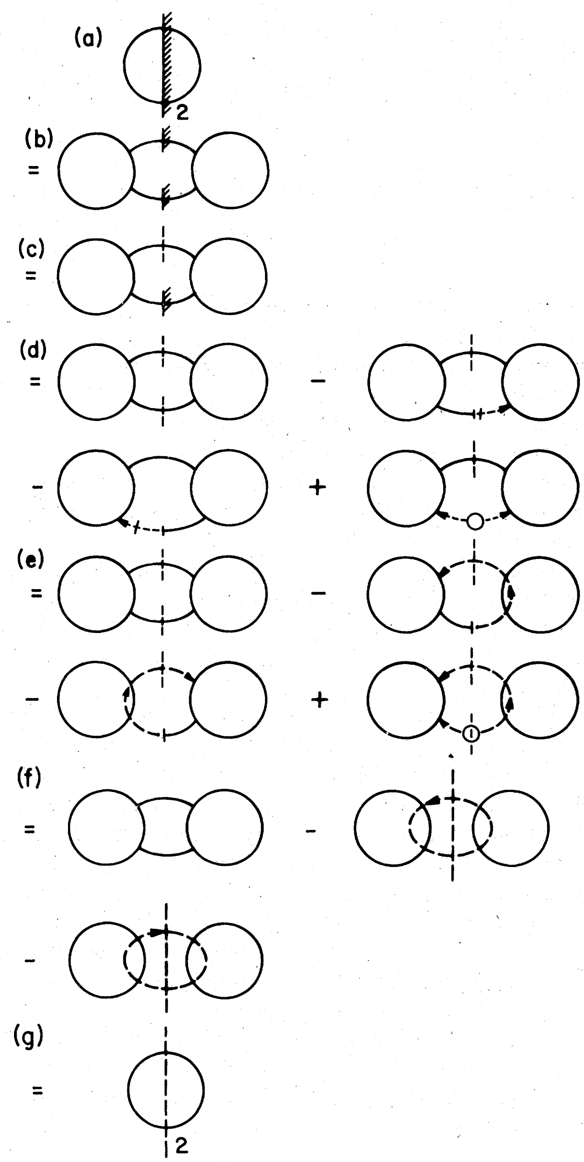

FIG. 9. Proof of Fig. 7 for the two-particle cut in the renormalizable gauge. All external lines are on-shell and have physical polarizations.

that we deal with "mixed" cuts, involving both ghost and gauge lines. As a result, a direct extension of the $M=2$ argument requires Ward identities for Green's functions with arbitrary numbers of external gauge and ghost lines. It is in fact possible to derive such identities using the methods of Sec. IV; unfortunately, they are not of a form which is directly applicable to the problem. To avoid this difficulty, we derived, following 't Hooft, the identities of Figs. 5 and 6. These are used to extend Fig. 7 to arbitrary N as follows.

Assume Fig. 7 holds for $M=N$. The proof for M

$=N+1$ is given in Fig. 10. (b) is simply (a) with a single spin-$\frac{3}{2}$ gauge line isolated. (c) follows by the assumption that Fig. 7 is true for $M=N$. (d) is the expansion of (c), using (5.5). To get (e), we have applied the Ward identities of Figs. 5 and 6; again the "extra" two degrees of freedom decouple independently.

In the unitary gauge the situation is much simpler. Here the dashed cut of Fig. 7 is the same as the shaded physical cut, except for contributions from the "unphysical" pole terms $4(p\cdot\bar{p})^{-1} \times p_\mu p_\nu(p^2)^{-1}$ in the spin-$\frac{3}{2}$ propagator. We can thus expand the sum over gauge states as in Fig. 11. By the identity of Fig. 3, all terms except the first vanish on the right-hand side of Fig. 11, so that Fig. 7, and hence unitarity, is verified for the unitary gauge.

VI. BRS INVARIANCE AND COVARIANT QUANTIZATION

We saw in Sec. IV that BRS invariance of the supergravity effective action provided a compact derivation of the Ward identities used to prove unitarity. In fact, the choice of this method is not simply a convenience but a necessity. Because of the well-known fact that the supergravity gauge transformations do not close to form a group, except on-shell, the conventional Slavnov-Taylor derivation leads to incorrect results. This is because the group property of gauge transformations is needed to establish the invariance of the product of functional integration measure with Faddeev-Popov determinant $(d\phi)\Delta(\phi)$. For example, for gravity where the generators of general coordinate transformations G_μ satisfy

$$[G_\mu, G_{\nu'}] = G_{\sigma''}C^{\sigma''}_{\mu\nu'} \tag{6.1}$$

and the primes indicate the spacetime points x', x'', the variation of $(d\phi)\Delta(\phi)$ is shown by Capper and Ramon-Medrano to be[20]

$$\delta(d\phi)\Delta(\phi) = C^{\lambda''}_{\lambda''\beta}\xi^\beta \tag{6.2}$$

for general coordinate parameters ξ^β and structure constants $C^{\lambda''}_{\rho'\beta}$. For supergravity (6.1) cannot be generalized, nor is it true that the trace of the structure constants vanishes. In addition, we shall see that the Faddeev-Popov prescription fails for supergravity, which invalidates the entire Slavnov-Taylor approach. Hence we turn to the BRS method.

In all other gauge theories BRS invariance is an expression of the original invariance of the classical action, *plus* the group property of gauge transformations. We illustrate this point with a Yang-Mills example. The effective action

$$-\tfrac{1}{4}(\vec{F}_{\mu\nu})^2 - \tfrac{1}{2}(\partial\cdot A)^2 + \vec{C}^*\cdot\partial^\mu D_\mu\vec{C} \tag{6.3}$$

is invariant under

$$\delta_{BRS}\vec{A}_\mu = (D_\mu\vec{C})\Lambda,\quad \delta_{BRS}\vec{C} = +\tfrac{1}{2}\vec{C}\times\vec{C}\Lambda,$$
$$\delta\vec{C}^* = -(\partial\cdot\vec{A})\Lambda. \tag{6.4}$$

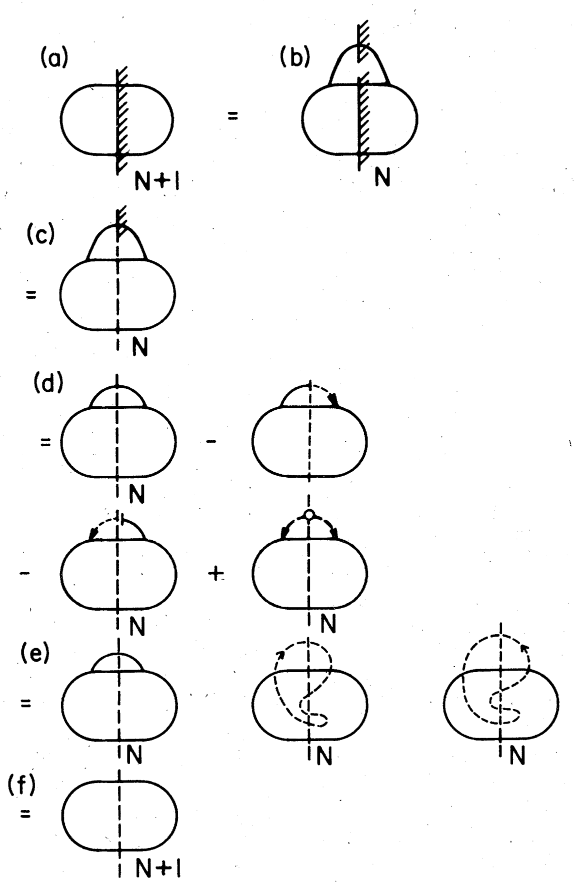

FIG. 10. Proof of Fig. 7 for the $(N+1)$-particle cut in the renormalizable gauge assuming the result for the N-particle cut.

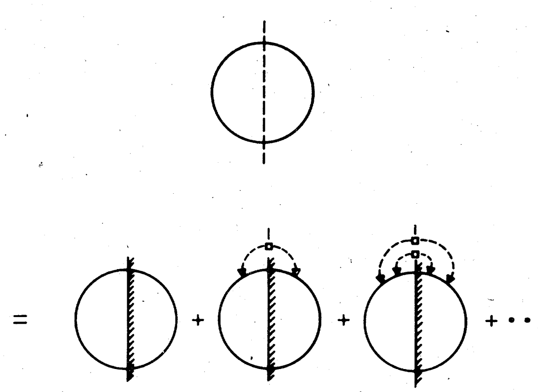

FIG. 11. Expansion of gauge state cut in the unitary gauge. The boxes represent the unphysical residue $4(p\cdot\bar{p})^{-1}p_\mu p_\nu$.

This invariance follows from the property that $\delta_{\rm BRS}(D_\mu \vec{C})=0$, which is in turn a consequence of the group property of gauge transformations. Because of the antisymmetry of $\Lambda_1\Lambda_2$, the product of two gauge transformations with parameters $\vec{C}\Lambda_1$ and $\vec{C}\Lambda_2$ is equal to half the commutator and hence *by the group composition law* to another gauge transformation with parameter $\frac{1}{2}\vec{C}\times\vec{C}\Lambda_1\Lambda_2$. The transformation of $\vec{C}$ in (6.4) is chosen to cancel this, so the transformation of the ghosts are determined by the gauge group composition law. Up to a point this is also true for supergravity.

We take the effective supergravity Lagrangian to be that of Eq. (1.4). Previously BRS transformations for supergravity were obtained in the first-order formulation.[11] Taking these results over to the second-order formulation is not difficult. We find

$$\begin{aligned}\delta_{\rm BRS}\, e_{a\mu} &= \kappa^2(C^\lambda e_{a\mu,\lambda} + C^\lambda{}_{,\mu} e_{a\lambda} + C_a{}^b e_{b\mu} - \kappa\bar\psi_\mu\gamma_a C)\Lambda\,,\\ \delta_{\rm BRS}\psi_\mu &= \kappa^2[-C^\lambda\psi_{\mu,\lambda} - C^\lambda_{,\mu}\psi_\lambda - \tfrac{1}{2}\sigma^{ab}C_{ab}\psi_\mu \\ &\quad + (2/\kappa)D_\mu C]\Lambda\end{aligned} \qquad (6.5)$$

and

$$\begin{aligned}\delta_{\rm BRS}\, C^\lambda &= \kappa^2(-C^\nu C^\lambda{}_{,\nu} + \hat{C}\gamma^\lambda C)\Lambda\,,\\ \delta_{\rm BRS}\, C_{ab} &= \kappa^2(-C^\nu C_{ab,\nu} - C_{ac}C^c{}_b + w_{\nu ab}\hat{C}\gamma^\nu C)\Lambda\,,\\ \delta_{\rm BRS}\, C &= \kappa^2(C^\nu C_{,\nu} + \tfrac{1}{2}\sigma^{ab}C_{ab}C - \tfrac{1}{2}\kappa\psi_\nu\hat{C}\gamma^\nu C)\Lambda\,,\\ \delta_{\rm BRS}\, C^{\nu *} &= -\kappa\,\tfrac{1}{2}(\partial_\mu \bar{h}^{\mu\nu})\Lambda\,,\\ \delta_{\rm BRS}\, C^*_{a\mu} &= \frac{-\alpha}{2\kappa}(h_{a\mu} - h_{\mu a})\Lambda\,,\\ \delta_{\rm BRS} B &= \frac{\kappa}{2}(\bar\psi\cdot\overline{\gamma\partial})\Lambda\,,\end{aligned} \qquad (6.6)$$

where $D_\mu C = [\partial_\mu + \frac{1}{2}w_{\mu ab}(e,\psi)\sigma^{ab}]C$, and the factors of κ are determined by the ghost dimensions, Eq. (1.8). $\delta_{\rm BRS}\, e_{a\mu}$ and $\delta_{\rm BRS}\psi_\mu$ are themselves particular gauge transformations, while $\delta_{\rm BRS}$ of the ghosts is determined by the (on-shell) group composition law. The only nontrivial commutator is for two supersymmetry transformations and is given by

$$\begin{aligned}[\delta_{\epsilon_1},\delta_{\epsilon_2}] &= \delta_{\rm gen\ coord}(2\bar\epsilon_2\gamma^\nu\epsilon_1) + \delta_{\rm loc\,Lorentz}(2\omega_{\nu ab}\bar\epsilon_2\gamma^\nu\epsilon_1)\\ &\quad + \delta_{\rm sup}(-\psi_\nu\bar\epsilon_2\gamma^\nu\epsilon_1)\\ &\quad + (\psi \text{ equation-of-motion terms}),\end{aligned} \qquad (6.7)$$

where the terms in brackets are the new transformation parameters. The supersymmetry variation of the ghosts in (6.6) can be deduced from (6.7). We now remark that $\delta^2_{\rm BRS}(e_{a\mu})=0$,[11] so that also

$$\delta^2_{\rm BRS}(\bar{h}^{\mu\nu}) = \delta^2_{\rm BRS}(h_{a\mu} - h_{\mu a}) = 0. \qquad (6.8)$$

However, $\delta^2_{\rm BRS}\psi_\mu \neq 0$ because of the equation-of-motion term in (6.7), and this leads to the failure of this form of BRS invariance. In fact, the variation of the effective action under (6.6) is

$$\begin{aligned}\delta\mathcal{L}_{\rm eff} &= \kappa^3[\hat{C}\gamma^\nu CB(-\tfrac{3}{8}\gamma_\lambda\gamma_\nu + \tfrac{1}{2}g_{\lambda\nu})\\ &\quad -\tfrac{1}{4}\hat{C}\sigma^{\alpha\beta}CB\gamma_\lambda\sigma_{\alpha\beta}]M^\lambda\Lambda\,,\end{aligned} \qquad (6.9)$$

where M^λ is the ψ_λ field equation, and a caret denotes Majorana conjugation,

$$M^\lambda = \epsilon^{\lambda\rho\mu\nu}\gamma_5\gamma_\rho D_\mu\psi_\nu. \qquad (6.10)$$

This is the previously obtained result[11] that *BRS invariance only holds modulo the equations of motion* and, as conjectured previously, is a direct consequence of the nonclosure of the algebra. In this case we only need the ψ_μ field equation because we work in second-order form. In fact, many of the results of Ref. 11 can now be understood. For example, the following cancellations were found on-shell for $\delta_{\rm BRS}$ of a gauge field ϕ:

$$\begin{aligned}&\delta^{\rm grav}_{\rm BRS}\delta^{\rm grav}_{\rm BRS}\phi = 0\,,\\ &(\delta^{\rm grav}_{\rm BRS}\delta^{\rm sup}_{\rm BRS} + \delta^{\rm sup\,(gauge)}_{\rm BRS}\delta^{\rm grav}_{\rm BRS})\phi = 0\,,\\ &(\delta^{\rm sup}_{\rm BRS}\delta^{\rm sup}_{\rm BRS} + \delta^{\rm sup(ghost)}_{\rm BRS}\delta^{\rm grav}_{\rm BRS})\phi = 0\,,\end{aligned} \qquad (6.11)$$

where we have split $\delta_{\rm BRS}$ into gravitational and supersymmetry parts and the word gauge (ghost) in parentheses is an injunction to transform only gauge (ghost) fields. If these equations are further separated into transformations on gauge fields and ghost fields they can be seen to be a consequence of the (on-shell) supergravity group composition law.

Apart from the new equation of motion terms of (6.9) there is another distinguishing feature of the supergravity BRS transformation which is that the Jacobian is not unity. In fact, the Jacobian J gets contributions from δC and the torsion terms in $\delta\psi_\mu$ which sum to

$$J = 1 - 4\kappa^3\int \hat{C}\gamma\cdot\psi\, d^4x\,\Lambda\delta^4(0)\,. \qquad (6.12)$$

This can be canceled if the functional measure includes an explicit factor of g^{-2}. But according to Fradkin and Vasiliev *this is exactly the measure required by Hamiltonian quantization.* Although one might argue that a measure gives rise only to $\delta^4(0)$ terms, which can be ignored if one uses dimensional regularization, we consider this agreement to be an indication of the consistency of both approaches. However, we still have the problem of the equation of motion term (6.9). If one attempts to use the rules (6.6) to derive the on-shell Ward identity of Fig. 2 following the method of Sec. IV, one obtains instead the relation shown in Fig. 11 where the circle at the $BCC\psi$ vertex indicates the equation-of-motion term $\epsilon^{\gamma\rho\mu\nu}\gamma_5\gamma_\rho D_\mu$. Clearly the third diagram in Fig. 12 will cause a violation of unitarity unless it can be canceled in some way. To this end we now attempt to generalize the BRS prescription to take into account the nonclosure of the algebra. As we shall see this leads to the appearance of quartic ghost couplings and the failure of the Faddeev-Popov prescription for covariant quantization of supergravity.

To cancel (6.9) we add to $\delta\psi_\mu$ the term

$$\delta'_{\rm BRS}\psi_\mu = -[\kappa^3(-\tfrac{3}{8}\gamma_\nu\gamma_\mu + \tfrac{1}{2}g_{\nu\mu})\hat{B}(\hat{C}\gamma^\nu C) - \tfrac{1}{4}\kappa^3\sigma_{\alpha\beta}\gamma_\mu\hat{B}(\hat{C}\sigma^{\alpha\beta}C)]\Lambda. \qquad (6.13)$$

We now make two simplifications. Firstly, we will restrict ourselves to an order-by-order expansion in κ which is so successful in matter coupling to supergravity. We will obtain BRS invariance to the $O(\kappa^3)$ level and we will determine the action to $O(\kappa^2)$ and the transformation rules to $O(\kappa^3)$. Secondly, we use an unweighted gauge $e_{a\mu} - e_{\mu a} = 0$, so that $\alpha \to \infty$ in (1.4) and we solve the C^*_{ab} field equation for C_{ab}, which gives Eq. (1.6). This does not cause any additional contributions to $\delta\mathcal{L}_{\rm eff}$ because $\delta_{\rm BRS}C_{ab}$ is the same whether (1.6) is used before or after the variation. This is a consequence of $\delta^2_{\rm BRS}(e_{a\mu} - e_{\mu a}) = 0$, which says that the BRS variation of the C^*_{ab} field equation *also* vanishes when the field equation is satisfied. As we saw previously, the elimination of C_{ab} and C^*_{ab} is a natural procedure since these ghosts are nonpropagating.

We now recall that the equation of motion term (6.10) has been canceled by an $O(\kappa^3)$ addition to $\delta_{\rm BRS}\psi_\mu$. This causes the appearance of a new $O(\kappa^3)$ term in $\delta\mathcal{L}_{\rm eff}$ coming from $\delta_{\rm BRS}\psi_\mu$ in the gauge-fixing term, $\frac{1}{4}\bar{\psi}\cdot\gamma\not{\partial}\gamma\cdot\psi$. This can be canceled most easily by the variation $\delta_{\rm BRS}B$ in a new $O(\kappa^2)$ addition to the action

$$\mathcal{L}'_{\kappa^2} = -\kappa^2\tfrac{5}{8}\hat{C}\gamma^\nu CB\gamma_\nu\hat{B}. \qquad (6.14)$$

This is a natural extension of the BRS cancellation mechanism and ensures BRS invariance to order (κ^3). In fact, we found it impossible to obtain BRS invariance without this addition to the action. To order κ^4 new terms appear from $\delta'_{\rm BRS}\psi_\mu$, from the Faddeev-Popov Lagrangian and from δC in (6.14). However, the action to $O(\kappa^2)$ is *unambiguous* and is

$$\mathcal{L}_{\rm eff} = \mathcal{L}_{\rm inv} + \mathcal{L}_{\rm fix} + \mathcal{L}_{\rm FP} - \kappa^2\tfrac{5}{8}(B\gamma^\nu\hat{B})(\hat{C}\gamma_\nu C) + O(\kappa^3). \qquad (6.15)$$

This agrees with the result of Fradkin and Vasiliev[12] when account is taken of our normalization of $\delta\psi_\mu$.[21]

The general recipe for obtaining generalized BRS invariance at higher orders is the same as in matter couplings to supergravity: One adds at each κ level extra terms both to action and transformation laws which effect invariance at that κ level. In all matter couplings, this iterative scheme leads to full invariance, and we expect the same result here.

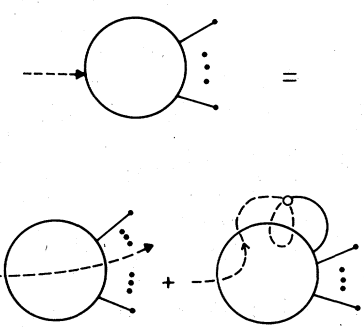

FIG. 12. Relation obtained from BRS rules (6.6). The circle indicates the equation-of-motion term at the $BCC\psi$ vertex.

In summary, unitarity requires BRS invariance which requires new rules for covariant quantization, which agree with those found from Hamiltonian quantization. We have obtained BRS invariance only to $O(\kappa^3)$, but we expect this invariance to exist to all orders. With this invariance the Ward identities derived in Sec. IV follow. In particular, the linear part of the generalized BRS rules is the same as for the "naive" rules since the extra terms are multilinear in fields. Moreover, further addition to these rules will be higher order in κ and must also be multilinear in fields. Hence *the linear terms of the full BRS rules will be the same as those of the "naive" rules*. This justifies our assumption of Sec. IV.

VII. SUMMARY AND CONCLUSIONS

We have studied the question of unitarity in supergravity, using a diagrammatic approach. We have shown that in the residues of two propagators, corresponding to the renormalizable and unitary gauges $\gamma\cdot\psi = 0$ and $\vec{\gamma}\cdot\vec{\psi} = 0$, there are extra unphysical modes present, in addition to the usual unphysical modes in the renormalizable propagator which are, as always, canceled by Faddeev-Popov ghosts. In order to show that these extra modes decouple by themselves, without the need to invoke further ghosts for this cancellation, we needed Ward identities for cut graphs as well as for uncut graphs. These were derived from the invariance of the complete effective action, thus including gauge-fixing and ghost terms, under BRS transformations. With these Ward identities we could then prove the decoupling of the extra unphysical modes. We first considered the two-particle cuts, where the decoupling is particularly clear, and subsequently we proved unitarity for a general n-

particle cut.

Previous work on BRS invariance of the effective supergravity action[11] had shown that invariance only holds when the classical field equations for the spin-$\frac{3}{2}$ field are satisfied. This is reminiscent of the gauge algebra, which closes in second-order formalism under the same restriction[8] (in first order one also needs the ω field equation), and indeed we have demonstrated that the former is a necessary consequence of the latter. We used that BRS transformations must be an explicit representation of the gauge invariance of the classical action *plus* the group composition rule of the classical gauge algebra. We obtained invariance of the effective action under BRS transformations by adding extra terms to action and transformation laws. In this way we discovered the need of an extra four-ghost interaction. This term had been found before by Fradkin and Vasiliev,[12] and Vilkovisky and Batalin,[13] who used a noncovariant Hamiltonian approach. We consider it nontrivial that our diagrammatic method yields the same results. We also find an extra factor g^{-2} in the functional measure in order that the BRS Jacobian be unity; its existence is also required in the Hamiltonian approach. However, for practical Feynman diagram calculations such a term seems of no importance.

Our method of adding order by order in κ new terms to action and transformation rules is the same method which was originally invented to couple matter to supergravity,[22] and which has since then been the standard procedure. Although our results are worked out and needed only to lowest nontrivial order, we certainly expect that it can be completed to all orders in κ, just as in cases of matter coupling. One important consequence is that we can no longer write the ghost action as a determinant, and thus it seems that the BRS procedure is the only path-integral method for obtaining Ward identities. Clearly, the BRS invariance of a quantum gauge theory is a much more important property than has been hitherto realized.

It should be stressed that we used mostly a diagrammatic approach, similar to 't Hooft and Veltman's Yang-Mills treatments. Whereas in principle it is equivalent to path-integral methods, the diagrammatic approach has in our opinion the advantage that at each stage it is perfectly clear what one is doing: One can verify each step by perturbation calculations. An added reason for using this diagrammatic approach is that we are intending to use these results for analyzing higher-loop diagrams in our ongoing investigation of the higher-loop renormalizability of supergravity.

It has been hoped for some time that the addition of enough auxiliary fields to the supergravity action would allow the closure of the algebra. For such a theory we would expect the "naive" BRS rules to leave the effective action invariant and the Faddeev-Popov prescription to be correct. It is then easy to see that additional ghost couplings could arise from the elimination of these auxiliary fields from the effective action, just as quartic ψ_μ couplings arise from the elimination of the spin connection $\omega_{\mu ab}$ in the transition from the first- to second-order form of supergravity.

Note added. After completion of this work an article by R. Kallosh, Zh. Eksp. Teor. Fiz. Pis'ma Red. 26, 575 (1977), was shown to us by J. Lukierski. She obtains complete BRS invariance for supergravity along the lines started in Ref. 11, thus providing the remaining step in our proof of unitarity. Our results of Sec. VI confirm those of Kallosh, except that we (and Fradkin and Vasiliev[12]) find the extra factor g^{-2} in the measure.

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