# Current algebra with a four-flavor effective Lagrangian

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(Received 2 September 1977)

A current-algebra treatment of the low-lying spin-zero meson mass spectrum (with four flavors) is given on the basis of a Lagrangian which includes not only the "quark mass" symmetry-breaking terms but also the simplest effective term which breaks the U(4) × U(4) down to SU(4) × SU(4) × "quark number." A term of this sort has been used previously in the linear $\sigma$ model and has been claimed by 't Hooft to arise from the effects of the pseudoparticle on the color-gauge-theory vacuum. In this way good agreement with what is currently known about the mass spectrum can be achieved. We also show that this term provides a source for violation of the Okubo-Zweig-Iizuka rule even without considering unitarity corrections. Predictions are made for the mass of the $F$ meson, the decay constants of the $D$ and $F$ mesons, the $\eta$-$\eta'$-$\eta''$ mixing angles, etc. All the parameters involved in the model—the "quark masses," the vacuum parameters, and the strength of the new effective term—are estimated. Generalizations of the model to include isospin breaking and more flavors are also discussed. Finally, throughout the discussion we have taken note of the points at which the conventional current-algebra treatment (which we use) are open to question and we have referred to different approaches.

## I. INTRODUCTION

One of the basic problems in strong-interaction physics is the decipherment of the symmetry structure of the fundamental Lagrangian. Our present ideas assume that an exactly conserved gauge group of color[1] is responsible for the strong forces. This leads to a chiral $U(n)\times U(n)$-invariant strong Lagrangian ($n$ = number of quark flavors). However, the effective (in the sense, for example, of doing current-algebra calculations) strong Lagrangian has a lower symmetry. It is convenient to identify the terms which break this symmetry. First, there are different mass terms for each flavor of quark. These are expected to arise from the unified weak-electromagnetic gauge theory. If one stops here and calculates the pseudoscalar mass spectrum in the conventional way one finds a bad result, known as the $U(1)$ problem.[2] This can be avoided by including an effective term which breaks $U(n)\times U(n)$ down to $SU(n)\times SU(n)\times U(1)$, where the U(1) corresponds to quark number. Such a term has long been utilized[3] in $\sigma$ models to solve the pseudoscalar-mass-spectrum problem and to give a satisfactory description of the decay $\eta\to 3\pi$. Remarkably, 't Hooft[4] has recently found that just this kind of term may arise when the pseudoparticle[5] contributions to the color-gauge vacuum are taken into account. This work[6] is still in its beginning stages and requires further clarification.[7] For our present purposes we shall just assume that this term exists and find its consequences. We are thus led to postulate that the following Lagrangian density is the effective one for doing current-algebra calculations:

$$\mathcal{L} = -\sum_{a=1}^{n} \bar{q}_a\gamma_\mu\partial_\mu q_a - \sum_{a=1}^{n} m_a\bar{q}_a q_a - U[\det\bar{q}(1+\gamma_5)q + \det\bar{q}(1-\gamma_5)q] + [\mathrm{U}(n)\times\mathrm{U}(n)\text{-invariant interaction}], \quad (1.1)$$

where the $q_a$ are the quark fields ($a$ is a flavor index; color indices are suppressed) and $m_a$ their "masses." $U$ is a new numerical parameter while

$$\det\bar{q}(1\pm\gamma_5)q \equiv \frac{1}{n!}\sum \epsilon_{a_1a_2\cdots a_n}\epsilon_{a_1'a_2'\cdots a_n'} \times \bar{q}_{a_1}(1\pm\gamma_5)q_{a_1'}\cdots\bar{q}_{a_n}(1\pm\gamma_5)q_{a_n'} \quad (1.2)$$

is the term which breaks[8] $\mathrm{U}(n)\times\mathrm{U}(n)$ down to $\mathrm{SU}(n)\times\mathrm{SU}(n)\times\mathrm{U}(1)$.

The case of three flavors ($n=3$) has been discussed previously[9] and found to give a numerically consistent mass spectrum. Here we shall mainly concentrate on the standard model with four quark flavors. This is quite a bit more complicated and leads to a consideration of certain allied questions such as the validity of the current-algebra approach and the origin of the pseudoscalar decay constants. We also raise the possibility that the term (1.2) not only solves the U(1) problem but is also the mechanism responsible for the breaking of the Okubo-Zweig-Iizuka (OZI) or quark-line rule.

From the present point of view, the goal of a fundamental theory should be to permit one to calculate the parameters $m_a$ and $U$ from first principles. Here by consideration of the pseudoscalar mass spectrum we cannot find the $m_a$ and $U$ by themselves for comparison but only their product with certain vacuum expectation values of fields. This occurs because of the assumed Goldstone nature of the symmetry breaking. Defining

$$e_a = \langle \bar{q}_a q_a \rangle_0 , \quad \tilde{m}_a = m_a e_a , \tag{1.3}$$

the quantities which we are able to estimate are

$$\tilde{m}_1, R = m_3/m_1, \quad R' = m_4/m_1, \quad w = e_3/e_1, \quad w' = e_4/e_1, \quad \tilde{U} = U \langle [\det \bar{q}(1+\gamma_5) q + \det \bar{q}(1-\gamma_5) q] \rangle_0 . \tag{1.4}$$

With further assumptions we will make crude estimates of $m_1$, $e_1$, and $U$ by themselves.

In Sec. II we give the general current-algebra mass formulas and discuss the way in which we approximate them. The connection between the U(1) problem and the new term is mentioned. It is noted that a somewhat similar mass matrix is postulated in various phenomenological models.

The linear $\sigma$ model is briefly discussed as an aid to intuition in Sec. III. This leads us to believe that the vacuum should be almost symmetric and that the decay constants should be correlated with the vacuum values. We fit the mass spectrum to an accuracy of about 20% using an exactly SU(4)-symmetric vacuum and equal decay constants. We also demonstrate that the value of the new parameter $\tilde{U}$ does not change drastically even if we were to consider fits with badly broken vacuums.

In Sec. IV we attempt to fit the mass spectrum exactly in our framework, allowing a broken vacuum and unequal decay constants. To reduce the number of parameters a testable ansatz is proposed correlating the vacuum values and decay constants. Also, the decay constant of the $\eta_c$ obtained from a charmoniumlike picture is used. Detailed predictions are made for masses, mixing angles, and decay constants. The associated algebraic and numerical work is described in the Appendix.

It is shown in Sec. V how the new term acts as a violator of the OZI rule. Hadronic decay modes of the $\eta_c$ meson may test this result. The order of magnitude of OZI-rule violation for the $\omega$-$\phi$ system is estimated and found to be reasonable.

In Sec. VI we attempt to get absolute values of the parameters, rather than just their ratios, by postulating that the "current" value of the charmed-quark mass is roughly the same as the "constituent" value. The parameter $U$ is estimated by making a semiclassical approximation.

Finally in Sec. VII, isospin breaking and the extension of this formalism to more quark flavors are discussed.

## II. MASS FORMULAS

We take the current-algebra point of view that the pseudoscalars (all $n^2$ of them) would be (zero-mass) Goldstone bosons[10] were it not for the U($n$) $\times$ U($n$)-violating terms in the strong Lagrangian. This holds explicitly in the linear $\sigma$ model. See for example Table I (with $V_4 = 0$) in the first of Ref. 3. Their masses are considered to arise as a first-order effect of the symmetry-breaking terms. This leads to the general mass formula[11]

$$\sum_{A,B} F_A^a F_B^b (M^2)_{AB} = -\tfrac{1}{2} \int d^3x\, d^3y \langle 0 | [P_4^a(\vec{x}, 0), [P_4^b(\vec{y}, 0), \mathcal{L}]] | 0 \rangle + (a \leftrightarrow b) , \tag{2.1}$$

where $(M^2)_{AB}$ is the mass squared matrix of the pseudoscalar fields in some basis, $P_\mu^a(\vec{x}, t)$ is the $a$th pseudovector current, while $F_A^a$ is the "decay constant" obtained by sandwiching the $P_\mu^a$ between the vacuum and state $A$. Note that only the U($n$) $\times$ U($n$)-violating part of $\mathcal{L}$ contributes to (2.1). Substituting (1.1) into (2.1) gives the squared masses of the $\pi$, $K$, $D$, and $F$ mesons:

$$\begin{aligned} m_\pi^2 &= -\frac{1}{F_\pi^2}(m_1+m_2)(e_1+e_2) , \\ m_K^2 &= -\frac{1}{F_K^2}(m_1+m_3)(e_1+e_3) , \\ m_D^2 &= -\frac{1}{F_D^2}(m_1+m_4)(e_1+e_4) , \\ m_F^2 &= -\frac{1}{F_F^2}(m_3+m_4)(e_3+e_4) . \end{aligned} \tag{2.2}$$

Our axial-vector current is normalized so that the pion decay constant $F_\pi \simeq m(\pi^0)$. The quantity $U$ does not appear in (2.2) since all the currents involved belong to the SU(4) $\times$ SU(4) $\times$ U(1) subgroup under which (1.2) is invariant. $U$ appears when we consider the mass-squared matrix for the zero-flavor objects. It is convenient to use a quark basis by defining

$$P_\mu^a(x) = i\, \bar{q}_a(x) \gamma_\mu \gamma_5 q_a(x) . \tag{2.3}$$

Using the quark-model commutation relations[12]

$$[P_4^a(\vec{x}, 0), \det \bar{q}(0)(1 \pm \gamma_5) q(0)] = \mp 2i \delta^3(x) \det \bar{q}(0)(1 \pm \gamma_5) q(0) \tag{2.4}$$

we find for the right-hand side of (2.1) specialized to the flavorless subspace

$$-4\begin{bmatrix} \tilde{m}_1+\tilde{U} & \tilde{U} & \tilde{U} & \tilde{U} \\ \tilde{U} & \tilde{m}_2+\tilde{U} & \tilde{U} & \tilde{U} \\ \tilde{U} & \tilde{U} & \tilde{m}_3+\tilde{U} & \tilde{U} \\ \tilde{U} & \tilde{U} & \tilde{U} & \tilde{m}_4+\tilde{U} \end{bmatrix}, \qquad (2.5)$$

where $\tilde{m}_a$ and $\tilde{U}$ are defined in (1.3) and (1.4). As it stands (2.5) is *not* the mass-squared matrix of the flavorless pseudoscalars. This is because first of all $F_A^a$ is not necessarily a square matrix so $M^2$ is not necessarily a $4\times 4$ matrix. Here, however, we shall assume that $F_A^a$ is a square matrix which amounts to identifying the labels $A$ and $a$. Physically this means we are neglecting the mixing of the lowest-lying pseudoscalars with their radial excitations[13] and with any states of gluonic matter.[14] Then (2.5) would be proportiona- to the squared mass matrix for the $\pi^0$, $\eta$, $\eta'$, and $\eta''$ [presumably $\eta_c(2830)$] when $F_A^a$ is proportional to the unit matix. (we will not assume this, exactly, in what follows).

If $U=0$ we see that (2.5) is diagonal so that (noting $\tilde{m}_1=\tilde{m}_2$ by isospin invariance) one other flavorless pseudoscalar is roughly degenerate with the $\pi^0$. This is essentailly the mass spectrum aspect of the U(1) problem.

We should mention that mass squared matrices of the general form (2.5) have been postulated by a number of authors with a different interpretation of the parameter $\tilde{U}$. These treatments[15] consider that the off-diagonal terms in (2.5) are generated by a two-gluon-exchange diagram as in Fig. 1. An argument against the importance of such a term, expecially for the first three flavors, is the success of current-algebra theory at low energies. Equation (2.5) is an "exact" consequence of the current-algebra approach and would give zero off-diagonal elements if the term (1.2) were not present. Also, there is no immediate way to explain $\eta\to 3\pi$ decay on the two-gluon exchange picture whereas it can be nicely accomodated in the present framework. We would like to leave open the possibility that diagrams such as Fig. 1 may nevertheless make small contributions (presumably more important for the fourth flavor) or that they may, when computed carefully in quantum chromodynamics (QCD), turn out to in fact be the same as our $\tilde{U}$ term.

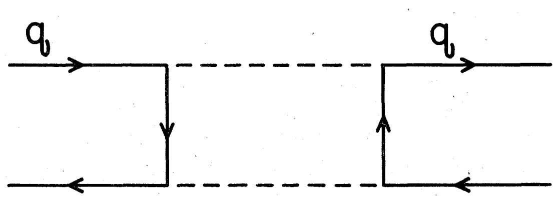


FIG. 1. Two-gluon-exchange diagram.

## III. SEVERAL APPROACHES TO THE MASS SPECTRUM

There are in fact a large number of unknown parameters in the formulas (2.1), (2.2), and (2.5) so we must, to go further, make some approximations or assumptions. As an aid to our intuition in this respect, we may consider a linear $\sigma$ model.[16] The pseudoscalar field $\phi_a^b$ is considered as an analog of the composite operator $i\bar{q}_b\gamma_5 q_a$ while the scalar field $S_a^b$ is considered as an analog of the composite operator $\bar{q}_b q_a$. The Lagrangian density with the transformation properties of (1.1) is

$$\begin{aligned}\mathcal{L} = &-\tfrac{1}{2}\sum_{a,b}(\partial_\mu\phi_a^b\,\partial_\mu\phi_b^a + \partial_\mu S_a^b\partial_\mu S_b^a) \\ &+ 2\sum_a A_a S_a^a - U_0[\det(S+i\phi)+\det(S-i\phi)] \\ &+ [\mathrm{U}(4)\times\mathrm{U}(4)\text{-invariant nonderivative} \\ &\quad \text{polynomial}]. \qquad (3.1)\end{aligned}$$

In addition, the following vacuum values are required:

$$\langle S_b^a\rangle_0 = \delta_b^a\alpha_a. \qquad (3.2)$$

We can derive the mass spectrum and pseudoscalar decay constants from (3.1) and (3.2). These turn out to be special cases of the formulas derived in the preceding section. Specifically,[17] one gets a consistent solution of the present equations by choosing the decay constants as

$$\begin{aligned} &F_A^a = 2\alpha_a\delta_A^a, \\ &F_\pi = 2\alpha_1, \\ &F_K = \alpha_1+\alpha_3, \qquad (3.3)\\ &F_D = \alpha_1+\alpha_4, \\ &F_F = \alpha_3+\alpha_4, \end{aligned}$$

and by setting

$$\begin{aligned} &m_a = \frac{1}{C}A_a, \\ &e_a = \langle\bar{q}_a q_a\rangle_0 = -2C\alpha_a, \qquad (3.4)\\ &\tilde{U} = 2U_0\alpha_1\alpha_2\alpha_3\alpha_4, \end{aligned}$$

where $C$ is an arbitrary constant of dimension (mass)$^2$.

One can learn several interesting things if the identifications in (3.3) and (3.4) are made. First, the experimental fact that $F_K/F_\pi$ is around unity implies by (3.3) that $\alpha_3/\alpha_1$ is around unity. Then (3.4) implies $e_3/e_1$ is around unity, so that the vacuum is approximately SU(3) symmetric. Fur-

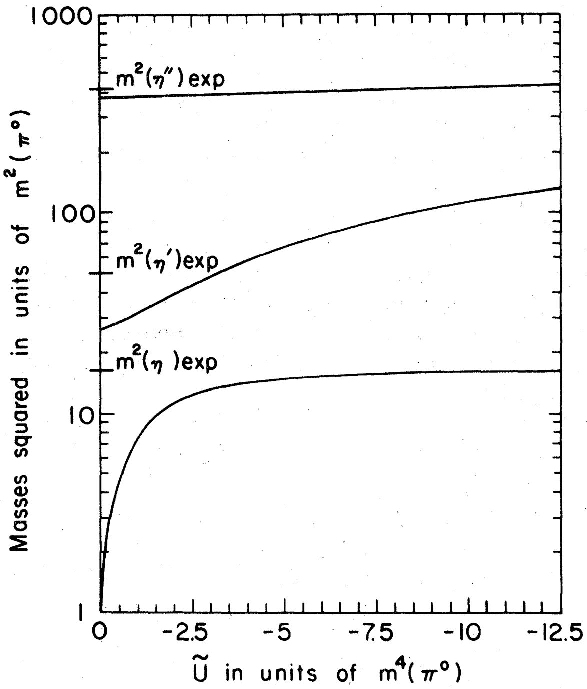


FIG. 2. $m_\eta{}^2$, $m_{\eta'}{}^2$, and $m_{\eta''}{}^2$ [in units of $m^2(\pi^0)$] plotted against $\tilde{U}$ for the case of an SU(4)-symmetric vacuum.

thermore, the formulas for $m_K{}^2$ and $m_\pi{}^2$ in (2.2) then show that $m_3/m_1$ is of the order $2(m_K/m_\pi)^2 \simeq 25$.

Here we shall not take the $\sigma$-model identifications literally,[18] but just use the above as a motivation for considering solutions with approximately symmetric vacuums. There is actually an ambiguity if we say that the fields in the $\sigma$ model are *fundamental* fields (rather than quark-antiquark composites) which exist in addition to the quark fields. Then one would have quark mass terms arising from a term in the Lagrangian of the form $\sum \bar{q}_a q_b S_a^b$. This would give $m_a \propto \alpha_a$ rather than $e_a \propto \alpha_a$ as in (3.4). Such a model would give $m_3/m_1$ of the order of unity rather than about 25. However, we shall not pursue this analogy since introducing quarks as well as mesons seems like double counting.[19]

With the above lesson in mind let us first attempt a fit to the mass spectrum in which the vacuum is exactly SU(4) symmetric,

$$e_1 = e_2 = e_3 = e_4\,, \tag{3.5}$$

and in which all decay constants are equal,

$$F_\pi = F_K = F_D = F_F\,, \quad F_A^a = F_\pi \delta_A^a\,. \tag{3.6}$$

[Eq. (3.6) would follow from (3.5) in the $\sigma$ model]. Equations (3.5) and (3.6) are the generalizations of the Gell-Mann–Oakes–Renner assumption[20] to SU(4). The mass formulas then become

$$\begin{aligned} m_\pi{}^2 &= -2(\tilde{m}_1+\tilde{m}_2)/F_\pi{}^2\,,\\ m_k{}^2 &= -2(\tilde{m}_1+\tilde{m}_3)/F_\pi{}^2\,,\\ m_D{}^2 &= -2(\tilde{m}_1+\tilde{m}_4)/F_\pi{}^2\,,\\ m_F{}^2 &= -2(\tilde{m}_3+\tilde{m}_4)/F_\pi{}^2\,, \end{aligned} \tag{3.7}$$

and for the flavorless subspace:

$$M^2 = -\frac{4}{F_\pi{}^2}\begin{bmatrix} -\frac{F_\pi{}^2}{4}m_\pi{}^2+\tilde{U} & \tilde{U} & \tilde{U} & \tilde{U}\\ \tilde{U} & -\frac{F_\pi{}^2 m_\pi{}^2}{4}+\tilde{U} & \tilde{U} & \tilde{U}\\ \tilde{U} & \tilde{U} & -\frac{F_\pi{}^2}{4}(2m_K{}^2-m_\pi{}^2)+\tilde{U} & \tilde{U}\\ \tilde{U} & \tilde{U} & \tilde{U} & -\frac{F_\pi{}^2}{4}(2m_D{}^2-m_\pi{}^2)+\tilde{U} \end{bmatrix}. \tag{3.8}$$

From (3.7) we have[21] the usual prediction[22]

$$m_F{}^2 = m_D{}^2 + m_K{}^2 - m_\pi{}^2\,. \tag{3.9}$$

The more interesting results follow from (3.8). There, everything but $\tilde{U}$ is known. The four eigenvalues of (3.8) are $m_\pi{}^2$, $m^2(\eta)$, $m^2(\eta')$, and $m^2(\eta'')$. It is most convenient to display the predicted values of the last three as a function of $\tilde{U}$ (see Fig. 2). We see that there is no value of $\tilde{U}$ for which all three masses are fitted exactly. However, a fit of about 20% accuracy is obtained for

$$\tilde{U} \simeq -4m_{\pi^0}{}^4\,. \tag{3.10}$$

The above results are not outstandingly accurate but are good enough to encourage us to search for a more accurate fit by using more realistic values of the parameters. For completeness, though, we first give the predicted mixing angles in this case. We define the physical $\pi^0$, $\eta$, $\eta'$, and $\eta''$

fields by

$$\begin{bmatrix} \pi^0 \\ \eta \\ \eta' \\ \eta'' \end{bmatrix} = Y(y)Z(z)X(x) \begin{bmatrix} \frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d}) \\ \frac{1}{\sqrt{6}}(u\bar{u} + d\bar{d} - 2s\bar{s}) \\ \frac{1}{\sqrt{3}}(u\bar{u} + d\bar{d} + s\bar{s}) \\ c\bar{c} \end{bmatrix}, \tag{3.11}$$

where

$$X(x) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos x & -\sin x & 0 \\ 0 & \sin x & \cos x & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$Y(y) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos y & 0 & -\sin y \\ 0 & 0 & 1 & 0 \\ 0 & \sin y & 0 & \cos y \end{bmatrix}, \tag{3.12}$$

$$Z(z) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos z & -\sin z \\ 0 & 0 & \sin z & \cos z \end{bmatrix};$$

$x, y$, and $z$ are $\eta$-$\eta'$, $\eta$-$\eta''$, and $\eta'$-$\eta''$ mixing angles, respectively. For small $x, y, z$ one finds (in radians)

$$\begin{aligned} x &\simeq -\frac{1}{\sqrt{2}} \frac{m_\eta^2 - m_\pi^2}{m_{\eta'}^2 - m_\eta^2} = -0.32 , \\ y &\simeq 0 , \\ z &\simeq \frac{1}{\sqrt{3}} \frac{m_{\eta'}^2 - m_\eta^2/2 - m_\pi^2/2}{m_{\eta''}^2 - m_{\eta'}^2} = 0.057 . \end{aligned} \tag{3.13}$$

[Note that the right-hand side of (3.13) is independent of $\tilde{U}$.]

Before going on to our more detailed discussion of the mass spectrum (in the next section) based on an approximately symmetric vacuum, we should mention that this assumption has recently been brought into question by some authors.[23] Their work is based on the use of baryon chiral-symmetry results and Melosh-transformation ideas but does not attempt to solve the U(1) problem. It may thus be worthwhile to reemphasize here that an approximately symmetric vacuum is an assumption not required by Eqs. (2.2) and (2.5). For example, we see that interchanging $m_a$ and $e_a$ leaves (2.2) and (2.5) invariant so one could get a fit with $m_1 = m_2 = m_3 = m_4$ but $e_3/e_1 \simeq 25$, $e_4/e_1 \simeq 400$, as an extreme example. The above authors advocate a somewhat milder case $m_3/m_1 \simeq 5$. In fact the present formalism can be used as is for any value of $R = m_3/m_1$ so long as the ratio of vacuum values $w = e_3/e_1$ is modified accordingly. From (2.2) and (1.4) we see that the relation is

$$w = \frac{4}{R+1}\left(\frac{m_K F_K}{m_\pi F_\pi}\right)^2 - 1 , \tag{3.14}$$

which gives $w \simeq 12$ for $R = 5$ (and $F_K/F_\pi = 1.2$). It is easy to carry out the exact calculation for all the parameters of the theory for a given value $R$ when we restrict ourselves to the SU(3) case, as in Ref. 9. Taking $F_A^a$ of (2.1) to be a $3\times 3$ diagonal matrix with components $(F_\pi, F_\pi, \epsilon F_\pi)$, where $\epsilon$ (assumed positive) is initially unknown, and examining the truncated version of (2.5) gives, after some calculation the quadratic equation for $\epsilon^2$,

$$(m_\eta m_{\eta'})^2\epsilon^4 + \tfrac{1}{2}[m_\pi^4 - m_\pi^2(m_\eta^2 + m_{\eta'}^2)(2Rw+1) + (m_\eta m_\eta')^2]\epsilon^2 + (m_\pi^2 Rw)^2 = 0 \tag{3.15}$$

The new parameter $\tilde{U}$ is given by

$$\tilde{U} = \frac{F_\pi^2}{4(2Rw+1)}\left[m_\pi^2 Rw - \left(\frac{\epsilon m_\eta m_{\eta'}}{m_\pi}\right)^2\right] \;[\text{SU(3) case}], \tag{3.16}$$

and the $\eta$-$\eta'$ mixing angle satisfies

$$\tan 2x = \frac{2\sqrt{2}\,[m_\pi^2 F_\pi^2(1 - Rw/\epsilon^2) - 4\tilde{U}(2 - 1/\epsilon - 1/\epsilon^2)]}{m_\pi^2 F_\pi^2(1 - Rw/\epsilon^2) - 4\tilde{U}(2 + 8/\epsilon - 1/\epsilon^2)} . \tag{3.17}$$

Solutions for various values of $R$ are illustrated in Table I. Equation (3.15) has two solutions, but one gives rise to an unacceptably large mixing angle. It is interesting that the new parameter $\tilde{U}$ does not change drastically as $R$ varies over a large range. Thus, although we shall use in the following the conventional assumption of an al-

TABLE I. SU(3) parameters with badly broken vacuum and $F_k/F_\pi = 1.2$

| $R$ | $W$ | $\epsilon$ | $\tilde{U}$ in units of $m_{\pi_0}^4$ | Mixing angle "$x$" in degrees |
|---|---|---|---|---|
| 2 | 25.1 | 1.48 | −4.5 | −5.4 |
| 5 | 12.1 | 1.69 | −4.9 | 0° |
| 10 | 6.1 | 1.71 | −4.9 | 0.4 |
| 15 | 3.9 | 1.65 | −4.8 | −0.8 |
| 25 | 2.0 | 1.48 | −4.5 | −5.3 |
| 34 | 1.24 | 1.24 | −3.7 | −15.8 |

most symmetric vacuum, this result shows that the value of the new parameter $\vec{U}$ should not change drastically even if a badly broken vacuum (and more nearly equal quark masses) were to be assumed.

## IV. DETAILED DISCUSSION OF MASS SPECTRUM

For orientation note that six masses (for $\pi, K, \eta, \eta', \eta'', D$) and two decay constants ($F_\pi$ and $F_K$) are known. We consider these as the reliable experimental inputs; they do not furnish enough information to enable us to calculate the relevant parameters. Hence we make the following ansatz:

$$(e_1+e_2)/F_\pi^2=(e_1+e_3)/F_K^2=(e_1+e_4)/F_D^2 = (e_3+e_4)/F_F^2 . \quad (4.1)$$

The first motivation for assuming (4.1) is that it leads [see (2.2)] to expressions for the $\pi$, $D$, $K$, and $F$ squared masses which are the same as one would get from a Gell-Mann–Okubo type of perturbation ($m_F^2-m_D^2=m_K^2-m_\pi^2$ is an immediate consequence). Secondly, as in the linear $\sigma$ model, (4.1) correlates the vacuum values $e_i$ with the appropriate decay constants. In fact it makes the prediction

$$F_F^2-F_D^2=F_K^2-F_\pi^2 . \quad (4.2)$$

(This is to be contrasted with the formula $F_F-F_D=F_K-F_\pi$ in the linear $\sigma$ model.) Note that (4.1) tends to give us an almost SU(3)-invariant vacuum ($e_3\simeq e_1$) since $F_K$ is around $F_\pi$. One way of testing for deviations from (4.1) would be to test for experimental deviations from (4.2).

Now (4.1) is still not sufficient to enable us to uniquely determine the parameters of the theory from the masses. We need some information on the matrix of decay constants $F_A^a$. On the basis of either the $\sigma$ model or a quark-model calculation we assume

$$F_A^a=\delta_A^a F_A , \quad (4.3)$$

where $F_1=F_2=F_\pi$. One thing more is needed. The success of nonrelativistic quark-model calculations for charmonium[24] leads us to feel that this method should also work for $\eta''[=\eta_c'(2830)]$. We have carried out this calculation[25] elsewhere and found that

$$F_4=F(\eta_c)\simeq(1.78 \text{ to } 1.97)F_\pi , \quad (4.4)$$

depending on the choice of some parameters. It seems remarkable that one gets on the basis of the quark model values just a little larger than $F_\pi$ and $F_K$. This is in accord with the simplest extrapolation of the existing world. As a check on the reasonableness of this picture we expect $F_D$ and $F_F$ to also come out in this range.

Equations (2.2) with (4.1) now lead to the predicted mass ratios

$$\begin{aligned} R&=m_3/m_1=(2m_K^2-m_\pi^2)/m_\pi^2=26 ,\\ R'&=m_4/m_1=(2m_D^2-m_\pi^2)/m_\pi^2=381 , \end{aligned} \quad (4.5)$$

which are the same as in the simple model of Sec. III.

For typical values of the input parameters (aside from masses[26])

$$F_K/F_\pi=1.204, \quad F(\eta_c)/F_\pi=1.86 ,$$

we have the predictions (see the Appendix)

$$\begin{aligned} F_D/F_\pi&=1.57 ,\\ F_F/F_\pi&=1.71 ,\\ x&=-5.7^\circ ,\\ y&=0.53^\circ ,\\ z&=2.11^\circ , \end{aligned} \quad (4.6)$$

and the parameter determinations

$$\begin{aligned} w&=1.9 ,\\ w'&=3.95 ,\\ F_3/F_\pi&=1.47 ,\\ \tilde{U}&=-4.50\,m_{\pi^0}^4 . \end{aligned} \quad (4.7)$$

In the Appendix (and the associated tables) the results for various values of the input parameters are given. $F_D$ and $F_F$ are relatively insensitive to the precise choice of inputs. $\tilde{U}$ and the $\eta$-$\eta'$ mixing angle $x$ depend somewhat on $F_K/F_\pi$ but not sensitively on $F_4/F_\pi$. Note that the estimate for the new parameter $\tilde{U}$ is similar to (3.10). Our predicted value of $\tilde{U}$ does not vary by more than 25% over the complete range of the input parameters. An attempt to disentangle $U$ from $\tilde{U}$ will be discussed later. Our predictions for $F_D/F_\pi$ and $F_F/F_\pi$ should be directly testable from the pure leptonic decay modes of these particles.

Note that the present formalism can be extended[27] to include scalar mesons if desired.

## V. VIOLATION OF OZI RULE

We raise here the possibility that the violation of the OZI rule[28] and the U(1) problem are related in that both are explained by the new term in (1.1).

We interpret the OZI rule as the forbidding of quark diagrams with "hairpins" for the strange quark and for the charmed quark. Originally it was formulated for vector mesons but recently it has also been considered for the pesudoscalars, where it of course does not hold so well. The evidence for the OZI rule for pseudoscalars is examined in a recent review article[29] by Okubo. He stresses that since unitarity[30] forces OZI-rule violation the rule can only be approximate. Even if unitarity is not taken into account, however, the present model provides some violation of the OZI rule.

Note that if the OZI rule were to hold exactly for the pseudoscalar *two-point* functions we would have for the physical states

$$\begin{aligned} \pi^0 &= 1/\sqrt{2}\,(u\bar{u} - d\bar{d})\,,\\ \eta &= 1/\sqrt{2}\,(u\bar{u} + d\bar{d})\,,\\ \eta' &= s\bar{s}\,,\\ \eta'' &= c\bar{c}\,. \end{aligned} \tag{5.1}$$

This is just what we would get from (3.8) in the case when $\tilde{U}=0$. Hence the $U$ term in (1.1) clearly gives OZI-rule breaking for pseudoscalar two-point functions.

We can sharpen this statement by considering, as discussed in Sec. III, the linear $\sigma$ model as a representation of the chiral-quark-model dynamics. If $U_\sigma$ in (3.1) vanishes we have an exact OZI rule to tree order for all $n$-point functions which of course get unitarity connections to one loop and higher orders. We may see this in the following way.[31] The fields involved in the $\sigma$ model are not traceless ($\mathrm{tr}\phi \neq 0$, $\mathrm{tr} s \neq 0$) when regarded as $4\times 4$ matrices since the group $[\mathrm{SU}(4)\times\mathrm{SU}(4)]$ is larger than SU(4). If only $\mathrm{U}(4)\times\mathrm{U}(4)$ polynomials are allowed in (3.1) we have terms such as $\mathrm{tr}(\phi\phi\phi\phi)$, etc. which always conserve the OZI rule since if one of the $\phi$'s is $\phi_4^4$, for example, there must be an index 4 in two other places at least. On the other hand, the $U_\sigma$ term will contain a piece, for example, like $\phi_1^1\phi_2^2\phi_3^3\phi_4^4$ which clearly violates the OZI rule.

Actually the $\sigma$ model provides an explicit example of Okubo's original ansatz[32] (extended to the present case) that $\mathrm{tr}(\phi)$ should not appear as a factor in any terms of the Lagrangian. $\mathrm{U}(4)\times\mathrm{U}(4)$-invarient polynomials will give terms such as $\mathrm{Tr}(\phi^2)$, $\mathrm{Tr}(\phi^4)$, $\mathrm{Tr}(S^2\phi^2)$, etc. but not $\mathrm{Tr}(\phi)$. The OZI-violating term involving $\det(S+i\phi)$ can be rewritten, using a well-known identity, as a function of the $\mathrm{U}(4)\times\mathrm{U}(4)$-invariant terms as well as $\mathrm{Tr}(S+i\phi)$, excluded by the original ansatz.

A possible test of this idea is the prediction of the decays $\eta_c \to$ (only hadrons) by a $\sigma$-model calculation. These decays would be zero if $U_\sigma$ were to vanish (or in the limit of exact OZI rule). Some of these calculations have already been done in variants of the SU(4) $\sigma$ model by Ueda[33] for $\eta'' \to \eta K^+K^-$, $\eta' K^+K^-$, $\eta\pi^+\pi^-$, $\eta'\pi^+\pi^-$ and by Singer[34] for $\eta'' \to \eta\eta\eta$, $\eta'\eta\eta$, $\eta'\eta'\eta$. The latter decays are the dominant ones in this model and give widths in the "several MeV" range. Such large widths may possibly explain the anomalously low[35] branching ratio $\psi \to \eta_c\gamma$ (as seen via $\psi \to 3\gamma$) if $\eta'$s and $\eta''$s were not identified as coming from an intermediate $\eta_c$.

All our considerations have been for the system of spin-zero mesons. It is natural to wonder if some of these ideas can also be applied to the vector-meson system (and others) for which the OZI rule is a more dominant feature. One possible approach might be to use a model like that of Caldi and Pagels[36] in which the vector mesons are treated in parallel manner to the pseudoscalars. Here, however, we will be content to make a very crude estimate to see if the order of magnitude of the OZI-rule violation for the vector-mesons two-point functions is reasonable (i.e., small). Let us expand the $U$ term in (1.1) as a function of the composite operators $\pi_a^b = i\bar{q}_b\gamma_5 q_a$ and $S_a^b = \bar{q}_b q_a - \langle \bar{q}_b q_a\rangle_0$. This yields

$$\begin{aligned} -U[\det\bar{q}(1+\gamma_5)q + \det\bar{q}(1-\gamma_5)q] \simeq & -Ue_1e_2e_3e_4\Big[\Big(\sum S_a^a/e_a\Big)^2 - \Big(\sum \pi_a^a/e_a\Big)^2 \\ & - \sum S_a^b S_b^a/e_a e_b + \sum \pi_a^b\pi_b^a/e_a e_b\Big] \\ & + \cdots\,, \end{aligned} \tag{5.2}$$

where only terms of second order have been kept and where $e_a$ is defined in (1.3). Assuming that it is meaningful to apply the Fierz transformation to (5.2) we get

$$-\tfrac{1}{2}Ue_1e_2e_3e_4 \sum_{a\neq b} \bar{q}_a\gamma_\alpha q_a \bar{q}_b\gamma_\alpha q_b/e_a e_b + \cdots\,, \tag{5.3}$$

where only the interesting vector $\times$ vector terms are written. Denoting $\rho_{a\alpha}^b$ as the vector-meson 16-plet and setting $\bar{q}_a\gamma_\alpha q_a = i\sqrt{2}\, m_\rho F_\pi \rho_{a\alpha}^a$ (field-current identity) we may consider (5.3) as giving an effective contribution to the vector-meson mass matrix. Probably the above arguments are better[37] for SU(3) rather than SU(4) so let us specialize to the $\omega$-$\phi$ mixing term:

$$\begin{aligned} \mathcal{L}(\text{OZI-violating}) &= \sqrt{2}\, U e_2 e_4 m_\rho^2 F_\pi^2 \omega_\alpha \phi_\alpha\,,\\ \omega_\alpha &= \frac{1}{\sqrt{2}}(\rho_{1\alpha}^1 + \rho_{2\alpha}^2)\,,\\ \phi_\alpha &= \rho_{3\alpha}^3\,. \end{aligned} \tag{5.4}$$

Using the estimates for $U$ and the $e_a$ given in the next section we find the coefficient of $\omega_\alpha\phi_\alpha$ in (5.4) to be $(-2.5\times 10^{-2})m_\rho^2$ which does seem reasonably small (considering $m_\rho \simeq 780$ MeV).

## VI. ESTIMATE OF BASIC PARAMETERS

For comparison with a possible fundamental theory of strong interactions it is interesting to try to estimate the parameters $m_a$, $e_a$, and $U$ by themselves from their ratios and combinations [see (1.4)] found above.

First consider the quark masses, $m_a$. In the most naive quark model one might expect $m_1 \simeq m_2 \simeq m_\rho/2 = 390$ MeV, $m_3 \simeq m_\phi/2 \simeq 510$ MeV, and $m_4 \simeq m_\psi/2 \simeq 1550$ MeV. One puzzle is that we also expect $m_1 \simeq m_2 \simeq m_\pi/2 \simeq 70$ MeV. Another puzzle is that the ratio $m_3/m_1$ as determined in the quark model is very different from $R=26$ given in (4.5). Various attempts have been made to resolve these difficulties.[38] A fascinating point of view, based on renormalization-group ideas,[39] is that the effective quark mass depends on the momentum scale, $M$ of the system. The "current-algebra" values computed here are considered to be the values as $M \to \infty$. The success of the charmonium picture leads us to believe that for the fourth (charmed) quark we are sufficiently close to $M=\infty$ at 3100 MeV to enable us to set $m_4 \simeq 1550$ MeV. Using this value in conjunction with the typical ratios determined in (4.5) gives

$$\begin{aligned} m_1 &\simeq m_2 \simeq 4.1 \text{ MeV}, \\ m_3 &\simeq 106 \text{ MeV}, \\ m_4 &\simeq 1550 \text{ MeV}. \end{aligned} \tag{6.1}$$

Since $m_1$ and $m_2$ turn out to be of characteristic "electromagnetic" strength the approximation $m_1 = m_2$ is clearly not appropriate. We should thus write $(m_1+m_2)/2 \simeq 4.1$ MeV. In fact it has even been speculated[40] that we might have $m_1 = 0$.

Now using the formula for the pion mass given in (2.2) we estimate $(e_1+e_2)/2 \simeq -2\times 10^7$ MeV$^3$. Hence with the typical values of the ratios $w$ and $w'$ computed in (4.7) we get

$$\begin{aligned} \tfrac{1}{2}(e_1+e_2) &\simeq -2\times 10^7 \text{ MeV}^3, \\ e_3 &\simeq -3.8\times 10^7 \text{ MeV}^3, \\ e_4 &\simeq -8\times 10^7 \text{ MeV}^3. \end{aligned} \tag{6.2}$$

Finally we estimate the new parameter $U$ by making the semiclassical approximation

$$\begin{aligned} \tilde{U} &= U\langle[\det \bar{q}(1+\gamma_5)q + \det \bar{q}(1-\gamma_5)q]\rangle_0 \\ &\simeq 2U\langle\bar{q}_1 q_1\rangle_0 \langle\bar{q}_2 q_2\rangle_0 \langle\bar{q}_3 q_3\rangle_0 \langle\bar{q}_4 q_4\rangle_0, \end{aligned} \tag{6.3}$$

which, using (6.2) and (4.7), gives the typical value

$$U \simeq -6.1\times 10^{-22} \text{ MeV}^{-8}. \tag{6.4}$$

Of course, if one makes the assumption of a badly broken vacuum as discussed at the end of Sec. III, the above estimates are modified in a straightforward way.

It is amusing to note that, assuming partial conservation of axial-vector current (PCAC) and the nonrelativistic quark model the quantity $U$ can be related[41] to the binding energy of $\eta_c$ as

$$BE(\eta_c) \simeq 4Ue_1e_2e_3 \simeq 37 \text{ MeV}. \tag{6.5}$$

This is, however, substantially lower than what one gets from a linear potential model.[24]

## VII. GENERALIZATION AND DISCUSSION

### A. Isospin breaking

Assuming that the origin of the $m_a$ is a unified weak-electromagnetic gauge theory one expects $m_1 \neq m_2$. This effect is generally considered to play an important role in $\Delta I = 1$ isospin violations such as the $K^+K^0$ mass splitting, the $D^+D^0$ mass splitting, and the $\eta \to 3\pi$ decay. Since our mass formulas (2.2) were derived in a quark (i.e., tensor) notation it is trivial to modify them to take this into account. For example, we get for the $K^+$ and $K^0$ masses

$$\begin{aligned} m_{K^+}^2 &= -\frac{1}{F^2(K^+)}(m_1+m_3)(e_1+e_3), \\ m_{K^0}^2 &= -\frac{1}{F^2(K^0)}(m_2+m_3)(e_2+e_3). \end{aligned} \tag{7.1}$$

Note that there are three sources of isospin violation in (7.1). In addition to $m_1 \neq m_2$ we must expect $e_1 \neq e_2$ and $F(K^0) \neq F(K^+)$. In fact all three sources are present in the $\sigma$ model. Usually only $m_1 \neq m_2$ is taken into account. It was shown,[42] though, that in a renormalizable SU(3) $\sigma$ model the correct rate for $\eta \to 3\pi$ can be obtained when all these sources of isospin violation are included.

The extra sources of isospin breaking may also play a role in the "nonelectromagnetic" contribution to the $D^+$-$D^0$ mass splitting. It was shown[43] that the sign of this contribution relative to that of the $K^+$-$K^0$ "nonelectromagnetic" splitting is reversed if the parameters are chosen as in a $\sigma$ model [see (3.3) and (3.4) modified to allow isospin violation]. This arises because the effects of unequal decay constants and vacuum values overcome the $(m_1-m_2)$ term. On the other hand, in general, we are not required to identify the parameters as in the $\sigma$ model. In fact if we were to take the generalization of the ansatz (4.1) to include electromagnetism, i.e.,

$$\frac{e_1+e_2}{F^2(\pi^+)} = \frac{e_1+e_3}{F^2(K^+)} = \frac{e_2+e_3}{F^2(K^0)} = \text{etc.}, \tag{7.2}$$

we would only keep, by *construction*, the $m_1 - m_2$ term in the "nonelectromagnetic" part of the mass splittings. Thus a more detailed study of the isospin violations may shed light on the relation between the vacuum values and the decay constants.

### B. More flavors

Recent experimental data seem to indicate a possible need for more quark flavors. Assuming that there is some sense in also considering these (heavy) new mesons as Goldstone bosons in zeroth order the formalism here can be easily extended. Suppose there are six flavors altogether (denote the new ones by 5 and 6) then, for example, the meson made of 5 and $\bar{6}$ would have the mass

$$m_{56}^2 = \frac{-1}{F_{56}^2}(m_5 + m_6)(e_5 + e_6). \qquad (7.3)$$

(Note that the present formalism does not require us to specify the electric charges of 5 and 6.) The generalization of the ansatz (4.1) would give a wealth of mass relations (in the isospin limit):

$$\begin{aligned} m_F^2 &= m_K^2 + m_D^2 - m_\pi^2, \\ m_{35}^2 &= m_K^2 + m_{15}^2 - m_\pi^2, \\ m_{45}^2 &= m_D^2 + m_{15}^2 - m_\pi^2, \\ m_{36}^2 &= m_K^2 + m_{16}^2 - m_\pi^2, \\ m_{46}^2 &= m_D^2 + m_{16}^2 - m_\pi^2, \\ m_{56}^2 &= m_{15}^2 + m_{16}^2 - m_\pi^2. \end{aligned} \qquad (7.4)$$

In addition every mass in (7.4) may be replaced by the decay constant $F$ for that particle and the resulting relations should still hold [assuming again the generalization of (4.1) to hold].

What about the new parameter $\tilde{U}$? In the case of three flavors it turned out[9] to be about $-3.5m^4(\pi^0)$ which is only slightly smaller in magnitude than the present value (see 4.7) of $-4.5m^4(\pi^0)$. Hence we would expect the magnitude of $\tilde{U}$ to increase slightly for two new flavors.

### C. Summary and discussion

In this paper we have shown by using the *conventional* current-algebra treatment how the determinant term (inspired by possible pseudoparticle contributions to the color-gauge-theory vacuum and previously postulated in the linear $\sigma$ model) solves the U(1) problem by giving a reasonable description of the four-flavor pseudoscalar mass spectrum. We have also speculated in general and shown in a simple case that this term may be responsible (in part, perhaps) for the violation of the OZI rule. The parameters in the model were all estimated. In particular the strength of the new term was estimated and predictions were made for the decay constants of the $D$ and $F$ mesons. At various places in the text we have indicated where the conventional current-algebra treatments are open to question and have given references and brief discussions of these points.

*Note added in proof.* An alternate ansatz to solve for the flavor-nonzero masses has been proposed by R. S. Oakes and P. Sorba, Fermilab Report No. 77/78-THY (unpublished). They consider (3.5) to hold but all $F$'s to be different. In the present formalism this gives results substantially similar to ours. For example, we get (with $F_K/F_\pi = 1.28$ and $\epsilon' = 1.97$) $F_D/F_\pi = 2.11$, $F_F/F_\pi = 1.96$, $\epsilon = 1.30$, $\tilde{U} = -4.0m_{\pi^0}^4$, $x = -12.2°$, $y = 0.6°$, and $z = 1.8°$. In arriving at these values we assumed $m_F = 2.03$ GeV corresponding to some preliminary experimental results. We should also mention similar work by Z. Maki and I. Umemura, Kyoto Report No. R1FP-302, 1977 (unpublished) and N. G. Deshpande (private communication).

### ACKNOWLEDGMENTS

The work of two of us (J. K. and J. S.) was supported in part by the U. S. Energy Research and Development Administration Contract No. EY-76-S-02-3533. The work of another (M.S.) was supported in part by the University of Wisconsin Research Committee with funds granted by the Wisconsin Alumni Research Foundation, and in part by the U. S. Energy Research and Development Administration under Contract No. E(11-1)-881, C00-613.

### APPENDIX: THE FLAVORLESS SUBSPACE

To treat the flavorless subspace we first assume that the matrix of decay constants is diagonal,

$$F_A^a = F_A \delta_A^a, \qquad (A1)$$

where $F_1 = F_2 = F_\pi$ by isotopic-spin invariance. Introduce the abbreviations

$$\begin{aligned} &\epsilon = F_3/F_1, \quad \epsilon' = F_4/F_1, \\ &\gamma = \frac{w}{\epsilon^2}(2m_K^2 - m_\pi^2), \quad \gamma' = \frac{w'}{\epsilon'^2}(2m_D^2 - m_\pi^2). \end{aligned} \qquad (A2)$$

Now with the assumption (4.1) the mass-squared matrix for the flavorless subspace in a quark basis becomes

$$M^2=\begin{bmatrix} m_\pi{}^2-(4\tilde U/F_\pi{}^2) & -4\tilde U/F_\pi{}^2 & -4\tilde U/F_\pi{}^2\epsilon & -4\tilde U/F_\pi{}^2\epsilon' \\ -4\tilde U/F_\pi{}^2 & m_\pi{}^2-(4\tilde U/F_\pi{}^2) & -4\tilde U/F_\pi{}^2\epsilon & -4\tilde U/F_\pi{}^2\epsilon' \\ -4\tilde U/F_\pi{}^2\epsilon & -4\tilde U/F_\pi{}^2\epsilon & \gamma-(4\tilde U/F_\pi{}^2\epsilon^2) & -4\tilde U/F_\pi{}^2\epsilon\epsilon' \\ -4\tilde U/F_\pi{}^2\epsilon' & -4\tilde U/F_\pi{}^2\epsilon' & -4\tilde U/F_\pi{}^2\epsilon\epsilon' & \gamma'-(4\tilde U/F_\pi{}^2\epsilon'^2) \end{bmatrix}. \tag{A3}$$

We can simplify our analysis by noting that (A3) is of the same form as the matrix for the case of the SU(4) $\sigma$ model which has already been treated.[44] To avoid confusion (since the symbols must be transposed) we give the results again in the present context. From the secular equation for (A3) one gets the sum rules

$$\begin{aligned} m_\eta{}^2+m_{\eta'}{}^2+m_{\eta''}{}^2 &= r_1+\frac{s_1\gamma'\epsilon'}{2}+\frac{2u_1\tilde U}{F_\pi{}^2\epsilon\epsilon'}, \\ (m_\eta m_{\eta'} m_{\eta''})^2 &= \frac{s_2\gamma'\epsilon'}{2}+\frac{2u_2\tilde U}{F_\pi{}^2\epsilon\epsilon'}+\frac{v_2\tilde U\gamma'}{F_\pi{}^2\epsilon}, \\ (m_\eta m_{\eta'})^2+(m_\eta m_{\eta''})^2+(m_{\eta'} m_{\eta''})^2 & \\ &= r_3+\frac{s_3\gamma'\epsilon'}{2}+\frac{2u_3\tilde U}{F_\pi{}^2\epsilon\epsilon'}+\frac{v_3\tilde U\gamma'}{F_\pi{}^2\epsilon}, \end{aligned} \tag{A4}$$

where

$$\begin{aligned} & r_1=\frac{\epsilon'}{2}s_3=m_\pi{}^2+\gamma, \quad s_1=\frac{2}{\epsilon'}, \\ & v_3=-4\left(2\epsilon+\frac{1}{\epsilon}\right); \quad u_1=\frac{\epsilon'}{2}v_3-\epsilon s_1, \\ & s_2=-\frac{1}{\epsilon}u_2=\frac{2}{\epsilon'}r_3=2m_\pi{}^2\gamma/\epsilon', \\ & v_2=-8\left(\frac{m_\pi{}^2}{2\epsilon}+\gamma\epsilon\right), \quad u_3=\frac{\epsilon' v_2}{2}-\frac{2\epsilon r_1}{\epsilon'}. \end{aligned} \tag{A5}$$

We solve for $\gamma'$ in two different ways to get the equations[45]

$$\begin{aligned} \frac{\gamma'}{2} &= a_1 m_{\eta''}{}^2+b_1+\frac{1}{\epsilon'^2}(c_1 m_{\eta''}{}^2+d) \\ &= \frac{a_2 m_{\eta''}{}^2+b_2}{c_2 m_{\eta''}{}^2+d_2}. \end{aligned} \tag{A6}$$

The constants $a_1$ through $d_2$ are given by

$$\begin{aligned} & a_1=\tfrac{1}{2}-(1+2\epsilon^2)c_2/[8\epsilon(m_\pi{}^2-\gamma)^2], \\ & b_1=\frac{1}{4}\left(\frac{m_\pi{}^2}{2\epsilon}+\gamma\epsilon\right)c_2\Big/(m_\pi{}^2-\gamma)^2, \\ & c_1=-\frac{\epsilon}{8}c_2\Big/(m_\pi{}^2-\gamma)^2, \\ & d_1=-\frac{\epsilon}{8}d_2\Big/(m_\pi{}^2-\gamma)^2, \\ & a_2=\tfrac{1}{2}d_2+\frac{c_2}{2}(m_\pi{}^2+\gamma), \\ & b_2=-\frac{m_\pi{}^2\gamma c_2}{2}, \\ & c_2=\frac{4}{\epsilon}\left(\frac{m_\pi{}^4}{2}+\gamma^2\epsilon^2\right)+(m_\eta m_{\eta'})^2\left(4\epsilon+\frac{2}{\epsilon}\right) \\ & \qquad -4(m_\eta{}^2+m_{\eta'}{}^2)\left(\frac{m_\pi{}^2}{2\epsilon}+\gamma\epsilon\right), \\ & d_2=\frac{4}{\epsilon}(m_\eta{}^2+m_{\eta'}{}^2)\left(\frac{m_\pi{}^2}{2}+\gamma^2\epsilon^2\right) \\ & \qquad -4(m_\eta m_{\eta'})^2\left(\frac{m_\pi{}^2}{2\epsilon}+\gamma\epsilon\right)-\frac{4}{\epsilon}\left(\frac{m_\pi{}^6}{2}+\gamma^3\epsilon^2\right). \end{aligned} \tag{A7}$$

We proceed as follows. Equation (A6) yields an equation for $\epsilon'^2$ in terms of $\epsilon$, $\gamma$, and known masses. But from (4.1) we have

$$w=2\left(\frac{F_K}{F_\pi}\right)^2-1, \tag{A8}$$

so that $\gamma$ is given in terms of known quantities. Hence the equation of interest gives $\epsilon'^2$ in terms of just $\epsilon$. By our previous discussion we have an estimate of $\epsilon'$ from the quark model. Thus $\epsilon$ can be found. Knowing $\epsilon$ and $\epsilon'$ we can find $\gamma'$ (and hence $w'$) by (A4). This completes the determination of parameters.

In practice we used the computer to carry out the numerical work. Calculations were made for

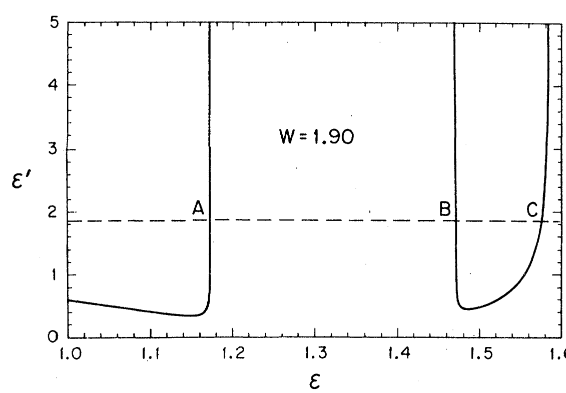


FIG. 3. $\epsilon'$ plotted as a function of $\epsilon$ for $F_K/F_\pi = 1.204$. The dotted line corresponds to $\epsilon'=1.86$. Of the three possibilities A, B, and C, case B is the physically reasonable one.

TABLE II: Predicted parameters for various values of $F_K/F_\pi$ and $\epsilon' = 1.78$.

| $F_K/F_\pi$ | $F_D/F_\pi$ | $F_F/F_\pi$ | $w$ | $w'$ | $\epsilon$ | $\tilde{U}$ in units of $m_{\pi^0}^4$ | Mixing angles in degrees $x$ | $y$ | $z$ |
|---|---|---|---|---|---|---|---|---|---|
| 1.140 | 1.520 | 1.616 | 1.6 | 3.6222 | 1.226 | −3.67 | −16.5 | 0.66 | 1.80 |
| 1.151 | 1.519 | 1.622 | 1.65 | 3.617 | 1.289 | −3.96 | −12.78 | 0.63 | 1.95 |
| 1.204 | 1.518 | 1.659 | 1.9 | 3.611 | 1.470 | −4.50 | −5.7[illegible] | 0.56 | 2.21 |
| 1.245 | 1.518 | 1.689 | 2.1 | 3.607 | 1.585 | −4.75 | −2.47 | .52 | 2.32 |
| 1.304 | 1.517 | 1.733 | 2.4 | 3.605 | 1.737 | −5.0 | 1.02 | .48 | 2.42 |
| 1.360 | 1.517 | 1.775 | 2.7 | 3.603 | 1.873 | −5.19 | +3.52 | 0.45 | 2.50 |

TABLE III. Predicted parameters for various values of $F_K/F_\pi$ and $\epsilon' = 1.86$.

| $F_K/F_\pi$ | $F_D/F_\pi$ | $F_F/F_\pi$ | $w$ | $w'$ | $\epsilon$ | $\tilde{U}$ in units of $m_{\pi^0}^4$ | Mixing angles in degrees $x$ | $y$ | $z$ |
|---|---|---|---|---|---|---|---|---|---|
| 1.140 | 1.574 | 1.666 | 1.6 | 3.957 | 1.226 | −3.67 | −16.5 | 0.63 | 1.73 |
| 1.151 | 1.574 | 1.674 | 1.65 | 3.954 | 1.289 | −3.95 | −12.79 | 0.61 | 1.87 |
| 1.204 | 1.573 | 1.710 | 1.9 | 3.947 | 1.470 | −4.5 | −5.72 | 0.53 | 2.11 |
| 1.245 | 1.572 | 1.738 | 2.1 | 3.944 | 1.585 | −4.74 | −2.47 | 0.50 | 2.21 |
| 1.304 | 1.572 | 1.781 | 2.4 | 3.941 | 1.737 | −5.00 | +1.02 | 0.46 | 2.32 |
| 1.360 | 1.571 | 1.822 | 2.7 | 3.939 | 1.873 | −5.18 | +3.52 | 0.43 | 2.39 |

TABLE IV. Predicted parameters for various values of $F_K/F_\pi$ and $\epsilon' = 1.97$.

| $F_K/F_\pi$ | $F_D/F_\pi$ | $F_F/F_\pi$ | $w$ | $w'$ | $\epsilon$ | $\tilde{U}$ in units of $m_{\pi^0}^4$ | Mixing angles in degrees $x$ | $y$ | $z$ |
|---|---|---|---|---|---|---|---|---|---|
| 1.140 | 1.650 | 1.738 | 1.6 | 4.444 | 1.226 | −3.67 | −16.5 | 0.60 | 1.63 |
| 1.151 | 1.649 | 1.745 | 1.65 | 4.441 | 1.289 | −3.95 | −12.79 | 0.57 | 1.76 |
| 1.204 | 1.648 | 1.78 | 1.9 | 4.434 | 1.470 | −4.49 | −5.72 | 0.50 | 1.99 |
| 1.245 | 1.648 | 1.807 | 2.1 | 4.431 | 1.585 | −4.74 | −2.48 | 0.47 | 2.09 |
| 1.304 | 1.647 | 1.847 | 2.4 | 4.428 | 1.737 | −4.99 | +1.02 | 0.43 | 2.18 |
| 1.360 | 1.647 | 1.888 | 2.7 | 4.426 | 1.873 | −5.17 | +3.60 | 0.40 | 2.25 |

TABLE V. Predicted parameters for the special case $\epsilon^2 = w$.

| $\epsilon'$ | $F_D/F_\pi$ | $F_F/F_\pi$ | $w'$ | $\tilde{U}$ in units of $m_{\pi^0}^4$ | Mixing angles in degrees $x$ | $y$ | $z$ |
|---|---|---|---|---|---|---|---|
| 1.78 | 1.519 | 1.621 | 3.617 | −3.93 | −13.16 | 0.63 | 1.94 |
| 1.86 | 1.574 | 1.673 | 3.954 | −3.92 | −13.16 | 0.61 | 1.85 |
| 1.97 | 1.649 | 1.744 | 4.440 | −3.92 | −13.16 | 0.57 | 1.75 |

TABLE VI. Predicted parameters with the (unphysical) choice A of Fig. 3 for $\epsilon' = 1.97$ and various values of $F_K/F_\pi$.

| $F_K/F_\pi$ | $F_D/F_\pi$ | $F_F/F_\pi$ | $w$ | $w'$ | $\epsilon$ | $\tilde{U}$ in units of $m_{\pi^0}^4$ | Mixing angles in degrees $x$ | $y$ | $z$ |
|---|---|---|---|---|---|---|---|---|---|
| 1.140 | 1.650 | 1.738 | 1.6 | 4.447 | 1.185 | −3.42 | −19.9 | 0.63 | 1.5 |
| 1.204 | 1.652 | 1.783 | 1.9 | 4.455 | 1.173 | −2.80 | −29.9 | 0.69 | 1.13 |
| 1.245 | 1.652 | 1.811 | 2.1 | 4.457 | 1.203 | −2.66 | −33.3 | 0.71 | 0.96 |
| 1.304 | 1.652 | 1.852 | 2.4 | 4.458 | 1.254 | −2.53 | −34.2 | 0.71 | 0.92 |
| 1.360 | 1.652 | 1.892 | 2.7 | 4.459 | 1.308 | −2.44 | −36.0 | 0.72 | 0.84 |

TABLE VII. Dependence of parameters on the mass of $\eta_c$ for $F_K/F_\pi = 1.225$ and $\epsilon' = 1.97$.

| Mass of $\eta_c$ (in MeV) | $F_D/F_\pi$ | $F_F/F_\pi$ | $\epsilon$ | $\tilde{U}$ in units of $m^4(\pi_0)$ | Mixing angles in degrees $x$ | $y$ | $z$ |
|---|---|---|---|---|---|---|---|
| 2800 MeV | 1.648 | 1.793 | 1.529 | −4.62 | −3.96 | 0.48 | 2.04 |
| 3085 MeV | 1.775 | 1.910 | 1.529 | −4.61 | −3.97 | 0.40 | 1.67 |
| 3455 MeV | 1.953 | 2.077 | 1.529 | −4.60 | −3.97 | 0.32 | 1.31 |

various values of $F_K/F_\pi$ and $\epsilon' = F(\eta_c)/F_\pi$. There is an ambiguity in that there are generally three solutions for $\epsilon$ corresponding to a given $\epsilon'$. This is illustrated in Fig. 3 where we have used the parameter values in (4.6). The three solutions for $\epsilon$ are denoted by A, B, and C. We can rule out C by noting that it leads to $w' < w$, which would lead to violation[46] of the positivity of the squared masses of the *scalar* particles of the theory. Furthermore, we can rule out A in most cases by calculating the mixing angles [see (3.11) and (3.12)] and noting that A leads usually to a rather large value[47] of $x$, the $\eta\eta'$ mixing angle. This can be understood in the following way. Solutions A and B occur when $c_2$ of (A7) is approximately equal to zero. The equation $c_2 = 0$ is the secular equation for the $\eta$ and $\eta'$ masses squared in the SU(3) model. From our knowledge of the SU(3) $\eta$ and $\eta'$ states we can rule out solution A since it leads there to a large value of the $\eta$-$\eta'$ mixing angle. The condition that $c_2 \simeq 0$ then tells us that we have a solution where the SU(3) states $\eta$ and $\eta'$ are slightly mixed with the $\bar{c}c$ state.

The numerical results are presented in Tables II, III, and IV, corresponding respectively to

$$\epsilon' = F(\eta_c)/F_\pi = 1.78,\ 1.86,\ \text{and}\ 1.97. \tag{A9}$$

These values correspond to our quark-model-calculation parameter $\alpha'$ (Regge slope parameter) equal to 1 $\text{GeV}^{-2}$ (rule of thumb), 0.89 $\text{GeV}^{-2}$ ($\rho$ trajectory), and 0.76 $\text{GeV}^{-2}$ (fit to $\psi$ mass spectrum). Alternatively these values may be interpreted as spanning a physically reasonable range. In each table the results are given for five values of $F_K/F_\pi$ in the range 1.14 to 1.36. Generally $F_K/F_\pi$ is estimated from experiment to be somewhere between 1.20 and 1.28. It is also interesting to note that there exists a solution where $\epsilon^2 = w$, which leads to the sum rule $(F_3)^2 + (F_\pi)^2 = 2(F_K)^2$ which is similar to the SU(4)-$\sigma$-model rule $F_3 + F_\pi = 2F_K$. As can be seen from Table V this solution gives rise to physically reasonable values of the $\eta$-$\eta'$-$\eta''$ mixing angles. There is no solution to the problem where both $\epsilon^2 = w$ and $(\epsilon')^2 = w'$.

To see why point A in Fig. 3 is ruled out we present for comparison the calculated values of decay constants and mixing angles in Table VI for the typical choice $\epsilon' = 1.97$.

In our calculation we have used 2830 MeV as the mass[35] of the $\eta''$. However, the results for $\tilde{U}$ are not very sensitive to the choice of mass in this range. This is illustrated in Table VII computed for the typical values $F_K/F_\pi = 1.225$ and $\epsilon' = 1.97$.

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[7]Three problems in connection with 't Hooft's effective term are (i) the necessity to maintain $T$ invariance, (ii) the possibility of an infrared divergence resulting from integration over all pseudoparticle sizes, and (iii) the necessity for extension to an SU(3) color gauge theory.

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[8]Actually in addition to $SU(n)\times SU(n)\times U(1)$ symmetry (1.2) is invariant under the discrete transformations $q_a \to e^{ik\pi\gamma_5/n} q_a$, where $k$ is a positive or negative integer. See S. Okubo, Phys. Rev. D 12, 3835 (1975).

[9]V. Mirelli and J. Schechter, Phys. Rev. D 15, 1361 (1977).

[10]Such a point of view is not of course immediate for those pseudoscalars containing charmed quarks. However, it is possibly reasonable since one really does not know the scale against which the pseudoscalars are to be considered small; e.g., the $D$ meson has a small mass compared to an intermediate vector boson presumably.

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[17]See, for example, the discussion in Ref. 9.

[18]Actually the SU(3)-$\sigma$ model results for the mass spectrum and decay constants are quite reasonable. However, the general (not necessarily renormalizable) SU(4) $\sigma$ model predicts somewhat too large masses for the $D$ and $F$ mesons if no "exotic" symmetry-breaking terms are added. An amusing feature of the ***renormalizable*** SU(4) $\sigma$ model is that it predicts, in addition, the $D$ and $F$ decay constants (leptonic decay amplitudes) to be about six times that of the pion. Hence the experimental determination of these decay constants will have an important bearing on chiral theories. A conservative point of view is that the results of the SU(4) $\sigma$ model should be regarded only as a qualitative guide.

[19]It may be possible, though, to justify such a procedure based on recent treatments of the Nambu–Jona Lasinio model. See T. Eguchi, Phys. Rev. D 14, 2755 (1976); H. Kleinert, Phys. Lett. 59B, 163 (1975); 62B, 429 (1976); N. J. Snyderman and G. S. Guralnik (unpublished); C. Bender, F. Cooper, and G. S. Guralnik, report, 1977 (unpublished); L. H. Chan, Phys. Rev. Lett. 39, 1124 (1977).

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[27]For example, the squared mass of the $\kappa$ meson can be computed, on the assumption that it is also a Goldstone boson, to be $m^2(\kappa) = -(m_3 - m_1)(e_3 - e_1)/F_\pi^2$. This gives an expression for the $\kappa$ decay constant $F_\kappa^2/F_\pi^2 = \frac{1}{4}(w-1)(R-1)(m_\pi/m_\kappa)^2$ which leads us to expect $F_\kappa \simeq 0.4 F_4$.

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$$\partial_\mu P^b_{a\mu} = i(m_a + m_b)\bar{q}_b\gamma_5 q_a + 2U\delta^b_a[\det\bar{q}(1+\gamma_5)q - \det\bar{q}(1-\gamma_5)q].$$

Using a semiclassical approximation for the second term and specializing to the charmed quark gives $\partial_\mu P_{4\mu}{}^4 = 2i\,(m_4 + 2U e_1 e_2 e_3)\bar{q}_4\gamma_5 q_4$. Equation (6.5) follows on sandwiching this between $\langle 0|$ and $|\eta_c\rangle$ and using the nonrelativistic approximation $\bar{q}\gamma_5 q \simeq \bar{q}\gamma_4\gamma_5 q$.

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[43]S. Borchardt, J. Kandaswamy, J. Schechter, and Singer, Phys. Lett. 66B, 95 (1977); also Y. Ueda (private communication).

[44]See Appendix of Schechter and Singer, Ref. 16.

[45]See also Ueda, Ref. 16.

[46]One can see this by referring to a generalization of the SU(3) case given in Ref. 9. The same result holds in the SU(4) $\sigma$ models, Ref. 16. See also Ref. 27.

[47]By a large value we mean the range 20° to 40°. Analysis of $\eta$-$\eta'$ mixing angles from the decays $\eta \to 2\gamma$, $\eta' \to 2\gamma$, $\phi \to \eta\gamma$, $\phi \to \eta'\gamma$, etc. by H. Nagai and A. Nakamura, Prog. Theor. Phys. 53, 523 (1975), is consistent with a mixing angle around $-10°$. We should mention that there have been proposals to modify the mixing formalism to include an admixture of some extra unspecified states by introducing effectively several angles. See C. Rosenzweig, Phys. Rev. D 13, 3080 (1976); T. Imani, K. Kawarabayashi, and S. Kitakado, Prog. Theor. Phys. 56, 1570 (1976), and Okubo, Ref. 29.