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Analytic and unitary representation for the pion form factor at all Q^2

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We propose an analytic parametrization of all data for the pion form factor which can explicitly accommodate, consistent with inelastic unitarity, both higher vector-meson states and a smooth inelastic continuum, in a rather economical way. This parametrization automatically gives the asymptotic behavior expected for a quark-antiquark bound state and is free of complex zeros. We find that the best fit to the data contains no $\rho'(1250)$ signal and a possible, broad, $\rho''(1600)$, but with rather small coupling to the photon and to the $I = J = 1$ π-π system.

I. INTRODUCTION

The last five years have seen a substantial improvement in our knowledge of the pion form factor $F_\pi(Q^2)$ and now a very large range of momenta, $10\ \mathrm{GeV}^2 \gtrsim Q^2 \gtrsim -4\ \mathrm{GeV}^2$, has become accessible to experimental investigation.[1-10] Such a dramatic increase in experimental information, both at timelike[1-8] and spacelike[9,10] momenta, has not been matched by an equal progress in our theoretical understanding of its detailed features. In the absence of a theory, the step next to the collection of experimental information is to attempt its classification via some phenomenological parametrization. This has of course already been attempted by several authors,[11-17] but in most cases either on smaller portions of the measured range[11-15] or in a language difficult to translate into the more familiar concepts of resonant contributions and underlying backgrounds.[16,17]

A few recent analyses[18,19] use almost the same experimental information we use here, but our analysis differes from those on two points which we think must be stressed. First, this analysis weighs separately the "elastic," ρ-meson peak region and the regions of timelike and spacelike Q^2, where the effects of higher inelastic channels should be mostly felt, in an attempt to separate the two effects. Second, we can rely on new, more accurate information in the timelike region, which allows us to put more severe limitations on the couplings for possible, higher *broad* vector mesons.

We can summarize our theoretical requirements by saying that $F_\pi(Q^2)$ has to be a real analytic function in the Q^2 plane cut from $4\mu^2$ to infinity, obeying the unitarity relations

$$\begin{aligned}\mathrm{Im}F_\pi(Q^2) &= A^*(Q^2)F_\pi(Q^2)+\sigma(Q^2)\\ &= A(Q^2)F_\pi^*(Q^2)+\sigma^*(Q^2)\end{aligned} \tag{1}$$

on the cut, where $A(Q^2)$ is the $J=I=1$ π-π partial amplitude and the inelasticity function $\sigma(Q^2)$, defined as

$$\sigma(Q^2)=\sum_{n\neq\pi\pi} A^*_{\pi\pi\to n}(Q^2)F_n(Q^2)\rho_n(Q^2) \tag{2}$$

[here $\rho_n(Q^2)$ is the phase-space factor for the nth intermediate state in the sum], vanishes below $Q^2=s_{\mathrm{in}}$, the first inelastic threshold.

Furthermore, general beliefs in the nature of hadronic constituents and of their interactions lead us to expect an asymptotic behavior[20] (up to powers of $\ln Q^2$, $\ln\ln Q^2$, etc.):

$$F_\pi(Q^2)\sim (Q^2/M^2)^{-1}, \tag{3}$$

with some "typically hadronic" mass scale $M \sim 1$ GeV.

Such a behavior will indeed be built in our parametrization. The result we obtain shows, in our opinion, that more "exotic" behaviors are for the moment unnecessary.

Despite the wealth of experimental data, our understanding of the detailed electromagnetic structure of the pion has not gone far beyond the initial attempts to solve, more[21,22] or less[23] successfully, the two-pion approximation to the unitarity equations when $\sigma\equiv 0$. But if we wish to account for the features of $F_\pi(Q^2)$ at least in the range of Q^2 already accessible to experiment, we have to try for a solution of Eq. (1), consistent in the same range with the available information on $\pi\pi$ scattering[24-26] and inelastic annihilation channels.[7,8,27-30] However, it must be noted that such information is not enough to construct the inelasticity $\sigma(Q^2)$ from its definition (2), but only to give the upper bound[31]

$$|\sigma| \leq \left[\frac{\sigma(e^+e^-\to \mathrm{hadrons})_{I=1}-\sigma(e^+e^-\to\pi^+\pi^-)}{\sigma(e^+e^-\to\mu^+\mu^-)}\right]^{1/2} \times(1-\eta_{11}{}^2)^{1/2}, \tag{4}$$

where η_{11} is the elasticity of the $I=J=1$ partial amplitude $A(Q^2)$.

Since our interest is purely phenomenological, we shall limit ourselves to building a model for

$F_\pi(Q^2)$ which could satisfy automatically relations (1) and (3) and to displaying explicitly possible higher vector-meson states. This will then give a test on the presence of such states consistent with general principles, unlike some which can be found, even in the most recent literature, which violate even the most elementary requirements of analyticity and unitarity.[32]

As is well known, very few problems exist for the solution of (1) and (2) if s_{in} is well above any strongly coupled resonance. The hypothesis that $s_{in} > m_\rho^2$ is so common in phenomenological analyses of the $\pi\pi$ $I=1$ channel that we mention it here only because it plays an essential role for our parametrization; we will also show in Sec. II how one must deal with inelasticity in an optimal way.

Of course, owing to the high inelasticity of the proposed states,[27,29,33] only a detailed study of Eq. (1) can ensure that their production phase relative to the ρ meson is consistent with unitarity and analyticity; this is what Sec. II will be dealing with.

II. INELASTIC UNITARITY AND ITS INFINITE TAUTOLOGIES

It is well known that a knowledge of both A and σ overdetermines the solutions to Eq. (1); it is much less known that such solutions can be written, apart from the well-known polynomial ambiguities of Omnes-Muskhelishvili equations,[34] in infinite tautological forms.

Let us begin rewriting Eq. (1) as

$$\begin{aligned} F_\pi(Q^2) &= S(Q^2)F_\pi^*(Q^2) + 2i\sigma^*(Q^2) \\ &= S^*(Q^2)^{-1}[F_\pi^*(Q^2) + 2i\sigma(Q^2)], \end{aligned} \tag{5}$$

where $S = 1 + 2iA = \eta_{11}\exp 2i\delta_{11}$, and introducing the arbitrary, complex phase α as

$$F_\pi(Q^2) = S_\alpha(Q^2)F_\pi^*(Q^2) + 2i\sigma_\alpha^*(Q^2), \tag{6}$$

with the following definitions:

$$S_\alpha = S\cos^2\alpha + (S^*)^{-1}\sin^2\alpha,$$

$$\sigma_\alpha^* = \sigma^*\cos^2\alpha + (S^*)^{-1}\sigma\sin^2\alpha.$$

Let us now introduce an arbitrary continuation ϕ of the phase shift δ_{11} from the elastic region into the inelastic one $Q^2 > s_{in}$, subject to the only limiting condition

$$\lim_{\eta_{11}\to 1} \phi = \delta_{11}\,(\mathrm{mod}\,\pi)$$

and the corresponding Omnès function $\Phi(Q^2)$, properly normalized at $Q^2 = 0$,

$$\Phi(Q^2) = \exp\left[\frac{Q^2}{\pi}\int_{4\mu^2}^\infty \frac{\phi(s)ds}{s(s-Q^2)}\right]. \tag{7}$$

Writing $F_\pi(Q^2) = \Phi(Q^2)\Omega_\phi(Q^2)$, where $\Omega_\phi(Q^2)$ is then real analytic in the Q^2 plane cut from s_{in} to infinity, we derive the unitarity equation for Ω_ϕ,

$$\begin{aligned} \Omega_\phi(Q^2) &= S_\alpha(Q^2)\Omega_\phi^*(Q^2)\exp[-2i\phi(Q^2)] \\ &\quad + 2i\Phi(Q^2)^{-1}\sigma_\alpha^*(Q^2). \end{aligned} \tag{8}$$

This equation clearly displays two classes of tautologies: the first class generated by the introduction of arbitrary complex phase $\alpha(Q^2)$ which does not even need to be continuous on the cut, and the second class generated by all possible continuous choices for $\phi(Q^2)$, obeying only the limiting condition for $\eta_{11}\to 1$ on the inelastic cut.

Equation (8) has not in general a simple solution; however, we can eliminate the tautologies of the first class fixing $\alpha = \alpha_0$ so that the first term on the right-hand side of Eq. (8) becomes simply Ω_ϕ^*, namely

$$\alpha = \alpha_0 = \tan^{-1}\left(\frac{\exp[2i(\phi-\delta_{11})] - \eta_{11}}{1 - \eta_{11}\exp[2i(\phi-\delta_{11})]}\eta_{11}\right)^{1/2},$$

for which choice Eq. (8) becomes then

$$\mathrm{Im}\,\Omega_\phi = 2\,\mathrm{Re}\frac{\sigma^*(Q^2)\{1-\eta_{11}\exp[2i(\phi-\delta_{11})]\}}{(1-\eta_{11}^2)\Phi(Q^2)} \tag{9}$$

and the most general solution to Eq. (1) will then have the form

$$F_\pi(Q^2) = \frac{\Phi(Q^2)P_z(Q^2)}{P_z(0)}\left[1 + \frac{2Q^2P_z(0)}{\pi}\int_{s_{in}}^\infty \mathrm{Re}\left(\frac{\sigma^*(s)\{1-\eta_{11}\exp[2i(\phi-\delta_{11})]\}}{(1-\eta_{11}^2)\Phi(s)P_z(s)}\right)\frac{ds}{s(s-Q^2)}\right],$$

where we have collected all zeros of F_π in the polynomial factor P_z, so that Ω_ϕ can then tend, without loss of generality, to a positive constant as Q^2 tends to infinity.

Tautologies of the second class are still present since ϕ is still completely free for $\eta_{11} \neq 1$. If $\sigma(Q^2)$ were known, we could choose a $\phi = \phi_0$ such as $\mathrm{Im}\,\Omega_\phi = 0$ everywhere, and obtain then the "Omnès solution" $F_\pi = \Phi(Q^2)P_z(Q^2)/P_z(0)$. Since there have recently been many attempts[11,12] (including one of our own[35]) to treat $\gamma = \arg\sigma(Q^2)$ as a very small "perturbation,"[12,36] let us have a closer look at the behavior of $\phi_0 = \arg F_\pi$ (modπ) for small γ. We have from Eq. (9)

$$\phi_0 = \tan^{-1}\frac{\cos\gamma - \eta_{11}\cos(2\delta_{11}+\gamma)}{\sin\gamma + \eta_{11}\sin(2\delta_{11}+\gamma)}$$

and ϕ_0 can differ arbitrarily, even for very small but nonvanishing γ, from $\arg A$ around any resonance or whenever δ_{11} approaches any multiple of $\pi/2$.

The hypothesis $\gamma \ll \delta_{11}$ for Eq. (9) is then bound to give highly unstable predictions whose local success may be purely accidental and whose failure is instead highly probable. However, we wish to point out that our problems are not limited to our ignorance about $\sigma(Q^2)$ outside of specific models, but also to our too limited knowledge of $A(Q^2)$, and in particular of its phase $\theta = \arg A$ in the inelastic region.

We shall then propose to use the tautologies still present in Eq. (9), not to simplify its formal solution, but to minimize the effects of our ignorance of A. If we regard the introduction of the Omnès function $\Phi(Q^2)$ as a way of separating the supposedly understood elastic channel from the mysteries of the high-energy inelastic contributions, we may expect that, in order to conserve the information contained in our measurements of $|F_\pi|$, we shall have to use that continuation ϕ of δ_{11} into the inelastic region which is less affected by the uncertainties on $|A|$ and θ.

ϕ may be related to A by the general linear transformation

$$\phi = \arg(Ae^{i\beta} + \rho e^{i\gamma});$$

when the parameters β, γ, and ρ are constants, the condition

$$\lim_{\eta_{11}\to 1} \phi = \delta_{11} (\mathrm{mod}\,\pi)$$

requires $\rho\cos\gamma = \sin\beta$ and $\rho\sin\gamma = 0$. If we are interested in the region $Q^2 \gtrsim 1\ \mathrm{GeV}^2$, where the phase θ of the partial amplitude is practically unmeasurable since $|A|$ is very close to zero, we have to fix β so that $d\phi/|d\theta|$ has an absolute minimum for small but nonzero $|A|$. This happens for $\beta = \pi/2$, which corresponds to the "old" Goldberger-Treiman choice[23] for ϕ,

$$\phi = \phi_{\mathrm{GT}} = \arg(1 + iA). \qquad (10)$$

In the case of the π-π P wave, it can be easily checked that almost all inelastic phase-shift analyses[24-26] indeed give values of ϕ_{GT} close to each other and to a simple ρ tail in the manner of Gounaris and Sakurai.[22] Note that stability of ϕ_{GT} at the ρ-meson is automatically ensured assuming $s_{\mathrm{in}} > m_\rho^2$: The rather good experimental bounds on the ρ-meson inelasticity (typically $< 2\times 10^{-3}$) corroborate the hypothesis, common to all analyses,[24-26] that no inelastic channel opens below the $\omega\pi$ threshold.

What if $s_{\mathrm{in}} = 16\mu^2$ and the four-pion continuum gives a small, nonvanishing contribution to Eq. (2)? Again the condition for ϕ to be stable with respect to uncertainties in $|A|$ and θ gives, for points close to $A = i$ in the Argand plot, the condition $\rho = \beta = 0$ and

$$\phi = \theta,$$

which is nothing other than what we had to choose using Watson's final-state-interaction theorem.

Of course keeping β, γ, and ρ constant we cannot accomplish maximum stability of ϕ everywhere on the Argand plot. But since for $Q^2 > s_{\mathrm{in}}$ ϕ is subject only to a condition for $\eta_{11} \to 1$, we can always find three continuous functions of Q^2, satisfying $\rho\cos\gamma = \sin\beta$ and $\rho\sin\gamma = 0$ anywhere η_{11} reaches unity, that will ensure stability of ϕ with respect to experimental uncertainties at least in a portion of the Argand circle.

With the choice $s_{\mathrm{in}} \geq (m_\omega + \mu)^2 > m_\rho^2$ and $\phi = \phi_{\mathrm{GT}}$, Ω_ϕ is defined in terms of the inelasticity function by the equation

$$\mathrm{Im}\,\Omega_\phi = \mathrm{Im}\,\Omega_{\mathrm{GT}} = \frac{\mathrm{Re}\,\sigma}{|\Phi_{\mathrm{GT}}|\,|1 + iA|}. \qquad (11)$$

Unfortunately, even the bound (4) soon becomes useless as new $I = J = 1$ channels open, such as $\rho^0\pi\pi$ (or $\rho^0\epsilon$) and $\rho^+\rho^-$.

However, constructing a phase ϕ_{GT} and its Omnès function Φ_{GT}, we can rescale the measurements for $|F_\pi|$ and thus obtain "experimental" information on $|\Omega_{\mathrm{GT}}|$. This can in turn be analyzed in terms of functions, analytic in the Q^2 plane cut from s_{in} to infinity and consistent with what we expect from Eq. (11), in order to gain some indications on possible resononant structures at c.m. energies from 1 to 3 GeV.

III. THE REPRESENTATION AND ITS FIT TO $|F_\pi|$ DATA

$\Phi(Q^2)$ has to be a solution to the elastic unitarity problem

$$\mathrm{Im}\,\Phi = \Phi^* A = A^*\Phi, \qquad \mathrm{Im}\,A = |A|^2, \qquad (12)$$

at $Q^2 \leq s_{\mathrm{in}}$. Since we have already observed the closeness of ϕ_{GT} to a simple Breit-Wigner tail (see Fig. 1), we shall assume Φ_{GT} to be a solution to elastic unitarity at all Q^2 and write a resonant N/D decomposition for A,

$$A = N(Q^2)/D(Q^2), \qquad (13)$$

where we recall that both D and N are real analytic functions in cut Q^2 planes, with cuts running respectively from $4\mu^2$ to ∞ and from 0 to $-\infty$. A solution for Φ_{GT}, properly normalized at $Q^2 = 0$,

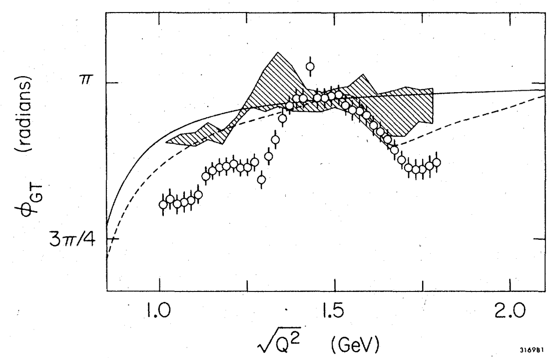

FIG. 1. The Goldberger-Treiman phase for $J=I=1$ $\pi\pi$ partial amplitude plotted versus c.m. energy. Here the dashed line is the energy-dependent fit of Ref. 22, the open circles are the results from Ref. 23, and the shaded area is the region covered by the ambiguities of Ref. 24. On this we superimpose as a full line our ansatz $\Phi_{GT}=D_\rho(0)/D_\rho(Q^2)$ where $D_\rho(Q^2)$ is given by Eq. (17).

is then

$$\Phi_{GT}=[D(0)P_z(Q^2)]/[D(Q^2)P_z(0)]\,, \tag{14}$$

with z complex zeros in the Q^2 plane. If Φ_{GT} has to satisfy the asymptotic condition (3), we must then have

$$\lim_{Q^2\to\infty}|D(Q^2)/(Q^2)^{z+1}|=\text{constant} \tag{15}$$

(up to powers of $\ln Q^2$, $\ln\ln Q^2$, etc.) and for $Q^2 \geq 4\mu^2$,

$$\operatorname{Im}D(Q^2)=-N(Q^2)\,, \tag{16}$$

so that in principle the left-hand cut discontinuity of $N(Q^2)$ will determine, together with the complex zeros in $D(Q^2)$, all the dynamics of the $\pi\pi$ system.

We shall then write a simple one-level resonant formula for D, parametrizing it as (fixing $z=0$, i.e., no zeros in F_π)

$$D(Q^2)=a+bQ^2+ch(Q^2)\,,$$

where

$$h(Q^2)=\frac{2}{\pi}\left(\frac{Q^2-t}{Q^2}\right)^{l^*}\left(1-\frac{R}{Q^2}\right)\times[f(Q^2)-\phi(Q^2)];$$

here f and ϕ are defined as

$$f(Q^2)=\left(\frac{Q^2-t}{Q^2}\right)^{1/2}\ln\frac{(t-Q^2)^{1/2}-(-Q^2)^{1/2}}{\sqrt{t}}$$

and

$$\phi(Q^2)=\sum_{n=0}^{l^*}\left(\frac{\partial^n f}{\partial x^n}\right)\frac{x^n}{n!}\,,\quad \text{for } x=\frac{Q^2}{t-Q^2}$$

where t is a threshold of the resonating two-body channel, l^* is an orbital angular momentum in that channel, and R is a skewness parameter to be fixed by the scattering length (for the elastic channel only).

$D(Q^2)$ then obeys the asymptotic constraint (15) automatically and has a resonance of mass M and width Γ if

$$\operatorname{Re}D(M^2)=0\,,$$

$$\operatorname{Im}D(M^2)=-M\Gamma;$$

requiring furthermore that

$$\mu^3 a_{11}=\lim_{Q^2\to 4\mu^2}\left(\frac{4\mu^2}{Q^2-4\mu^2}\right)^{3/2}\frac{N(Q^2)}{D(Q^2)}$$

fixes all parameters in $D(Q^2)$ up to an arbitrary normalization.[22]

Owing to the high inelastic threshold $s_{\text{in}}\geq(m_\omega+\mu)^2$, we can directly fit the formula

$$D(Q^2;M,\Gamma;t,R;l^*)=M^2-Q^2-M\Gamma\frac{h(Q^2)-\operatorname{Re}h(M^2)}{\operatorname{Im}h(M^2)-(\partial\operatorname{Re}h/\partial Q^2)_{M^2}M\Gamma} \tag{17}$$

to the unnormalized $e^+e^-\to\pi^+\pi^-$ cross section at the ρ-meson peak. Including ρ^0-ω mixing and constraining R to give $\mu^3 a_{11}\simeq 0.048$ (i.e., the "current algebraic" value), a fit to the results of Benaksas *et al.*[1] gives for the parameters in $D_\rho(Q^2)=D(Q^2;m_\rho,\Gamma_\rho;4\mu^2,R;1)$

$$m_\rho=772\text{ MeV}\,,$$

$$\Gamma_\rho=136\text{ MeV}\,,$$

$$R/4\mu^2=+0.85\,,$$

reproducing the results of the original paper.[1]

Note that $R/4\mu^2$ can be varied considerably without spoiling the fit on the ρ peak: Only the region from just above threshold down to very low spacelike Q^2 is really sensitive to this parameter (or alternatively to $\mu^3 a_{11}$). However, this region has data coming from four sources with different systematic uncertainties, i.e., e^+e^- annihilation, inverse electroproduction, electroproduction at threshold, $\pi^\pm e^-$ scattering. This latter source has, in most recent fits, too large an importance owing to the narrow binning and the very low statistical errors.

As we can see from Fig. 1, $\Phi_{GT}(Q^2)$ gives also a good fit to ϕ_{GT} values from the recent analysis by Hyams *et al.*,[26] and we shall then use it as the "elastic" contribution to F_π to derive, from the measurements of Ref. 2–10, $|\Omega_{GT}(Q^2)|$ outside the ρ-meson peak.

A plot of $|\Omega_{\mathrm{GT}}|$ versus the variable $x = m_\rho^2(Q^2 - m_\rho^2)$ shows marked, systematic deviations from unity which we choose to explain as inelastic effects. Since all expected vector mesons have to be highly inelastic, from both the analysis of the elastic channel[24-26] and their detection in inelastic channels,[37-39] we decompose Ω_{GT} into the sum of one or more resonant terms $\Phi_i(Q^2)$ and a smooth "background" $B(Q^2)$ and write, to enforce normalization at $Q^2=0$,

$$\Omega_{\mathrm{GT}}(Q^2) = 1 + \beta[B(Q^2) - B(0)] + \sum_i \alpha_i[\Phi_i(Q^2) - \Phi_i(0)]\,. \tag{18}$$

Rescaling the variable x to $\bar{x} = x(Q^2)/x(s_{\mathrm{in}}) = (s_{\mathrm{in}} - m_\rho^2)/(Q^2 - m_\rho^2)$, we will parametrize a "background" from s_{in} to infinity as

$$B(\bar{x}) = -\bar{x}[(1-\bar{x})^m \ln(1 - 1/\bar{x}) - Q_m(\bar{x})]\,, \tag{19}$$

which has the smooth discontinuity

$$\mathrm{Im}B(Q^2) = \pi\left(\frac{s_{in} - m_\rho^2}{Q^2 - m_\rho^2}\right)\left(\frac{Q^2 - s_{\mathrm{in}}}{Q^2 - m_\rho^2}\right)^m$$

across the inelastic cut, and where Q_m is a polynomial of degree $m-1$ fixed by the condition $\lim_{x\to\infty} B(x) = \text{constant}$, and both the scale β and the threshold behavior can be accomodated to fit the data.

Recalling the definition (2) for $\sigma(Q^2)$, we expect an inelastic resonance ρ_i to appear in the shape of a Breit-Wigner structure (over some background) in $\mathrm{Re}\sigma$, and taking formula (17) and imposing $R=0$, we can write

$$\Phi_i(Q^2) = \frac{D_i(0)}{x D_i(Q^2)}, \tag{20}$$

where $D_i(Q^2) = D(Q^2; M_i, \Gamma_i; t_i, 0; l_i)$. Note that we must then have the inequality

$$\lim_{Q^2\to\infty} \Omega_{\mathrm{GT}} = 1 - \sum_i \alpha_i[D_i(0)/m_\rho^2 - 1] - \beta B(0) \geq 0$$

if complete absence of complex zeros in F_π has to be guaranteed.

Since formulas (17)–(20) introduce a wealth of free parameters, let us restrict our search to those effects whose existence may be inferred from other processes. Two higher vector mesons have been claimed, a $\rho'(1250)$ claimed in both $\bar{p}p \to \omega\pi^+\pi^-$ annihilation[38] and in a compilation of $e^+e^- \to \pi^+\pi^-\pi^0\pi^0$ data,[29,33] and a $\rho''(1600)$ found in $\pi^+\pi^-\pi^+\pi^-$ photoproduction[37,39] and in $e^+e^- \to \pi^+\pi^-\pi^+\pi^-$ (Ref. 27) and shown to be mainly in a $\rho^0\pi^+\pi^-$ state.[27,39] Of the two, only the $\rho''(1600)$ shows up in π-π phase-shift analyses[24-26] where it seems necessary to satisfy backward dispersion relations[40] with an inelasticity of at least 75%. Note that recent measurements of $e^+e^- \to 4\pi$ at Novosibirsk[7] do not contain the strong $\rho'(1250)$ signal claimed by Refs. 29 and 33, while a previous CERN-Frascati experiment[28] failed to see any clear indication of either $\rho'(1250)$ or $\rho''(1600)$. Looking only for these two effects, since the interferences we are looking for in the "elastic" $\pi\pi$ channel will not be very sensitive to the masses, we can fix $M_1 = 1.25\ \mathrm{GeV}/c^2$ (with $\sqrt{t_1} = m_\omega + \mu$ and $l_1 = 1$) and $M_2 = 1.60\ \mathrm{GeV}/c^2$ (with $\sqrt{t_2} = m_\rho + 2\mu$ and $l_2 = 0$), since the two main decay channels are claimed to be $\rho' \to \omega\pi$ and $\rho'' \to \rho\pi\pi$, while for s_{in} in $B(Q^2)$ we shall try either choice $s_{\mathrm{in}} = t_1$ or $s_{\mathrm{in}} = t_2$.

The only free parameters left are then the threshold exponent m in $B(Q^2)$, the widths Γ_1 and Γ_2, and the scale factors β, α_1, and α_2.

Our data selection for $|\Omega_{\mathrm{GT}}|$ includes: (a) at $Q^2 > m_\rho^2$, three points from ACO (Ref. 6) at $Q^2 > 0.8\ \mathrm{GeV}^2$, 23 preliminary data from VEPP-2M (Ref. 7) up to $Q^2 \simeq 1.69\ \mathrm{GeV}^2$, 13 points from the Bologna-CERN-Frascati collaboration[4] from 1.44 to 9.0 GeV^2 and the SPEAR measurement at the ψ resonance,[8] and (b) at $Q^2 < m_\rho^2$, the lowest energy point from ACO, (Ref. 5) three points from analysis of inverse electroproduction[2] $\pi^-p \to e^+e^-n$, one from an electroproduction sum rule at threshold,[3] four obtained at very low $Q^2 < 0$ rebinning the original πe elastic scattering results[9] and 17 points, down to $-4\ \mathrm{GeV}^2$, from electroproduction isovector contributions in the t channel[10] (including a reassessment of previous CEA[41] and Cornell[42] results).

From Fig. 2 we may easily isolate the main features of $|\Omega|$: It tends to be systematically above unity at positive, sufficiently large x (at least for $x > 0.1$) and below it in the interval $0 > x > -1$ [implying then a mean-square radius larger

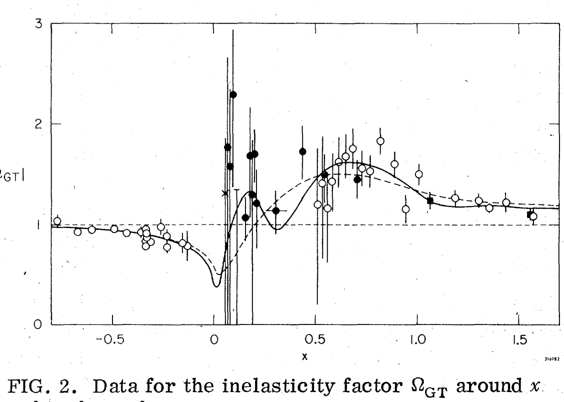

FIG. 2. Data for the inelasticity factor Ω_{GT} around $x = m_\rho^2/(Q^2 - m_\rho^2) = 0$, compared with our two models of a pure background (dashed line) and of a background plus higher vector mesons (the full line shows the best fit, with the ρ'' only). Solid circles are data from Ref. 4, the squares come from Ref. 6, the cross is the SLAC point at the ψ resonance, Ref. 8, open circles at $x<0$ are electroproduction results, Ref. 10.

TABLE I. Pion mean-square electromagnetic radius.

Q^2 range (F^{-2})	$\langle r_\pi^2\rangle^{1/2}$ (F)	Ref.
(−3.0, −1.0)	$0.74^{+0.11}_{-0.13}$	45
(−0.9, −0.3)	0.78 ± 0.03	9
	0.71 ± 0.05	46
(0, 1.1)	0.98 ± 0.24	3
(1.7, 2.9)	0.75 ± 0.14	2
ρ-meson dominance only	0.676	1
Our inelastic fits (all Q^2)	0.695	1–10

than expected from the approach followed by Gounaris and Sakurai in Eqs. (14)–(17)—see Table I for a comparison with experiments at low Q^2—and an asymptotic scale smaller than $D_\rho(0) \simeq m_\rho^2$]. Note that these effects are correlated by analyticity to predict an essentially non-negative discontinuity for Ω across the inelastic cut $Q^2 > s_{in}$ in Eq. (9); their relative size and shape are further useful to constrain the size and (less) the shape of such a discontinuity.

Furthermore, data from Adone[4] suggest the presence of at least one strong dip in $|\Omega|$ at $x \simeq 0.3$ (the region where this experiment has the highest integrated luminosity), or $Q^2 \simeq 2.5$ GeV^2.

The best fit to the whole set of 66 points with a pure smooth background as given by formula (19) is obtained with $m = 3$ and $s_{in} = t_2 = (m_\rho + 2\mu)^2$ for a value $\beta \simeq 2.1$, and gives a very low probability of 3.6×10^{-3}; however, the elimination from the fit of the points at $0 > Q^2 > -1.5$ GeV^2 (where there seem to be inconsistencies between data at very close values of Q^2) produces the more acceptable probability of 2.6×10^{-2}. At a purely statistical level, we do not have compelling evidence for additional timelike structures beyond $Q^2 = s_{in}$ since most of the χ^2 for the previous fit on all the 66 points came from data at $Q^2 < 0$.

However, such a fit does not follow the detailed features of $|F_\pi|$ at $Q^2 > 1.8$ GeV^2. We then insert the ρ' and ρ'' states at their "claimed" masses of 1.25 and 1.60 GeV/c^2, in addition to the same background.[21] We find that the data reject any appreciable content of $\rho'(1250)$, but the dip in $|\Omega|$ displayed by the Adone data[4] requires the inclusion of a $\rho''(1600)$ in the fit with a marked preference toward a rather broad state, $\Gamma_2 \simeq 750$ MeV, much broader than the ρ'' seen in the $\pi\pi$ phase shifts.[24-26]

The best fit for the width is reached [independent of Γ_1 as long as it is not as large as Γ_2, but no one has ever claimed a ρ' much broader than the ρ [Refs. (29, 33, 37, 39)] for the "coupling constants"

$$\alpha_1(\simeq -1.14 g_{\rho'\pi\pi}/f_{\rho'}) \simeq 0.00 ,$$

$$\alpha_2(\simeq -1.12 g_{\rho''\pi\pi}/f_{\rho''}) \simeq -0.15$$

(where finite-width effects have been included in parantheses to translate our α_n into the coupling-constant ratio g_n/f_n used in "extended" vector-meson dominance models), with a "background strength" $\beta \simeq 5.5$ which has, however, a strong, negative correlation to α_2 as a consequence of analyticity and our ignorance of $\arg\Omega$. The probability of such a fit is rather high, reaching 5.1×10^{-2} on all the 66 points (despite the decrease in the number of degrees of freedom); if only the data outside the region $0 > Q^2 > -1.5$ GeV^2 are considered, the probability of the fit reaches the rather satisfactory value 0.23.

This remarkable improvement comes mainly from the timelike region $Q^2 > s_{in}$: from Figs. 2–4, one can see that data in the "elastic region" $Q^2 < s_{in}$ (both timelike and spacelike) do not constrain strongly the behavior of $|\Omega|$ on the inelastic cut. Therefore, any claims of the presence of higher vector mesons based on analytic extrapo-

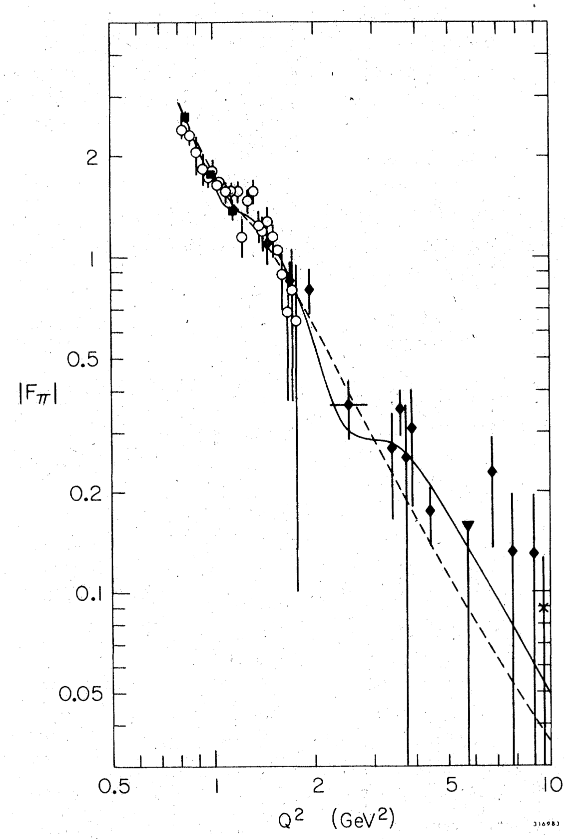

FIG. 3. $|F_\pi|$ versus Q^2 (see caption of Fig. 2 for the meaning of the two lines) for $Q^2 \geq m_\rho^2$. Circles are from Ref. 7, squares from Ref. 6, diamonds from Ref. 4, and the cross from Ref. 8.

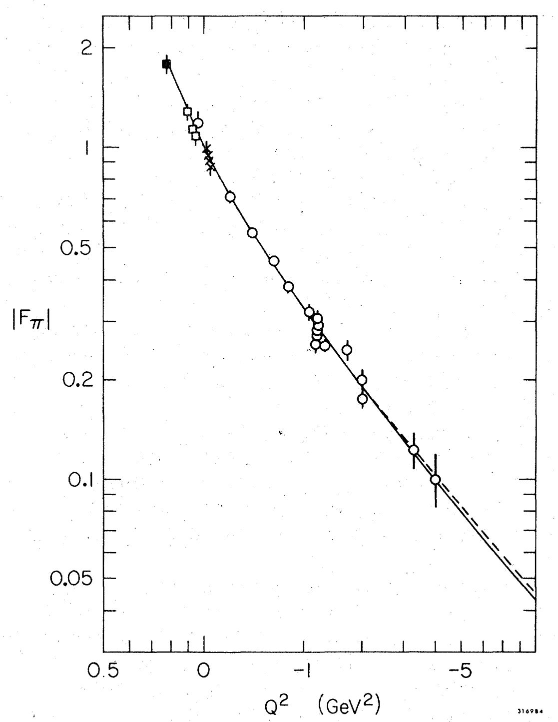

FIG. 4. $|F_\pi|$ versus Q^2 (lines have the same meaning as in Figs. 2 and 3) for $Q^2 \leq m_\rho^2$. Note that to represent all data we had to shift the origin of the logarithmic scale to $Q^2 = 1$ GeV^2. Circles at spacelike Q^2 are from Ref. 10, the solid square from Ref. 5, open squares from Ref. 2, crosses from Ref. 9, and the circle at timelike Q^2 from Ref. 3.

lation techniques,[16,17] which expand either F_π or Ω in a series of functions of some variable $z(Q^2)$ (which converge everywhere but on the inelastic cut) are particularly unstable since only convergence in the mean exists on the cut,[43] and the shape of Ω on the cut will be critically dependent on the particular truncation criterion used.

It becomes particularly difficult to decide if a rapid variation in ImΩ has to be associated with structures in the data or has to be ascribed to such a truncation. Furthermore, in such an analysis, it is hard to constrain the production phase of a possible higher inelastic resonance to the value expected from unitartiy.

The present approach has evidently the drawback of automatically associating sharp structures in the data with such resonances. Despite this, we feel it presents two main advantages: First, it yields in a very simple way the essential parameters of a possible higher resonance ρ_n, namely mass M_n, width Γ_n, and coupling ratio $g_{\pi\pi\pi}/f_n$, without any conflict with general principles or drastic approximations. Last, but not least, the model, at variance with more sophisticated expansions[16,17] can be built free of both unusual, far-away zeros in the Q^2 plane[17,44] and of unexpected asymptotic behaviors,[17,18] different from what quark-gluon theory[23] dictates.

We gladly point out that the good probability level reached with the present model ($\simeq$23% at timelike and high spacelike Q^2) shows, much to our taste, that none of such features is required by present data.

It is also to be noted that a quite satisfactory value for the pion radius $\langle r_\pi^2\rangle \simeq 0.483$ F^2, close to the estimate by Dubnicka and Dumbrajs,[46] 0.50 $\pm$ 0.07 F^2, has been found with a small scattering length, much smaller indeed than the one advocated by Ref. 19, indicating that finite-width effects and the treatment of inelastic contribution may explain most, if not all, of the discrepancy between $\langle r_\pi^2\rangle$ and the simple ρ-meson-dominance prediction $6/m_\rho^2$.

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