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Creation of particles by singularities in asymptotically flat spacetimes

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The creation of massless scalar particles by naked singularities in asymptotically flat spacetimes is investigated within the geometrical-optics approximation. To avoid the need to impose boundary conditions on the singularity, we consider models in which a curvature singularity arises at a finite time in the past. We consider two particular types of models. One is a shell-crossing singularity formed in the gravitational collapse of a dust cloud. The energy flux of the created particles remains finite up to the time of formation of the singularity. In the particular case when the singularity forms on the event horizon, geometrical optics yields the exact flux, in spite of the high curvature of spacetime. The radiation obtained is identical to the thermal Hawking radiation emitted by black holes. The other models considered are those of charged shells for which the charge exceeds the mass. If these shells collapse to form naked singularities (which is possible if the proper mass is negative or if Einstein's equations are not imposed), an infinite flux of created particles results. In the cases examined here, the flux is negative for two-dimensional models and for the minimally coupled scalar field in four-dimensional models, whereas it is positive for the conformally coupled scalar field in four-dimensional models. In either case, the back reaction from particle creation will be large and may prevent formation of a naked singularity.

I. INTRODUCTION

One of the fundamental problems facing general-relativity theory is the occurrence of singularities in spacetime, examples of which are the initial singularity of cosmological models and the curvature singularities on the interior of black holes. General theorems have been proven which demonstrate that singularities are inevitable in classical relativity theory provided that certain conditions are imposed on the energy-momentum tensor.[1] These conditions are reasonable for classical matter, but cannot be expected to hold in general for the energy-momentum tensor associated with quantized matter fields.[2] This holds out the hope that quantum effects associated with the matter fields can lead to the avoidance of singularities.

A related question of interest is that of calculating the spontaneous particle creation by the very strong gravitational fields in the vicinity of singularities, or almost singular regions. For many spacetimes possessing singularities, this is not possible in an unambiguous manner because of the uncertainty in the boundary conditions to be imposed on the singularity. If the singularity evolves from regular initial data, however, it is possible to calculate unambiguously the energy radiated by spontaneous particle creation up to the time of formation of the singularity. In such a situation one can attempt to answer the question of whether the back reaction from particle creation is sufficiently large to be able to prevent formation of the singularity. One might expect to find that the rate of particle creation will become infinite if back reaction is not taken into account. It is also possible for quantum effects to be significant even in the absence of particle creation since there can still be a nonzero vacuum energy and pressure.

In this paper models of singularity formation from regular initial configurations will be considered. One model is a shell-crossing singularity formed in the gravitational collapse of a dust cloud.[3] In this case there is no evidence for very large or infinite rates of particle creation as a result of the formation of the singularity. The other models considered are charged shells with charge greater than their mass, which collapse to form a naked singularity. In one model this is made possible by not requiring that Einstein's equations be satisfied; in another model the shell has negative proper mass so that a naked singularity forms in accordance with Einstein's equations. In these models it is found that the radiated flux becomes infinite as the singularity forms.

Each model may be regarded either as a four-dimensional spherically symmetric spacetime or as the corresponding two-dimensional spacetime obtained by removing the angular degrees of freedom. A solution of the four-dimensional version by use of the geometrical-optics approximation is equivalent to the exact solution of the two-dimensional version. We consider massless scalar particles, but the same techniques may be applied for other types of particles.

In Sec. II, we derive an approximate expression for the energy flux radiated in the geometrical-optics limit to infinity in a four-dimensional, spherically symmetric, asymptotically flat spacetime. In Sec. III the gravitational collapse of dust clouds and the formation of shell-crossing singularities is discussed. In Sec. IV the use of the geometrical-optics approximation to analyze the energy radiated by such a singularity is considered. The case of the collapsing charged shell is taken up in Sec. V.

II. PARTICLE PRODUCTION AND ENERGY FLUX IN THE GEOMETRICAL-OPTICS APPROXIMATION

The considerations of this section are rather general, and apply to any spherically symmetric, asymptotically flat spacetime, or region of spacetime, in which the radial null rays define a one-to-one mapping between past null infinity ($\mathscr{I}^-$) and future null infinity ($\mathscr{I}^+$). Using the geometrical-optics approximation, we derive a simple expression for the power radiated to infinity in the lower angular momentum modes, when no field quanta are present at early times. Our approach is analogous to that used by Fulling and Davies[4] in their analysis of radiation by a moving mirror in two dimensions, but also applies to four-dimensional spacetimes. For simplicity we consider a scalar field. The same technique should also apply to higher-spin fields.

In the asymptotic region, let r, θ, ϕ, t denote the usual quasi-Minkowskian spherical coordinates and time, which are asymptotically related to null coordinates u and v by $u=t-r$ and $v=t+r$. An incoming null ray $v=\text{const}$, originating on $\mathscr{I}^-$, propagates through the geometry becoming an outgoing null ray $u=\text{const}$, and arriving on $\mathscr{I}^+$ at a value $u=F(v)$. Conversely, one can trace a null ray from u on $\mathscr{I}^+$ to $v=G(u)$ on $\mathscr{I}^-$, where the function G is the inverse of F. In the geometrical-optics approximation, a wave packet formed at early times from incoming plane waves $\exp(-i\omega v)$ will become at late times an outgoing wave packet formed from plane waves $\exp[-i\omega G(u)]$. It is assumed that $|G(u)|$ is small for large r only at late times, so that the outgoing wave packet arrives in the asymptotic region at late times [for example, in Minkowski space $G(u)=u$].

Let $F_{\omega lm}$ be the solution of the massless scalar wave equation, which in the asymptotic region has the form

$$F_{\omega lm} \sim (4\pi\omega)^{-1/2}(e^{-i\omega v}+e^{-i\omega G(u)})r^{-1}Y_{lm}(\theta,\phi)\,. \qquad (2.1)$$

This asymptotic form is independent of whether the field obeys the minimally coupled equation, $\Box\phi=0$, or the conformally coupled one, $\Box\phi+\frac{1}{6}R\phi=0$. Wave packets formed from the $F_{\omega lm}$ are incoming at early times and outgoing at late times in accordance with propagation by geometrical optics. Any positive-frequency solution of the scalar wave equation which is incoming on $\mathscr{I}^-$ can be written as a wave packet formed from the $F_{\omega lm}$. Therefore we can write the Hermitian field operator in the form

$$\phi=\sum_{lm}\int_0^\infty d\omega(a_{\omega lm}F_{\omega lm}+a^\dagger_{\omega lm}F^*_{\omega lm})\,. \qquad (2.2)$$

The normalization of the $F_{\omega lm}$ is such that

$$(F_{\omega lm}, F_{\omega' l' m'})=\delta(\omega-\omega')\delta_{ll'}\delta_{mm'}\,, \qquad (2.3)$$

where the conserved scalar product is defined by

$$(h,f)=-i\int_s (h\partial^\mu f^* - f^*\partial^\mu h)(-g)^{1/2}ds_\mu\,, \qquad (2.4)$$

where ds_μ is a future-directed surface element of the spacelike three-dimensional hypersurface s. [In the Appendix, the normalization of Eq. (2.1) is obtained by considering a wave packet formed from the $F_{\omega lm}$ at early times, and taking s as a constant t hypersurface with $ds_0=drd\theta d\phi$.] As a consequence of Eq. (2.3) and the canonical commutation relations for the field and its conjugate momentum, the creation and annihilation operators satisfy

$$[a_{\omega lm}, a^\dagger_{\omega' l' m'}]=\delta(\omega-\omega')\delta_{ll'}\delta_{mm'}\,,$$
$$[a_{\omega lm}, a_{\omega' l' m'}]=0\,. \qquad (2.5)$$

The state of the field is specified by the state vector $|0\rangle$ containing no particles of the field at early times, i.e.,

$$a_{\omega lm}|0\rangle=0 \quad \text{for all } \omega, l, m\,. \qquad (2.6)$$

The energy-momentum tensor of the massless, minimally coupled, scalar field is (the effect of conformal coupling is considered below)

$$T_{\mu\nu}=\phi_{,\mu}\phi_{,\nu}-\tfrac{1}{2}g_{\mu\nu}\phi_{,\alpha}\phi^{,\alpha}\,, \qquad (2.7)$$

where a comma denotes the ordinary derivative, and symmetrization is understood. The average energy flux of particles radiated to $\mathscr{I}^+$ is given formally by the expectation value of

$$T^r{}_t=\tfrac{1}{2}(\phi_{,t}\phi^{,r}+\phi^{,r}\phi_{,t})\,. \qquad (2.8)$$

However, this operator is not well defined because it is quadratic in fields evaluated at a single point. It can be replaced by a well-defined operator having finite expectation values by separating the points at which the fields appearing in the quadratic product are evaluated, and taking the limit as the separation of the two points vanishes. No infinite renormalization is necessary when this procedure is applied to $T^r{}_t$, although that is not the case for the diagonal components of $T^\mu{}_\nu$. Because we work in the asymptotically flat region, possible ambiguities coming from the local curvature do not arise.

From Eqs. (2.2), (2.6), and (2.8) one finds that

$$\langle 0|T^r{}_t|0\rangle = \frac{1}{2}\sum_{lm}\int_0^\infty d\omega\,(F_{\omega lm,t}F^*_{\omega lm}{}^{,r} + F_{\omega lm}{}^{,r}F^*_{\omega lm,t})\,. \qquad (2.9)$$

We evaluate this in the asymptotic region (large r) using Eq. (2.1) and replacing u, v by $u+\epsilon$, $v+\epsilon$ in the function $F^*_{\omega lm}$ appearing as the right member of each product of F's in Eq. (2.9). Thus one obtains the following for the point-separated expression:

$$\langle 0|T^r{}_t|0\rangle_\epsilon = (4\pi r^2)^{-1}\sum_{lm}|Y_{lm}|^2\int_0^\infty d\omega\,\omega[G'(u)G'(u+\epsilon)\,e^{i\omega[G(u+\epsilon)-G(u)]} - e^{i\omega\epsilon}]\,, \qquad (2.10)$$

where a prime denotes the derivative with respect to the argument. One has

$$\int_0^\infty d\omega\,\omega e^{i\omega\epsilon} = -\epsilon^{-2}\,,$$

and

$$G'(u)G'(u+\epsilon)\int_0^\infty d\omega\,\omega e^{i\omega[G(u+\epsilon)-G(u)]} = -\frac{G'(u)G'(u+\epsilon)}{[G(u+\epsilon)-G(u)]^2} \sim -\epsilon^{-2} + \frac{1}{4}\left(\frac{G''}{G'}\right)^2 - \frac{1}{6}\left(\frac{G'''}{G'}\right) + O(\epsilon)\ \text{ as } \epsilon\to 0\,,$$

where we have made the integrals well defined by introducing convergence factors $e^{-\omega\alpha}$ into the integrands and taking the limit as α vanishes. Introducing these expressions into Eq. (2.10), one finds in the limit as ϵ approaches zero that

$$\langle 0|T^r{}_t|0\rangle = \frac{1}{4\pi r^2}\sum_{lm}|Y_{lm}|^2\left[\frac{1}{4}\left(\frac{G''}{G'}\right)^2 - \frac{1}{6}\,\frac{G'''}{G'}\right]. \qquad (2.11)$$

The total power radiated across a sphere of radius r at late times is

$$P = \int \langle 0|T^r{}_t|0\rangle\, r^2 \sin\theta\, d\theta d\phi$$
$$= \frac{1}{4\pi}\sum_{lm}\left[\frac{1}{4}\left(\frac{G''}{G'}\right)^2 - \frac{1}{6}\,\frac{G'''}{G'}\right]. \qquad (2.12)$$

We infer that the power radiated in the mode (l,m) across a large sphere is

$$P_{lm}(u) = \frac{1}{24\pi}\left[\frac{3}{2}\left(\frac{G''}{G'}\right)^2 - \frac{G'''}{G'}\right]. \qquad (2.13)$$

The sum of l in Eq. (2.12) is divergent; however, that is because we have neglected backscattering by the curved spacetime, which in a geometry like that of a collapsing body reduces the radiated flux, especially for large l. For the cases of interest, such as a collapsing body, we expect Eq. (2.13) to be a good approximation for sufficiently small l, while for large l the radiated power P_{lm} will effectively vanish.

This can be checked for radiation by a body of mass M collapsing to form a black hole. In that case, Hawking[5] showed that

$$G(u) = -C\exp[(4M)^{-1}u] + v_0\,, \qquad (2.14)$$

where C and v_0 are constants. Then Eq. (2.13) gives, for the power radiated in mode (l, m) at late times,

$$P_{lm} = (768\pi M^2)^{-1}\,. \qquad (2.15)$$

The expression given for this case in Ref. 6, Eq. (136) leads to

$$P_{lm} = \frac{1}{2\pi}\int_0^\infty d\omega\,|B_l(\omega)|^2\left[\exp\left(\frac{\omega}{kT}\right) - 1\right]^{-1}, \qquad (2.16)$$

where $|B_l(\omega)|^2$ is the probability that a photon of energy ω in angular mode (l, m) will reach $\mathscr{I}^+$ from $\mathscr{I}^-$. If $|B_l(\omega)|^2$ is approximated by unity, one finds by direct integration that

$$P_{lm} = \pi[12(kT)^2]^{-1}\,, \qquad (2.17)$$

which agrees with Eq. (2.15) when the temperature $T = (8\pi kM)^{-1}$ of the black hole is substituted. The approximation of Eq. (2.13) is not good in this case for large l because of the neglect of backscattering into the black hole, which reduces the actual value of P_{lm}.

One can also write Eq. (2.13) in terms of the function F which is the inverse of G [i.e., $v = G(u)$ implies $u = F(v)$]. One finds that

$$P_{lm}(v) = \frac{1}{24\pi}\left[\frac{F'''}{(F')^3} - \frac{3}{2}\left(\frac{F''}{F'^2}\right)^2\right], \qquad (2.18)$$

where now the prime denotes the derivative with respect to v. Here P_{lm} is given as a function of v, but can be expressed as a function of u by writing $v = G(u)$.

The energy-momentum tensor given in Eq. (2.7) is that for the minimally coupled field. Although the asymptotic form of the solutions of the wave equation, Eq. (2.1), is the same for the minimally coupled and the conformally coupled scalar field, the energy-momentum tensors differ even in flat space. That for the conformal field is,[7] in the flat region,

$$\Theta_{\mu\nu} = \phi_{,\mu}\phi_{,\nu} - \tfrac{1}{2}g_{\mu\nu}\phi_{,\alpha}\phi^{,\alpha} - \tfrac{1}{6}(\phi^2)_{,\mu\nu} + \tfrac{1}{6}g_{\mu\nu}\Box\phi^2\,. \qquad (2.19)$$

The flux is the expectation value of

$$\Theta^r{}_t = T^r{}_t - \tfrac{1}{6}(\phi^2)^{;r}{}_t \,. \tag{2.20}$$

A calculation analogous to that above[8] reveals that the analog of Eq. (2.13) for the power radiated in mode (l, m) is

$$\hat{P}_{lm} = (48\pi)^{-1}\left(\frac{G''}{G'}\right)^2 .$$

Thus the power is different for the two fields; however, the total energy radiated over all time is the same in both cases since

$$\int_{-\infty}^{\infty}\left(\frac{G''}{G'}\right)^2 du = \int_{-\infty}^{\infty}\frac{G'''}{G'}\,du\,,$$

provided G'' and $G''' \to 0$ as $u \to \pm\infty$. Furthermore, both expressions yield the same value, Eq. (2.15), for the Hawking flux. It is interesting to note that $\hat{P}_{lm} \geq 0$, whereas P_{lm} can become negative for some choices of G. Such negative fluxes are due to quantum coherence effects associated with states containing an indefinite number of particles and also arise in the radiation from moving mirrors.[4]

In the case of two-dimensional spacetimes, the minimally coupled and the conformally coupled scalar field are equivalent, and the energy-momentum tensor is given by Eq. (2.7). The net flux is given exactly by the right-hand side of Eq. (2.13), the labels l and m no longer having any significance. The fact that this expression is exact arises from the conformal flatness of two-dimentional spacetimes, so the modes which have the form of $e^{-i\omega v}$ on $\mathscr{I}^-$ and $e^{-i\omega G(u)}$ on $\mathscr{I}^+$ are exact solutions of the wave equation. For a more detailed discussion of the calculation of the energy flux in two-dimensional spacetimes, see the papers of Davies, Fulling, and Unruh[9] and of Davies.[10]

III. COLLAPSING DUST CLOUDS AND SHELL-CROSSING SINGULARITIES

We shall calculate the function $F(v)$ for the spacetime of a spherically symmetric collapsing pressureless fluid or dust in which a shell-crossing singularity forms. The metric can be written in comoving coordinates[11] as

$$ds^2 = dt^2 - \Gamma^{-2}(r)[R'(r,t)]^2 dr^2 - R^2(r,t)\,d\Omega^2\,, \tag{3.1}$$

where a prime denotes $\partial/\partial r$. Each particle of the collapsing dust falls along a radial geodesic characterized by constant values of the comoving coordinates r, θ, ϕ. The dust cloud or fluid can be thought of as made up of concentric shells labeled by r. The shell at r has proper circumference $2\pi R(r,t)$. The motion of each shell as a function of time is governed by the equation

$$\tfrac{1}{2}\dot{R}^2(r,t) - m(r)R^{-1}(r,t) = \tfrac{1}{2}[\Gamma^2(r) - 1]\,. \tag{3.2}$$

which is a consequence of the Einstein equations, and resembles the Newtonian equation for a given shell r. Here

$$m(r) = \mu\int_0^r \Gamma(r')dr'\,, \tag{3.3}$$

where μ is an arbitrary constant giving the proper energy of a fluid element, and $r=0$ labels the "innermost" shell, which need not have zero circumference $2\pi R(0,t)$. A fluid element is defined by the condition that it have rest energy μ, and that the total number of fluid elements in all shells labeled by $r' \leq r$ is constant. As in Ref. 11, we take the comoving coordinate r to be the number of fluid elements contained in all shells with $r' \leq r$. The quantity $m(r)$ can be interpreted as the total energy of the matter contained in those shells, with the factor $\Gamma(r)$ needed to take into account the kinetic energy and gravitational binding energy of the various shells. The rest energy contained in the shells in the range r to $r+dr$ is μdr. The proper volume of those shells found from Eq. (3.1) is $|4\pi R^2 R' \Gamma^{-1} dr|$, so that the proper energy density of the shells near r is

$$\epsilon = \frac{\mu\Gamma}{|4\pi R^2 R'|}\,. \tag{3.4}$$

We will be dealing with solutions such that initially the matter is very dispersed and the shells start to fall from infinity [i.e., $\lim_{t\to-\infty}R(r,t)=\infty$] with no kinetic energy. Then we have

$$\Gamma(r) = 1 \tag{3.5}$$

for all the shells. That this condition is independent of time indicates that the effect of the kinetic energy and gravitational binding energy cancel at later times. We also take $R'(r,t) > 0$ initially for all shells. As the system evolves it can happen that $R'(r,t)$ changes sign, indicating that shells with $r' > r$ have fallen past the shell labeled by r and now have smaller circumferences. During such a process R' passes through zero, and the proper energy density of Eq. (3.4) becomes infinite. Such an event is called a shell-crossing singularity. It is a naked singularity in the sense that components of the Riemann curvature tensor become infinite and are not hidden from infinity by an event horizon. The nature of such singularities has been discussed by Yodzis *et al.*[12] Apart from questions of how realistic shell-crossing singularities are, problems concerning the energy-flux and particle creation occurring up to the formation of the singularity can be answered independently of the boundary conditions at the singularity.

In the present case, the total mass of the collapsing dust cloud is given by Eq. (3.3) as

$$m(r_0) = \mu r_0\,, \tag{3.6}$$

where r_0 is the largest value of r labeling a shell of matter. Outside the dust cloud one will have an exterior Schwarzschild geometry of mass $m(r_0)$, and beyond the innermost shell one has a flat spacetime (if the circumference of the inner surface of the dust cloud is not zero). The solution of Eq. (3.2) with $\Gamma(r) = 1$ is

$$R(r,t) = \{\tfrac{3}{2}[f(r) - (2\mu r)^{1/2}t]\}^{2/3}, \tag{3.7}$$

where $f(r)$ is an arbitrary function of r. The g_{rr} component of the metric is

$$g_{rr} = (R')^2 = [f' - \tfrac{1}{2}(2\mu r^{-1})^{1/2}t]^2/R. \tag{3.8}$$

We will be interested here in the case when the function $f(r)$ is a positive constant, i.e.,

$$f(r) = a. \tag{3.9}$$

Then we can write

$$R(r,t) = 3Br^{1/3}(Ar^{-1/2} - t)^{2/3}, \tag{3.10}$$

where

$$A = a(2\mu)^{-1/2}, \quad B = (\mu/6)^{1/3}. \tag{3.11}$$

Also

$$R' = -Btr^{-2/3}(Ar^{-1/2} - t)^{-1/3}. \tag{3.12}$$

For $t<0$, increasing r corresponds to increasing proper circumference ($R'>0$). At $t=0$, R' vanishes so that the proper energy density ϵ becomes infinite, and

$$R(r,0) = 3BA^{2/3} = (3a/2)^{2/3}, \tag{3.13}$$

for all r, so that all the shells have met at $t=0$ at a single circumference. One can formally continue the solution (3.10) past the shell-crossing singularity up to the time $Ar^{-1/2}$ when the shell labeled by r reaches zero circumference, but we will only require the solution for $t<0$, before the time of formation of the shell-crossing singularity.

A Penrose diagram of the solution is shown in Fig. 1, in the case when the shell-crossing singularity forms at the Schwarzschild radius. This is the most advantageous case for using geometrical optics, as a ray which reaches $\mathscr{I}^+$ with finite frequency at late times has a very high frequency when it passes through the region of high curvature. (We will investigate the range of frequencies observed at $\mathscr{I}^+$ for which geometrical optics is valid later.)

Next, we match the metrics at the inner and outer boundaries of the collapsing dust, and trace null rays from $\mathscr{I}^-$ to $\mathscr{I}^+$ to find the function $F(v)$ giving the value of u at which the ray originating at v on $\mathscr{I}^-$ arrives on $\mathscr{I}^+$. In these considerations, the inner radius R_i of the cloud, at which the singularity forms, can be outside or at the Schwarzschild radius.

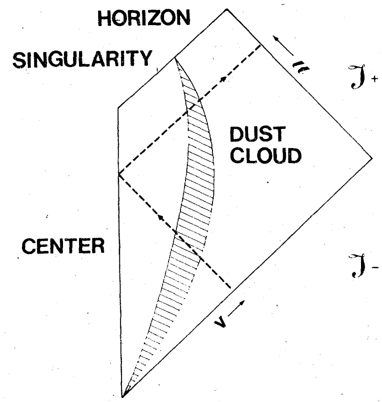

FIG. 1. The spacetime of a collapsing dust cloud which forms a singularity at the Schwarzschild radius. The dashed line is a null ray which passes through the cloud.

IV. PARTICLE CREATION BY SHELL-CROSSING SINGULARITIES

Exterior to the outer shell ($r=r_0$), one has the Schwarzschild metric

$$ds^2 = \left(1 - \frac{2M}{R}\right)dT^2 - \left(1 - \frac{2M}{R}\right)^{-1}dR^2 - R^2d\Omega^2, \tag{4.1}$$

where R denotes the Schwarzschild coordinates. This must be matched to the metric of Eq. (3.1), with Γ, R, and R' given by Eqs. (3.5), (3.10), and (3.12), respectively. The circumference of the outer boundary of the dust is proportional to

$$R(r_0,t) \equiv R_0(t). \tag{4.2}$$

On the three-dimensional hypersurface Σ in the spacetime traced out by the collapsing outer boundary, the intrinsic geometry expressed in terms of the metrics of Eqs. (3.1) and (4.1) must agree. From the angular part of the line elements one finds that the Schwarzschild coordinate R has the value $R_0(t)$ on the boundary. Therefore the matching condition on Σ is

$$1 = \left[1 - \frac{2M}{R_0(t)}\right]\left[\frac{dT}{dt}\right]^2 - \left[1 - \frac{2M}{R_0(t)}\right]^{-1}\left[\frac{dR_0}{dt}\right]^2. \tag{4.3}$$

The integral of this equation, relating the exterior Schwarzschild time T to the proper time t of a dust particle at the surface, is

$$T = t - (2M)^{-1/2}a - 2(2M)^{1/2}R_0^{1/2}(t) - 2M\ln\left|\frac{R_0^{1/2}(t) - (2M)^{1/2}}{R_0^{1/2}(t) + (2M)^{1/2}}\right|. \tag{4.4}$$

Here the condition

$$M = \mu r_0, \tag{4.5}$$

which comes from matching the extrinsic curvature of the hypersurface Σ, has been used. In the case that the shell-crossing singularity forms at the Schwarzschild radius one has the further condition that $R(r, 0) = 2M$, which requires that

$$a = \tfrac{2}{3}(2M)^{3/2} . \quad (4.6)$$

The matching at the inner boundary of the dust is trivial because the circumference of the shell at $r = 0$ is constant for $t < 0$, as there is no matter present at smaller radii to attract the inner shell. One has from Eq. (3.10) that

$$R(0, t) = (3a/2)^{2/3} \equiv R_i \quad (t < 0) . \quad (4.7)$$

One finds that the metric at smaller radii than that of the inner shell is

$$ds^2 = dt^2 - dR^2 - R^2 d\Omega^2 , \quad (4.8)$$

where the Minkowski coordinate t is the same as the comoving coordinate t of Eq. (3.1) on the inner boundary of the cloud, and the coordinate R has the value in Eq. (4.7) on that boundary.

In the Schwarzschild region [metric of Eq. (4.1)] the well-known null coordinate which is constant along outgoing null rays is

$$u = T - R - 2M \ln\left(\frac{R - 2M}{2M}\right) . \quad (4.9)$$

In the region occupied by the dust [metric of Eq. (3.1) with $\Gamma = 1$ and R given by Eq. (3.10)], null rays are determined by

$$\frac{dt}{dr} = \pm R'(r, t) , \quad (4.10)$$

where R' is given by Eq. (3.12). We are particularly interested in the outgoing null rays which pass through the cloud when t is small and negative, just before formation of the shell-crossing singularity. For $-t \ll A r_0^{-1/2} = a(2M)^{-1/2}$, one has the following, to lowest order in $-t/(Ar^{-1/2})$ for outgoing rays:

$$\frac{dt}{dr} = -A^{-1/3} B r^{-1/2} t , \quad (4.11)$$

which gives

$$\ln(-t) = -2A^{-1/3} B r^{1/2} + \text{const} . \quad (4.12)$$

Hence, a null coordinate which is constant along outgoing rays is [for $-t \ll a(2M)^{-1/2}$]

$$\tilde{u} = t \exp(2A^{-1/3} B r^{1/2}) = t \exp[(2\mu r/R_i)^{1/2}] , \quad (4.13)$$

where R_i is the inner radius of the cloud given by Eq. (4.7). The incoming null ray which passes through the center and out of the cloud just before formation of the singularity at $t = 0$, enters the cloud at too early a time to make use of the same approximation as in Eq. (4.11). For example, in order to pass through the flat region interior to the inner shell before $t = 0$, it must pass in through that shell at $t < -2R_i$, which can be shown to be earlier than the time at which that approximation becomes valid. As we are interested in the effect of the singularity, and the cloud is relatively tenuous as the incoming ray passes through it, we will make the simplification of imagining that the incoming ray passes through flat spacetime on its way from $\mathscr{I}^-$ to the center. Thus we use the Minkowski null coordinate to characterize the incoming ray. The contribution to the particle production coming from the formation of the singularity occurs as the ray is outgoing, and will not be affected by this simplification. Alternatively, one could use the incoming Schwarzschild null coordinate to characterize the incoming ray up to some average radius R_a of the cloud as the ray passes in through it, and then match it to the Minkowski null coordinate for $R < R_a$. (One can show that R_a should be greater than or of order $3M$ for $R_i \geq 2M$.) This procedure does not alter our conclusions. To avoid unnecessary complication we use the simpler approach here.

To find the function $F(v)$ giving the value of u on $\mathscr{I}^+$ of a radial null ray originating at v on $\mathscr{I}^-$, consider an incoming null ray on the path $v = \text{const}$. In the flat region the outgoing null coordinate is

$$U = t - R , \quad (4.14)$$

and one has, with the above simplification, $v = t + R$, so that after the ray passes through the center ($R = 0$) it moves on the path

$$U = v . \quad (4.15)$$

When the outgoing ray reaches the inner boundary of the cloud, the t coordinates are identical, and one has $R = R_i$ and $r = 0$, so that

$$\tilde{u} = U + R_i = v + R_i . \quad (4.16)$$

When the outgoing null ray reaches the outer boundary of the dust cloud, T is given as a function of t by Eq. (4.4), and one has $r = r_0$ and $R = R_0(t)$. Then Eq. (4.9) gives

$$u = T(t) - R_0(t) - 2M \ln\left(\frac{R_0(t) - 2M}{2M}\right) , \quad (4.17)$$

and Eqs. (4.13) and (4.16) give t as a function of v:

$$t = (v + R_i) \exp[-(2M/R_i)^{1/2}] . \quad (4.18)$$

Since it is the ray that goes through when $|t|$ is small that is of interest, one can use in Eq. (4.4) and Eq. (4.17) the first-order expansion of $R_0(t)$:

$$R_0(t) \approx R_i - (2M/R_i)^{1/2} t . \quad (4.19)$$

Finally, Eqs. (4.17) and (4.18) give u as a function of v. One finds that

$$u=F(v)=(1+\alpha)e^{-\alpha}(v+R_i)-R_i(1+2\alpha+\tfrac{2}{3}\alpha^{-1})$$
$$-2M\ln\left|\frac{\alpha^2R_i^{-2}(v+R_i)^2-(3-2\alpha-\alpha^2)\alpha R_i^{-1}(v+R_i)+2(1-\alpha)(1-\alpha^2)}{2(1+\alpha)\alpha^2}\right|, \quad (4.20)$$

where

$$\alpha=(2M/R_i)^{1/2}, \quad (4.21)$$

and the term involving t^2 has been retained in the logarithm because when the singularity forms on the Schwarzschild horizon one has $\alpha=1$, so only this term is nonzero. The ray which gets out of the cloud at $t=0$, just as the shells cross, corresponds to $v=-R_i$. The power radiated in each mode, as calculated using either Eqs. (2.18) or (2.21), does not diverge for $v\leqslant -R_i$. As these expressions approximate the actual P_{lm} for small l, and can be used as an upper bound on P_{lm} for large l, we conclude that there is no indication from this geometrical-optics result that any particularly intense particle production will be produced just prior to formation of this shell-crossing singularity, or that there will be a significant back reaction which might influence its formation.

In general, geometrical optics does not apply at all frequencies, so the energy flux calculated from Eq. (2.18) will contain some finite error. However, since the geometrical-optics approximation does apply for sufficiently short wavelengths, the error should be finite. That is, a finite result in the geometrical-optics approximation suggests that the actual flux is also finite even though some error may be introduced into the lower frequency modes.

The case when the shell-crossing singularity forms at the Schwarzschild horizon, that is, when

$$R_i=2M, \quad (4.22)$$

is of special interest. In that case, one finds that for t near zero or v near $-2M$,

$$T\approx -2M\ln\left(-\frac{t}{8M}\right), \quad (4.23)$$

and

$$u=F(v)\approx -4M\ln\left[-\frac{(v+2M)}{4M}\right]. \quad (4.24)$$

This expression is valid for t in the range $0<-t/M \ll 1$. The effect of the Schwarzschild exterior metric on the incoming null ray will only alter the constants appearing in the argument of the logarithm, which has no effect on the observed flux. This function $F(v)$ is of the same form as that obtained by Hawking for the spacetime of a body collapsing to form a black hole in the absence of a shell-crossing singularity. Thus one expects a thermal spectrum at late times (large u) of the same temperature,

$$T=(8\pi kM)^{-1}, \quad (4.25)$$

as for a black hole of mass M (here k is Boltzmann's constant). This is in fact the case. The analysis of the limitations of geometrical optics given below shows that for all frequencies reaching $\mathscr{I}^+$, geometrical optics is valid. This is true in spite of the fact that the curvature is increasing without bound.

The range of validity of geometrical optics in a local inertial frame L comoving with the dust depends on the components of the Riemann curvature tensor in that frame. One finds the following for the nonvanishing independent components in the comoving inertial frame L at a point r,t in the dust cloud:

$$\begin{aligned} &R^r{}_{ttr}=\ddot{R}'/R',\\ &R^\phi{}_{rr\phi}=R^\theta{}_{rr\theta}=-\dot{R}\dot{R}'/(RR'),\\ &R^\theta{}_{tt\theta}=R^\phi{}_{tt\phi}=\ddot{R}/R,\\ &R^\phi{}_{\theta\theta\phi}=-\dot{R}^2/R^2, \end{aligned} \quad (4.26)$$

where R is given in Eq. (3.10), and the dot and prime refer to derivatives with respect to the coordinates t and r, respectively. Just before formation of the singularity (for $-t$ small) these components have the form

$$\begin{aligned} R^r{}_{ttr}&\approx -R^\theta{}_{rr\theta}=-R^\phi{}_{rr\phi}\\ &\approx(2\mu r/R_i)^{1/2}(R_it)^{-1},\\ R^\theta{}_{tt\theta}&=R^\phi{}_{tt\phi}\approx -\tfrac{1}{2}R^\phi{}_{\theta\theta\phi}\approx(\mu r/R_i)R_i^{-2}. \end{aligned} \quad (4.27)$$

As is well known, the metric in L can be regarded as Minkowskian over a region of spacetime of linear dimension l as measured in the coordinates of L, where $l\lesssim |R^\alpha{}_{\beta\gamma\delta}|^{-1/2}$ for all curvature tensor components in L. In particular, light will propagate by geometrical optics if its frequency $\hat{\omega}$ as measured in L satisfies

$$\hat{\omega}\gtrsim |R^\alpha{}_{\beta\gamma\delta}|^{1/2} \quad (4.28)$$

for all components. The curvature tensor components of Eq. (4.27) are largest at the outer boundary r_0 of the cloud. Then geometrical optics is valid for frequencies as measured in L such that

$$\hat{\omega}\gtrsim \alpha^{1/2}(-R_it)^{-1/2}, \quad (4.29)$$

where α is defined in Eq. (4.21). This range of frequencies becomes narrower as $t\to 0$, but it is the range of frequencies as observed at infinity, and not in L, that determines over what range the previous results are valid.

Therefore, we need the relation between the frequency ω observed at rest at infinity, and the frequency $\hat{\omega}$ relative to a local inertial frame L, instantaneously comoving with the dust at the outer boundary of the cloud just before the shells cross at $t=0$. The four-velocity u^μ of a freely falling dust particle at the outer boundary of the cloud is, in the Schwarzschild coordinates of Eq. (4.1),

$$u^\mu = \left(\frac{dT}{dt}, \frac{dR_0}{dt}, 0, 0\right)$$
$$= \left(H^{-1}\left[H + \alpha^2\left(1 - \frac{3}{2}\frac{\alpha t}{R_i}\right)^{-2/3}\right]^{1/2}, -\alpha\left(1 - \frac{3}{2}\frac{\alpha t}{R_i}\right)^{-1/3}, 0, 0\right), \tag{4.30}$$

where

$$H = 1 - \frac{2M}{R_0(t)}; \tag{4.31}$$

the coordinate t is the proper time of the dust particle; $R_0(t)$ is given in Eq. (3.10); and dT/dt is found from Eq. (4.3). Let p^μ be the four-momentum of a radially outgoing photon. In the Schwarzschild coordinates at the outer boundary of the cloud one finds from $p^\mu p_\mu = 0$ that

$$p^R = Hp^T. \tag{4.32}$$

Thus the frequency $\hat{\omega}$ observed in the instantaneously comoving inertial frame L of the dust particle is

$$\hat{\omega} = p^\mu u_\mu = p^T\left\{\left[H + \alpha^2\left(1 - \frac{3}{2}\frac{\alpha t}{R_i}\right)^{-2/3}\right]^{1/2} + \alpha\left(1 - \frac{3}{2}\frac{\alpha t}{R_i}\right)^{-1/3}\right\}. \tag{4.33}$$

Near formation of the singularity, one has

$$\hat{\omega} \approx p^T[(H+\alpha^2)^{1/2} + \alpha]. \tag{4.34}$$

To relate this to the frequency ω observed at rest in the asymptotic region we use the conserved quantity $p^\mu \xi_\mu$, where ξ^μ is the timelike Killing vector, $\xi^\mu = \delta^\mu_0$ in the Schwarzschild coordinates.

One has for the frequency at infinity,

$$\omega = p^\mu \xi_\mu = p^T H. \tag{4.35}$$

(The usual red-shift factor of $H^{1/2}$ is obtained if one notes that $p^T = H^{-1/2}\omega'$, where ω' is the frequency relative to an inertial frame instantaneously at rest in the Schwarzschild coordinates.) Thus for small $|t|$ one has

$$\omega \approx [(H+\alpha^2)^{1/2} + \alpha]^{-1} H \hat{\omega}. \tag{4.36}$$

It follows from Eq. (4.29) that geometrical optics is valid for small $|t|$ for the following frequency range observed at rest at infinity:

$$\omega \gtrsim [(H+\alpha^2)^{1/2} + \alpha]^{-1} H \alpha^{1/2} (-R_i t)^{-1/2}. \tag{4.37}$$

Therefore, when $R_i > 2M$, any given frequency ω will pass out of the above range of validity as $|t| \to 0$. However, when the shell-crossing singularity forms on the Schwarzschild horizon $R_i = 2M$, one has $\alpha = 1$ and $H \approx -t/(2M)$ for small $|t|$, so that geometrical optics is valid in the range

$$\omega \gtrsim \frac{1}{4M}\left(-\frac{t}{2M}\right)^{1/2}. \tag{4.38}$$

Thus, when the singularity forms at the Schwarzschild horizon, geometrical optics is valid for the entire spectrum in the limit $t \to 0$.

V. THE REISSNER-NORDSTRÖM SINGULARITY

An example of a spacetime which possesses a naked singularity is the Reissner-Nordström solution with the charge Q greater than the mass M. The singularity is timelike and of a more serious nature than the shell-crossing singularities discussed above. One suspects that this singularity will be unstable in a theory in which quantum effects are included. However, in the case of a static singularity there is an ambiguity in the choice of boundary conditions to be imposed at the singularity itself. This question can be circumvented for spacetimes in which the singularity has an origin in time, as is the case for a charged shell which collapses to zero radius.

A shell with $Q < M$ will form a black hole as seen by an external observer; after it crosses its event horizon it may, depending upon initial conditions, reach zero radius.[13] A shell with $Q > M$ and having positive proper mass which collapses in accordance with Einstein's equations will rebound without forming either a singularity or an event horizon. There are two ways to overcome this difficulty and obtain spacetimes in which the singularity forms at a finite time. One is to suspend Einstein's equations and allow the shell to have any convenient trajectory. The other is to endow the shell with negative proper mass; such shells have been discussed by Boulware,[14] who finds that it is then possible to form a naked singularity in a spacetime which satisfies Einstein's equations. We will consider both possibilities below.

Spacetimes which are not solutions of Einstein's equations are still of interest for the purpose of understanding quantum field theory in curved spacetime. All two-dimensional models are in this category and, as was noted previously, the geometrical-optics solution of a four-dimensional, spherically symmetric model may be reinterpreted as the exact solution of a two-dimensional model. The simplest collapsing charged shell is that for which the radius decreases linearly in the proper time, that is,

$$R(\tau) = -\gamma\tau\,, \tag{5.1}$$

where $\gamma > 0$ is a constant and τ is the proper time along the shell. If $Q > M$, this shell forms a naked singularity at $\tau = 0$. The spacetime is given by the exterior Reissner-Nordström metric

$$ds_1^2 = \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)dT^2 - \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 - r^2 d\Omega^2\,, \quad r > R \tag{5.2}$$

and the interior flat metric

$$ds_2^2 = dt^2 - dr^2 - r^2 d\Omega^2\,, \quad r < R\,. \tag{5.3}$$

The relation between t and T is determined by the requirement that the timelike hypersurface Σ formed by the history of the shell have the same intrinsic geometry in both metrics, i.e.,

$$ds_1^2 = ds_2^2 \tag{5.4}$$

at $r = R$. This yields the relation

$$\left(\frac{dt}{dT}\right)^2 = C + (1 - C^{-1})\left(\frac{dR}{dT}\right)^2, \tag{5.5}$$

where

$$C = 1 - \frac{2M}{R} + \frac{Q^2}{R^2}\,. \tag{5.6}$$

The metric of the hypersurface Σ is

$$ds_3^2 = d\tau^2 - R^2 d\Omega^2\,. \tag{5.7}$$

The requirement that $ds_3^2 = ds_1^2$ on Σ yields

$$\left(\frac{dT}{d\tau}\right)^2 = C^{-1}\left[1 + C^{-1}\left(\frac{dR}{d\tau}\right)^2\right]. \tag{5.8}$$

Equations (5.5) and (5.8) imply that

$$\left(\frac{dt}{d\tau}\right)^2 = \left(\frac{dR}{d\tau}\right)^2 + 1\,. \tag{5.9}$$

The outgoing and ingoing null coordinates in the exterior $(r > R)$ region are, respectively,

$$u = T - r^*\,, \tag{5.10a}$$

and

$$v = T + r^*\,, \tag{5.10b}$$

where

$$\frac{dr^*}{dr} = \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1}. \tag{5.11}$$

The corresponding null coordinates in the interior $(r < R)$ region are

$$U = t - r\,, \tag{5.12a}$$

and

$$V = t + r\,. \tag{5.12b}$$

These equations contain all the information necessary for matching the null coordinates across the shell and obtaining the function $F(v)$. Thus far, Eq. (5.1) has not been imposed. Combined with Eq. (5.9), it implies that

$$t = a\tau\,, \tag{5.13}$$

where $a = (\gamma^2 + 1)^{1/2}$. Thus at $r = R$ one has

$$U = (a + \gamma)\tau\,, \tag{5.14}$$

and

$$V = (a - \gamma)\tau\,. \tag{5.15}$$

We are interested in null geodesics which pass through the shell shortly before formation of the singularity, so we may make expansions in powers of τ. Equation (5.8) implies that

$$T \sim -\gamma Q^{-1}\left(\tfrac{1}{2}\tau^2 - \tfrac{1}{3}MQ^{-2}\gamma\tau^3\right), \tag{5.16}$$

as $\tau \to 0$. As $r \to 0$, one has

$$r^* \sim (3Q^2)^{-1} r^3\,, \tag{5.17}$$

so that the relation between the u coordinate of an outgoing ray and the proper time at which it passes through the shell is

$$u \sim T(\tau) - (3Q^2)^{-1}R^3(\tau) \sim -\frac{\gamma}{2Q}\tau^2 + \frac{\gamma^2}{3Q^2}\left(\gamma + \frac{M}{Q}\right)\tau^3\,. \tag{5.18}$$

Thus we have

$$u(U) \sim -\frac{\gamma}{2Q}(a + \gamma)^{-2}U^2 + \frac{\gamma^2}{3Q^3}(\gamma Q + M)(a + \gamma)^{-3}U^3\,. \tag{5.19}$$

Similarly, the relation between the v coordinate of an incoming ray and the proper time at which it enters the shell is

$$v(\tau) \sim -\frac{\gamma}{2Q}\tau^2 - \frac{\gamma^2}{3Q^2}\left(\gamma - \frac{M}{Q}\right)\tau^3\,, \tag{5.20}$$

so that

$$v(V) \sim -\frac{\gamma}{2Q}(a - \gamma)^{-2}V^2 - \frac{\gamma^2}{3Q^3}(\gamma Q - M)(a - \gamma)^{-3}V^3\,. \tag{5.21}$$

Note that all of the null coordinates have been chosen to have the value of zero at the point where the singularity forms. The inverse of Eq. (5.21) is, as $v \to 0$,

$$V(v) \sim (a - \gamma)\left[-(-2\gamma^{-1}Qv)^{1/2} + \frac{2}{3Q}(\gamma Q - M)v\right]. \tag{5.22}$$

As usual, the matching condition at the origin is $U=V$. Hence one obtains the result that, as $v\to 0$,

$$u=F(v)\sim\beta[v+\alpha(-v)^{3/2}], \qquad (5.23)$$

where

$$\beta=\left(\frac{a-\gamma}{a+\gamma}\right)^2, \qquad (5.24)$$

and

$$\alpha=-\frac{4\gamma}{3Q^2}(2\gamma Q)^{1/2}(a+\gamma)^{-1}(aQ-M). \qquad (5.25)$$

From Eq. (2.18) one finds that the flux radiated is

$$P_{lm}\sim\frac{\alpha}{64\pi\beta^2}(-v)^{-3/2}\sim\frac{\alpha}{64\pi\beta^{1/2}}(-u)^{-3/2}, \qquad (5.26)$$

as $u\to 0$. Since $\alpha<0$, this diverging flux is negative. One may also express P_{lm} as a function of R. The u coordinate of an outgoing ray and the radius of the shell at which it leaves are related by

$$u\sim-(2\gamma Q)^{-1}R^2, \qquad (5.27)$$

so that

$$P_{lm}\sim-\frac{\gamma^3(aQ-M)}{12\pi(a-\gamma)R^3}, \qquad (5.28)$$

as $R\to 0$.

Recall that P_{lm} is the flux in a given mode for the minimally coupled scalar field in a four-dimensional mode, and is also the net flux for a scalar field in two dimensions (where minimal and conformal coupling are equivalent).[15] The flux per mode for the conformal field in four dimensions is given by Eq. (2.21); in the present model, we have

$$\hat{P}_{lm}\sim-\frac{3\alpha^2}{256\pi\beta u}, \qquad (5.29)$$

as $u\to 0$. Since u is negative, $\hat{P}_{lm}$ is positive. Thus both fields yield a diverging flux for this form of $R(\tau)$, although the sign of the flux is different.

Another set of models are those in which the shell has negative proper mass. In four dimensions such a shell can collapse in accordance with Einstein's equations to form a naked singularity. In general, the equation of motion for a collapsing charged shell is[16,17]

$$\left[1+\left(\frac{dR}{d\tau}\right)^2\right]^{1/2}=c-\frac{b}{R}, \qquad (5.30)$$

where

$$c=M/M_0, \qquad (5.31)$$

and

$$b=\frac{c^2Q^2-M^2}{2cM}=\frac{Q^2-M_0^2}{2M_0}. \qquad (5.32)$$

The gravitational mass M is positive, but the proper mass M_0 may have either sign. A negative proper mass is unphysical in that it requires a comoving observer to assign a negative mass density to the shell. However, it is still of interest to study the effect of the resulting singularity on the quantized field. If $c<0$ and $b<0$ ($M_0<0$ and $Q>|M_0|$) as well as $Q>M$, then the shell collapses inward from a finite initial radius to produce a naked singularity. As $\tau\to 0$, the solution of Eq. (5.30) may be given as an expansion in powers of $\xi=(-\tau)^{1/2}$:

$$R\sim a_0\xi+a_1\xi^2+a_2\xi^3+\cdots, \qquad (5.33)$$

where

$$a_0=(-2b)^{1/2}. \qquad (5.34)$$

We do not need the explicit expressions for a_1 and a_2.

One finds that, as $\xi\to 0$,

$$t(\xi)\sim-a_0\xi-a_1\xi^2-\left(\frac{2+3a_0a_2}{3a_0}\right)\xi^3, \qquad (5.35)$$

and

$$T(\xi)\sim A\xi^3, \qquad (5.36)$$

where

$$A=-\tfrac{1}{3}a_0Q^{-1}(4+a_0^4Q^{-2})^{1/2}. \qquad (5.37)$$

Thus along the shell one has

$$u(\xi)\sim\left(A-\frac{a_0^3}{3Q^2}\right)\xi^3, \qquad (5.38)$$

and

$$U\sim-2a_0\xi, \qquad (5.39)$$

so that

$$u(U)\sim-(24a_0^3Q^2)^{-1}(3Q^2A-a_0^3)U^3. \qquad (5.40)$$

Similarly, one has

$$V\sim-\tfrac{2}{3}a_0^{-1}\xi^3, \qquad (5.41)$$

and

$$v\sim\left(A+\frac{a_0^3}{3Q^2}\right)\xi^3, \qquad (5.42)$$

so that

$$V(v)\sim-2a_0^{-1}Q^2(3Q^2A+a_0^3)^{-1}v. \qquad (5.43)$$

Hence we find that

$$u=F(v)\sim Bv^3, \qquad (5.44)$$

where

$$B=\frac{Q^4}{3a_0^6}(3Q^2A-a_0^3)(3Q^2A+a_0^3)^{-3}. \qquad (5.45)$$

Since $|A|>a_0^3/3Q^2$, one has $B>0$.

The radiated flux (minimal field) is given by

$$P_{lm}(u) \sim -(54\pi u^2)^{-1}\,. \tag{5.46}$$

Thus the flux is negative and diverges as the singularity forms. It is of interest that this result is independent of any of the parameters associated with the shell. The flux associated with the conformal scalar field is given by

$$\hat{P}_{lm} \sim (108\pi u^2)^{-1}\,, \tag{5.47}$$

i.e., of the opposite sign from that for the minimal field.

In all cases the flux becomes infinite as the singularity forms. In the case of the four-dimensional versions of these models, one must recall that geometrical optics becomes a poor approximation near the singularity. Here there is no large compensating red-shift as there is in the case of singularities near the horizon of a black hole. Thus one should regard these results as only suggestive of the presence of a large flux in four dimensions. For the two-dimensional versions of the collapsing-shell models there is no such difficulty, and the fluxes given by Eqs. (5.26) and (5.46) are exact. Regardless of whether one prefers an exact solution of a less realistic model, or a heuristic treatment of a more realistic model, these results indicate that an attempt to form a naked singularity of the Reissner-Nordström variety will lead to a strong back reaction. The final effect of this back reaction can only be determined by a self-consistent calculation.

We are considering only a neutral scalar field, so the coupling is entirely to the spacetime geometry. A charged field will also couple to the electromagnetic field and produce an even greater back reaction.[18]

A diverging positive flux may be interpreted as the shell radiating away all of its mass before a singularity can form. A diverging negative flux is more surprising, but it may be interpreted as the shell forming a black hole rather than a naked singularity. That is, if a shell for which $Q > M$ initially can radiate enough negative energy to infinity, its mass will increase sufficiently that a horizon will form and avoid a naked singularity.[19] Although neither of these scenarios can be verified without a self-consistent treatment of the back reaction, an infinite flux of either sign may be taken as demonstrating the instability of the singularity.

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APPENDIX: NORMALIZATION OF WAVE FUNCTIONS

Let $F_{\omega lm}$ be the solution of the scalar wave equation which in the asymptotic region (r large) has the form

$$F_{\omega lm} \sim N(\omega)(e^{-i\omega v} + e^{-i\omega G(u)})r^{-1}Y_{lm}(\theta,\phi)\,, \tag{A1}$$

where $N(\omega)$ is a normalization factor which is to be found. Any positive-frequency solution of the scalar wave equation which is incoming on $\mathscr{I}^-$ can be written as a wave packet formed from solutions of the form (A1). It is assumed that $|G(u)|$ is small for large r only at late times, so that the term involving $G(u)$ does not contribute to the wave packet at early times. The normalization factor $N(\omega)$ is determined to within a phase factor by the requirement that

$$(F_{\omega lm}, F_{\omega' l' m'}) = \delta(\omega-\omega')\delta_{ll'}\delta_{mm'}\,, \tag{A2}$$

where the conserved scalar product is defined in Eq. (2.4). This normalization is required so that the canonical commutation relations will imply that the coefficients $a_{\omega lm}$ in the expansion of the field ϕ are annihilation operators obeying the correct commutation rules.

To find $N(\omega)$ consider a wave packet

$$Q_{\omega_1 lm} = \int_0^\infty d\omega f_1(\omega) F_{\omega lm}\,, \tag{A3}$$

where $f_1(\omega)$ is peaked near $\omega=\omega_1$. Using Eq. (A2) we have

$$\begin{aligned}(Q_{\omega_1 l_1 m_1}, Q_{\omega_2 l_2 m_2}) &= \int_0^\infty d\omega \int_0^\infty d\omega' f_1(\omega) f_2^*(\omega') \\ &\quad\times (F_{\omega l_1 m_1}, F_{\omega' l_2 m_2}) \\ &= \delta_{l_1 l_2}\delta_{m_1 m_2}\int_0^\infty d\omega f_1(\omega) f_2^*(\omega')\,. \end{aligned} \tag{A4}$$

Alternatively, one can evaluate the scalar product of Eq. (A4) by integrating over a constant-t hypersurface at early times, when only the first term of Eq. (A1) contributes to the wave packet. Thus, one finds that at early times

$$(Q_{\omega_1 l_1 m_1}, Q_{\omega_2 l_2 m_2}) = \delta_{l_1 l_2}\delta_{m_1 m_2} \times \int_0^\infty dr \int_0^\infty d\omega \int_0^\infty d\omega' f_1(\omega) f_2^*(\omega') N(\omega) N^*(\omega')(\omega+\omega') e^{-i(\omega-\omega')(t+r)}\,. \tag{A5}$$

Because $-t$ is large, the lower limit of integration over r can be extended to $-\infty$ without affecting the result (i.e., the wave packets are located at large r). Then one can interchange the order of integration, performing the integrations over r first, without affecting the convergence of the result. Thus, using $\int_{-\infty}^{\infty} dr \exp(-ikr) = 2\pi\delta(k)$, one finds that

$$(Q_{\omega_1 l_1 m_1}, Q_{\omega_2 l_2 m_2}) = \delta_{l_1 l_2}\delta_{m_1 m_2} 4\pi \times \int_0^\infty d\omega\, \omega |N(\omega)|^2 f_1(\omega) f_2^*(\omega)\,. \quad \text{(A6)}$$

Comparison with Eq. (A4) implies that, to within a phase factor,

$$N(\omega) = (4\pi\omega)^{-1/2}\,. \quad \text{(A7)}$$

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[19]We are working here in the context of the semiclassical theory in which the spacetime metric is a classical object coupled to the expectation value of the energy-momentum tensor, so any quantum effects associated with gravity itself are ignored. In addition, we do not attempt to deal with the question of the operational significance of the energy-momentum tensor apart from its coupling to gravity. One may show that a detector obeying the uncertainty principle is unable to make an accurate measurement of a flux such as that of Eq. (5.46) [L. H. Ford (unpublished)].