PHYSICAL REVIEW D VOLUME 17, NUMBER 5 1 MARCH 1978

Form factors of states bound by attractive potentials

Ira Herbst*
Physics Department, Princeton University, Jadwin Hall, Princeton, New Jersey 08540

S. Nussinov†
Center for Theoretical Physics, Department of Physics and Astronomy, University of Maryland, College Park, Maryland 20742
(Received 29 July 1977)

Various positivity and monotonicity properties are proved for form factors of ground states bound by attractive potentials. In particular the experimentally observed monotonicity and positivity of the electric form factor of the proton (and a similar property predicted for the π form factor) can be correlated with the possible purely attractive nature of the $q\bar{q}$ and $3q$ forces.

INTRODUCTION

Traditionally dispersion relations dominated most theoretical investigations of form factors, leading—among other results—to the prediction of the low-lying vector mesons (ω and ρ).[1] With the emergence of quark theories, form factors are now often studied within the framework of various bound-state models.[2] In particular it has been suggested that the approximate Q^{-4} asymptotic falloff of the nucleon electromagnetic form factor is related to the three-quark picture of the nucleon.[3,4] Very likely the two approaches (the dispersion-theoretic and quark-parton methods) are complementary. A simple vector-dominance-model saturation of dispersion relations appears useful for discussing features of the form factors at low Q^2; the second, quark-parton, approach is generally applicable in the asymptotic regime.

In the following we utilize mainly the second approach for proving some general properties of (nonasymptotic) form factors which follow from certain assumptions about the binding potentials, and also discuss some mutual constraints which follow from the application of both approaches. We will employ the simple nonrelativistic Schrödinger-equation framework where the proof of most of our results is essentially trivial. These results often appear to have model-independent characteristics suggesting a generality exceeding what is implied by the derivation, which makes them rather interesting.

I. CHARGE DENSITY AND SPECTRAL FUNCTION

The electric form factor of a particle with charge e bound in a bound state $\psi_\alpha(\vec{r})$ is given by

$$F_E(\vec{q}) = e\int d\vec{r}\,\psi_\alpha{}^2(\vec{r})e^{i\vec{q}\cdot\vec{r}}. \tag{1}$$

(The bound-state wave function can be chosen real, and complex-conjugation signs avoided.) Similarly if $\psi_\alpha(\vec{r}_1,\ldots,\vec{r}_n)$ is the bound state of n particles with charges $e_1,\ldots,e_n$,

$$F_E(\vec{q}) = \sum e_i \int d\vec{r}\,\rho_i(\vec{r})e^{i\vec{q}\cdot\vec{r}},$$

$$\rho_i(\vec{r}) = \int d\vec{r}_1\cdots d\vec{r}_{i-1}d\vec{r}_{i+1}\cdots d\vec{r}_n\psi_\alpha(\vec{r}_1\ldots\vec{r}_n)^2.$$

Consider first a two-particle (say $\bar{q}_a q_b$) bound state. If $m_a = m_b$, the particles are symmetric with respect to the center-of-mass system and hence $\rho_a(\vec{r}) = \rho_b(\vec{r})$. Thus $\rho_E(\vec{r})$ and $F_E(\vec{q})$ will be proportional to the probability density and corresponding form factor [as in Eq. (1)]. If $m_a \neq m_b$, as is the case for K_0, a form factor different from zero may occur.[5] Also in the symmetric quark model[6] it can be readily shown that the electric form factor of (say) the nucleon is proportional to the product of its total charge ($e = \sum e_i$) and the form factor of total probability density ($\rho_{\rm tot} = \sum \rho_i$). Thus for $e = 0$ we expect vanishing electric form factors, as is roughly the case for the neutron.

The first observation that we would like to make is that when $\rho_E(\vec{r}) \simeq \rho_{\rm tot}(\vec{r})$ the configuration-space charge density is everywhere non-negative. This density can be expressed in terms of the spectral function in a dispersion relation

$$F(\vec{q}^{\,2}) = \int_{\mu_0{}^2}^{\infty} \frac{d(\mu^2)\sigma(\mu^2)}{\mu^2 + \vec{q}^{\,2}} \tag{2}$$

as a corresponding superposition of Yukawa-type distributions

$$\rho(|\vec{r}|) = \int \sigma(\mu^2)\frac{e^{-\mu r}}{r}d(\mu^2). \tag{3}$$

$\sigma(\mu^2)$ need not be positive. In the case of a nucleon, the dipole asymptotic behavior (and also an elegant treatment of much earlier cruder data[7]) does indeed indicate alternation of signs in $\sigma(\mu^2)$. Demanding that the right-hand side (RHS) of Eq. (3) be positive for all r imposes a nontrivial constraint on the models.

As an example, consider a simple case where the dispersion relation is saturated by the contribution of just two narrow resonances (say, ρ and ρ' for the pion form factor)

$$\sigma(\mu^2) = g_1\delta(\mu^2 - \mu_1^2) + g_2\delta(\mu^2 - \mu_2^2),$$

$$\rho(|\vec{r}|) = g_1 \frac{e^{-\mu_1 r}}{r} + g_2 \frac{e^{-\mu_2 r}}{r}.$$

We cannot allow g_2 to be negative and larger in magnitude than g_1. ($g_2 = -g_1$ would in particular give dipole-type asymptotic behavior.)

Further constraints on the spectral function $\sigma(\mu)$ follow in more specific cases when more information on $\rho(r)$ is available. Thus for potentials $V(r)$, which are monotonically decreasing with r, one can show that $\psi_0(r)$, the ground state wave function, and hence also $\rho_0(r)$, are monotonically decreasing with r (see below). Taking the derivative of Eq. (3) and insisting that it is negative for all r yields then the requirement

$$\int \sigma(\mu^2)\left[e^{-\mu r}\left(\frac{1}{r}+\mu\right)\right]d(\mu^2) \geq 0, \text{ for all } r. \quad (4)$$

II. POSITIVITY OF THE FORM FACTOR

Rather than deal with the conventional configuration-space Schrödinger system, let us go over to momentum space where the equation reads

$$E_\alpha\psi_\alpha(\vec{p}) = \int d\vec{k}\, V(\vec{p},\vec{k})\psi_\alpha(\vec{k}) + \vec{p}^{\,2}\psi_\alpha(\vec{p}) \quad (5)$$

with $V(\vec{p},\vec{k})$ a general real symmetric function. We can use for the purpose of the present section this general nonlocal form, though conventionally one restricts attention (in the nonrelativistic case) to local potentials

$$V(\vec{p},\vec{k}) = V(\vec{p}-\vec{k}) \quad (6)$$

and often spherically symmetric ones

$$V(\vec{p}-\vec{k}) = V(|\vec{p}-\vec{k}|^2). \quad (6')$$

The probability density form factor is given in terms of the momentum-space wave function by a convolution

$$F_\alpha(\vec{q}) = \int \psi_\alpha(\vec{k})\psi_\alpha(\vec{k}+\vec{q})d\vec{k}. \quad (7)$$

Now the following statement can be readily demonstrated. If $V(\vec{p},\vec{k})$ is everywhere attractive (in momentum space)

$$V(\vec{p},\vec{k}) < 0 \text{ for all } \vec{p},\vec{k}, \quad (8)$$

then the ground-state form factor $F_0(\vec{q}^{\,2})$ is positive for all $\vec{q}$.

Proof. The ground-state wave function $\psi_0(\vec{p})$ minimizes

$$\langle\psi|H|\psi\rangle \equiv \int d\vec{p}\,d\vec{k}\,\psi(\vec{p})V(\vec{p},\vec{k})\psi(\vec{k}) + \int d\vec{p}\,\frac{\vec{p}^{\,2}}{2m}\psi^2(\vec{p}) \quad (9)$$

subject to the normalization condition

$$\langle\psi|\psi\rangle \equiv \int d\vec{p}\,\psi^2(\vec{p}) = 1. \quad (10)$$

If $\psi_0(\vec{p})$ flips sign and is (say) negative in some domain D of momentum space we can generate a better (i.e., lower-energy) trial function which is positive everywhere by choosing $\psi_0'(\vec{p}) = |\psi_0(\vec{p})|$. Clearly $\psi_0'(\vec{p})$ satisfies the normalization (10), and yields the same kinetic (second) term as $\psi_0(\vec{p})$ in the RHS of Eq. (9). Also this replacement will make the integrand in the potential (first) term in Eq. (9) everywhere negative. Thus $\langle\psi_0'|H|\psi_0'\rangle \leq \langle\psi_0|H|\psi_0\rangle$ and, barring degeneracy of the ground state, we conclude that $\psi_0' = \psi_0$ and $\psi_0(\vec{p}) \geq 0$ for all $\vec{p}$. Using then Eq. (7) we find

$$F_0(\vec{q}) \geq 0 \text{ for all } \vec{q}. \quad (11)$$

That result generalizes also to the case when we are dealing with the form factor for probability density of a many-particle bound state. We now view ψ as $\psi(K)$, a function of a $3n$-component "supervector" $K = (\vec{k}_1, \ldots, \vec{k}_n)$. The density form factor is given in this case by a sum of n terms

$$F^{(n)}(\vec{q}) = \sum_{i=1}^{n} \int \psi(K+\Delta_i)\psi(K)dK,$$
$$\Delta_i = (\vec{0}, \ldots, \vec{0}, \vec{q}, \vec{0}, \ldots, \vec{0}) \quad (12)$$
$$\qquad 1, \ldots, i-1, i, \ldots, n.$$

If the potential $V(P,K)$ is everywhere attractive in momentum space, we conclude by the same argument as above that the ground-state wave function $\psi_0(K)$ and hence from (12) also $F_0^{(n)}(\vec{q})$ are positive everywhere. Note that the "potential" $V(P,K)$ is really very general—not only is it nonlocal, but it could also include genuine many-body interactions, i.e., we need not have

$$V(P,K) = \sum_{i\neq j} V(\vec{p}_i,\vec{k}_j). \quad (13)$$

Also we could allow more general wave functions with components of a varying number of particles[8] so that, as long as we do not allow the electromagnetic current to create particles ($\bar{q}q$ pairs), we have

$$F_0(\vec{q}) = \sum_n w_n F_0^{(n)}(\vec{q}) \quad (14)$$

with $w_n \geq 0$ ($\sum w_n = 1$) being the weights of the various n-particle components, and the positivity of $F_0(\vec{q})$ is still maintained.

III. CHOICE OF POTENTIAL

Since the momentum-space attractive potential is the key element in proving the positivity of the density form factors, we would like now to discuss possible motivations for assuming such potentials.

Let us first focus on just the ground ($\bar{q}q$) state—the pion. If we approximate the $\bar{q}q$ potential by various single-particle exchanges, the force is always attractive—unlike the particle case (say, NN) where vector exchanges tend to be repulsive. The resulting potential is of the form

$$\tilde{V}(r) = -\sum g_i^2 e^{-\mu_i r}/r, \quad V(\vec{q}) = -\sum g_i^2/(\mu_i^2 + \vec{q}^2). \tag{15}$$

Clearly such potentials—or more general Yukawa superpositions

$$\tilde{V}(r) = -\int \sigma(\mu)(e^{-\mu r}/r)\,d\mu^2, \qquad V(q) = -\int \frac{\sigma(\mu)}{\mu^2+q^2}\,d\mu^2, \quad \sigma(\mu) > 0 \tag{15'}$$

are of the momentum-attractive type. They also happen to be monotonic in *both* momentum and configuration space. Finally the spin-dependent forces are also attractive for the singlet π state so as to split it down from the triplet ρ state.[9]

Most recently there has been considerable interest in confining potentials [e.g., harmonic oscillator, $\tilde{V}(r) \approx r^2$, and linear $\tilde{V}(r) \simeq r$, potentials] so as to achieve permanent quark confinement. This has been motivated by the experimental evidence (with one exception) against free nonintegrally charged quarks and the possibility that the complex infrared and/or topological structure of quantum chromodynamics (QCD) will indeed yield such confining which in particular in a nonrelativistic heavy-quark limit will yield a linear potential. Such linear potentials have been quite successfully and extensively applied to the charmonium system.[10]

Because of the infrared (long-distance) singular nature of the confining potentials they cannot be readily included in the above [Eqs. (15) and (15′)] class of positive Yukawa superpositions and require a more careful discussion. To be specific we will restrict ourselves to

$$\tilde{V}_\lambda(r) = cr^\lambda, \quad c \geq 0, \quad 0 \leq \lambda \leq 2, \tag{16}$$

or positive superpositions of these. From a certain point of view [concentrating the (color) line of force into a zero-width string] the linear growth of $v(r)$ is maximal but we would like to include the harmonic-oscillator case as well because of the large amount of work done with it (and Gaussian propagators and distributions).[10,11]

The Fourier transform of (16) is[12]

$$V_\lambda(p) = 2^{\lambda+3}\pi^{3/2}\frac{\Gamma((\lambda+3)/2)}{\Gamma(-\lambda/2)}p^{-\lambda-3}. \tag{16'}$$

The spherically symmetric inverse power of p on the RHS has to be interpreted as a generalized function[12] and as such has added "residue" corrections for $\lambda = 0, 2, 4$. In particular for $\lambda = 2$ (i.e., the harmonic-oscillator case) the expression on the RHS of (16′) vanishes (because of the Γ-function pole in the denominator) and only the correction residue $[= -\vec{\nabla}_p^2\delta^3(\vec{p})]$ survives. Indeed in this case we have a complete $\vec{r} \leftrightarrow \vec{p}$ symmetry and, as expected, we have the same Schrödinger equation in both $\vec{r}$ and $\vec{p}$ space. The positivity of $\psi_0(\vec{p})$ could now be deduced from the general well-known theorem that the Schrödinger ground state with a local potential has no nodes. Indeed both wave functions and form factors are simple Gaussians.

In the interval $0 < \lambda < 2$ there are no correction terms and because of the sign of $\Gamma(-\lambda/2)$, $V_\lambda(p)$ is negative, which suggests that our proof of the positivity of the ground-state wave function will hold. Amusingly $V_\lambda(p)$ alternates in sign as λ varies between $2, 4, 6, \ldots,$ etc. so that this property is not shared by all potentials r^λ. This is quite gratifying since for an infinitely deep square-well potential of radius 1 [which is the formal limit of $V_\lambda(r)$ as $\lambda \to \infty$], $\tilde{\psi}_0(r) = \sin\pi r\,\theta(1-r)\theta(r)$, and $\psi_0(q)$ is not nodeless.

Unfortunately the proof of the theorem cannot be carried through directly because, while $\langle\psi_0|H|\psi_0\rangle$ exists both in configuration and in momentum space, the singularity of $V(p-k)$ prevents us from exhibiting the latter as the first term in Eq. (9).

This difficulty is circumvented by letting

$$V_\lambda(p) = -\left(\frac{1}{p^2}\right)^{-\lambda-3/2} \to -\left(\frac{1}{p^2+\mu^2}\right)^{-\lambda-3/2} = V_\lambda^{(\mu)}(p),$$

a natural infrared regularization which is suggested if screening occurs at a distance $1/\mu$.[14] For the specific case of $\lambda = 1$, $V_\lambda(p) = -[1/(p^2+\mu^2)]^{-2}$, and the corresponding $\tilde{V}_\lambda^{(\mu)}(r) = -e^{-\mu r}/\mu$. This potential, which $\to 0$ at ∞ is nonconfining, has depth $\approx \mu^{-1}$, and a linear portion extending to a distance $O(\mu^{-1})$ which for sufficiently small μ^{-1} exceeds the localization distance of low-lying wave functions. The formal $-1/\mu$ divergence of

$$\int dp\,dq \frac{1}{[(p-q)^2+\mu^2]}\psi_0(p)\psi_0(q)$$

reflects the depth of the potential viewed from a reference point $\tilde{V}(r=\infty) = 0$ which is appropriate for nonconfining potentials and can be avoided by adding a constant $1/\mu$. This adds $(1/\mu)\delta(p)$ to

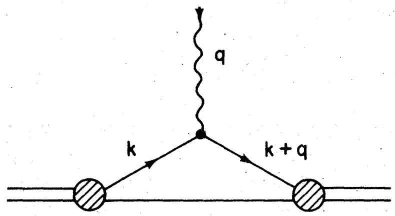

FIG. 1. A diagrammatic illustration of the convolution formula.

$V_\lambda(p)$ and, as expected, does not effect the proof of $\psi_0(p) > 0$. Indeed the proof of Sec. II goes through if instead of the nonrelativistic kinetic term $p^2/2m$ in Eq. (9) we use any positive function which in particular would allow for some relativistic corrections.

Also the convolution formula (7) which corresponds to the NR diagram (Fig. 1) can be corrected by introducing propagators as extra positive weights so as to incorporate some recoil effects, without changing the conclusion that a nodeless $\psi_0(p)$ implies a positive $F_0(\vec{p})$.

All these considerations and our above result therefore suggest that even for light-quark systems the ground-state electromagnetic pion form factor will never vanish and we expect that

$$F_\pi(q^2) \geq 0 \quad \text{for all } q. \tag{17}$$

Within the framework of QCD in which the interactions are purely color independent there is considerable similarity between the $\bar{q}q$ (35) mesons and the qqq (56) baryons. The color singlet qqq state is a symmetric superposition of the three possible diquark-quark configurations, where in each diquark the (3 + 3) quarks couple to a $\bar{3}$. Each of the diquark-quark configurations is therefore a color $\bar{3}$-3 just like a $\bar{q}q$ meson and we might expect again a purely attractive interaction. Coupling this with the argument of Sec. I for the proportionality (within the symmetric quark model) of charge and probability density we would then predict the positivity of the proton electric form factor

$$F_E^p(Q^2) \geq 0, \tag{18}$$

which appears to be the case experimentally.

In spite of their humble nonrelativistic Schrödinger-equation origin, the results (17) and (18) are rather model independent and should hold for any (momentum) positive potential, and some of the generalizations above suggest that they may actually carry into the relativistic domain.

It should be pointed out that, whereas the momentum positivity of the potential does guarantee the positivity of $\psi_0(\vec{p})$ and hence of $F(\vec{q}) = \int \psi(\vec{p})\psi_0(\vec{p}+\vec{q})d\vec{p}$, the reverse is not true. The convolution integral may be always positive even when $\psi_0(\vec{p})$ has nodes, and also $\psi_0(\vec{p})$ can be always positive when $V(p,q)$ is not. While this is an aesthetic drawback, it strengthens our belief in the predictions (17) and (18).

IV. MONOTONIC DECREASE OF THE FORM FACTOR

Another striking feature of $F_E^p(Q^2)$ is its monotonic decrease. We would now like to show that the density form factor [Eq. (1)] for the ground state decreases monotonically with Q^2 if $|V((\vec{K}-\vec{P})^2)|$ does. Note that in this section we will consider explicitly only the two-body case with local spherically symmetric potentials though we believe that the results can be more general.

The assumption of monotonic decrease of $V(\vec{Q}^2)$ is motivated by the fact that all the positive $V(Q^2)$ discussed above actually do have this property.

Again it suffices to show the monotonicity for the ground-state wave function $\psi_0(q)$. Since if $\psi_0(q)$ is positive and monotonically decreasing, so is

$$F_0(\vec{K}^2) = \int d\vec{q}\,\psi_0(\vec{q}^{\,2})\psi_0((\vec{q}+\vec{K})^2), \tag{19}$$

where we exhibited explicitly the spherical symmetry of the ground state.

Proof: Let r_i, r_{i+1} be the (unique) radii (in momentum space) between which $\psi_0|(\vec{q})|$ decreases from $i\Delta$ to $(i-1)\Delta$ [where $n\Delta = \psi(0)$] (Fig. 2). We can approximate $\psi_0|\vec{q}|$ as $\sum_n f_i|\vec{q}|$, with $f_i(|\vec{q}|) = \Delta\Theta(-|\vec{q}|+r_i)$. [This approximation becomes exact when $n\to\infty$, $\Delta\to 0$, and $n\Delta = \psi(0)$.] The convolution integral in Eq. (19) then breaks down into the sum of n^2 $(f_i^* f_j)$ terms. Each of these convolutions is proportional to the geometrical overlap volume between two spheres of radius r_i and r_j which clearly either stays constant or is mono-

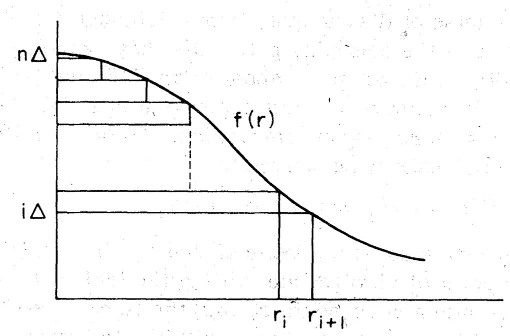

FIG. 2. A (one-dimensional) illustration of the "slicing" of $\psi(k)$ into a sum of many equal-height rectangles involved in the proof that the convolution of two monotonic functions is monotonic.

tonically decreasing when the distance ($|k|$) between the centers is increased. Hence follows the monotonic decrease of $F_0(\vec{K})$ of Eq. (19).

Now we only need to show that $\psi_0|\vec{P}|$ is monotonic. This can be done by using monotonic or "spherical rearrangement" techniques—a classical tool[15] in proofs of inequalities most recently used and improved by Brascamp, Friedberg, Luttinger, and Lieb (see Ref. 16 and references given therein).

To illustrate this method in a simpler context we prove our claim [see Sec. I prior to Eq. (4)] that if $(-\tilde{V}(r))$ is monotonic, so is $\tilde{\psi}_0(r)$, the configuration-space wave function. Consider first a one-dimensional case with $V(x)=V(-x)$ and $\psi_0(x)=\psi_0(-x)$. If the positive $\psi_0(x)$ is not monotonic along $(0,\infty)$ we can rearrange it as follows: imagine dividing the area under $\psi_0(x)$ into small rectangles and permuting those so that they are decreasing in magnitude. If the function $\psi_0(x)$ is smooth there will be a unique limit $\psi^R(x)$, the monotonically rearranged function which is monotonic with all "p norms" the same as those of $\psi(x)$,

$$\int_0^\infty [\psi_R(x)]^p dx = \int_0^\infty \psi^p(x)dx. \tag{20}$$

The ground state minimizes

$$\int V(x)\psi_0^2(x) + \int \left(\frac{d\psi}{dx}\right)^2, \text{ with } \int \psi_0^2(x) = 1. \tag{21}$$

Because of the monotonicity of V the first term is evidently smaller when $\psi_R(x)$ is substituted for $\psi_0(x)$, since more of the normalization of the wave function is shifted towards the region where the potential is more attractive. Also in the discrete version, since we put [in $\psi_R(x)$] next to ψ_i a ψ_{i+1} (or ψ_{i-1}) which are closer to it in magnitude, $\sum(\psi_i-\psi_{i+1})^2$, which in the limit is proportional to the second term in (21), is smaller for ψ_R than for ψ. Hence from the variational property of $\psi_0(x)$, $\psi_0(x)=\psi_R(x)$ and $\psi_0(x)$ is monotonous. The proof for the three-dimensional spherically symmetric case is very similar; we permute into a monotonic decreasing sequence the function values ψ_i on shells of equal volumes $\delta v = 4_\pi r_i^2(\Delta r_i)$. The normalization $\sum\psi_i^2\delta v$ is invariant, the potential term $\sum\psi_i^2 V(r_i)\delta v$ and the kinetic term $\approx\sum(\psi_i-\psi_{i+1})^2\delta v$ both decrease. The radial part of the kinetic energy is

$$\int r^2 dr\,\psi(r)\left(-\frac{1}{r^2}\frac{\partial}{dr}r^2\frac{\partial}{dr}\right)\psi(r) = \int\left(\frac{\partial\psi}{\partial r}\right)r^2\left(\frac{\partial\psi}{\partial r}\right)dr.$$

Returning now to the momentum-space case let us examine the effect of a similar spherical rearrangement on

$$\langle\psi_0|H|\psi_0\rangle = \int d\vec{p}\int d\vec{k}\psi_0(k)\psi_0(p)\,V|\vec{k}-\vec{p}| + \int d\vec{p}\,p^2\psi^2(p).$$

The second term clearly decreases. The rearrangement also decreases (i.e., increases the absolute value of) the first term.

To prove this we note that this term represents the interaction energy of a fictitious system of density $\psi^0(p)$. [Remember that $\psi^0(p)>0$] The permutation (in the process of rearrangement) of $\psi(r_i)$ and $\psi(r_j)$ with $r_i<r_j$ and $\psi(r_i)<\psi(r_j)$ is equivalent, then, to a shift of a mass element $\delta_m\approx\psi(r_i)-\psi(r_j)$ from a larger to a smaller radius ("fall towards the center").

Starting with an initial ψ we can do partial rearrangements, i.e., any number of permutations of the above type, until the potential energy is minimized and $\psi\to\psi_f(r)$. We claim that this occurs only when the new function is completely monotonic, i.e., when rearrangement has been completed.

In general, mass points far out (at the end of the distribution) will be attracted towards the origin. Let r_0 be the radius [for the $\psi_f(r)$ distribution] where this tendency reverses for the first time and a particle at $r=r_0-\epsilon$ is pulled out by the attraction of exterior shells. (The monotonicity of the potential means $V'>0$ for all r, i.e., a pure attractive central "force".) (See Fig. 3.)

$\psi_f(r)$ must then be monotonic to the right of r_0 since otherwise we could still improve $\langle V\rangle$ by appropriate rearrangement. Also, for the same reason, $\psi_f(r_0)$ must not exceed $\min_{0\le r\le r_0}\psi_f(\vec{r})$. In this case, however, the force exerted on a particle at r_0 due to concentric shells around some $\vec{r}_0(|\vec{r}_0|=r_0)$ is nonzero. There will be a net attractive force towards the origin. To show this we divide three-space by a plane at $\vec{r}_0$ perpendicular

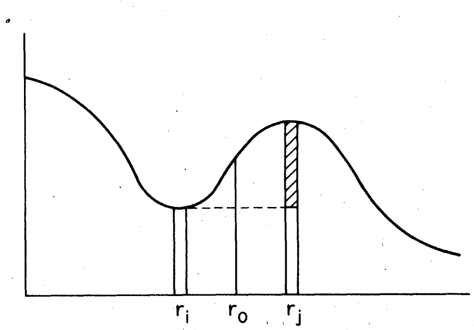

FIG. 3. A (one-dimensional) illustration of the rearrangement method which involves transfer of matter, the crosshatched region, into location closer to the origin.

to $\vec{r}_0$. There will be more "matter" on the origin side of this plane than on the other side, leading to attraction towards the origin. This, however, contradicts our assumption about reversal of sign of the net force at r_0. Hence r_0 must vanish and $\psi_f(q) \equiv \psi_0(q)$ must be always monotonic. This completes the proof of the statement made at the beginning of Sec. IV.

While this proof has not been made [like the momentum-attractive $V \to F_0(q)$ positive, proof] for the most general conceivable (nonrelativistic) case, we believe that it could be extended to at least the case of a symmetric many-particle system with spherically symmetric pairwise local potentials and, with appropriate definition of monotonicity of the potential, perhaps much further. We thus believe that the experimentally observed $F_E^p(q^2)$ monotonicity could also be traced to the general monotonic nature of the basic interconstituent interactions. We also predict then that the π form factor will be monotonic in the spacelike region.

ACKNOWLEDGMENT

This work began when both authors were at Princeton University. Helpful conversations with R. Amado, W. Greenberg, D. Soper, and J. Sucher are gratefully acknowledged. The work of one of us (S.N.) was supported in part by the U. S. National Science Foundation and the United States-Israel Binational Science Foundation.

---

*Present address: University of Virginia, Charlottesville, Virginia.

†On leave of absence from Tel Aviv University, Ramat Aviv Tel Aviv.

[1]S. D. Drell and F. Zachariasen, *Electromagnetic Structure of Nucleons* (Oxford Univ. Press, London, 1961).

[2]An early attempt along this line is by S. D. Drell, A. C. Finn, and Michael H. Goldhaber, Phys. Rev. 157, 1402 (1967).

[3]Stanley Brodsky and Glennys Farrar, Phys. Rev. Lett. 31, 1153 (1973); Phys. Rev. D 11, 1309 (1975).

[4]V. Matveev, R. Muradyan, and A. Tavkhelidze, Lett. Nuovo Cimento 7, 719 (1973).

[5]The issue of the separation of the center-of-mass motion has been avoided here. This does not affect our conclusions for equal mass (e.g., π) and for completely symmetric three-quark systems such as the nucleon. The effect of mass asymmetry in the K^+-K^0 system has been discussed in detail by O. W. Greenberg, J. Sucher, and S. Nussinov, Phys. Lett. (to be published).

[6]O. W. Greenberg, Phys. Rev. Lett. 13, 598 (1964); O. W. Greenberg and M. Resnikoff, Phys. Rev. 163, 1866 (1967).

[7]See, e.g., the review of J. Bernstein and M. L. Goldberger, Rev. Mod. Phys. 30, 465 (1958).

[8]See, e.g., J. D. Bjorken and E. A. Paschos, Phys. Rev. 185, 1975 (1969).

[9]If the spin splitting is attributed to axial-vector–meson exchanges then nonrelativistically one tends to have $m_\pi > m_\rho$. But this reverses for confining potentials.

[10]See in particular B. J. Harrington, S. Y. Park, and A. Yildiz, Phys. Rev. Lett. 34, 168 (1975); E. Eichten *et al.*, Phys. Rev. Lett. 34, 369 (1975), and the forthcoming review article.

[11]See, e.g., G. Preparata, lectures given at the Ettore Majorana XII School of Subnuclear Physics, Erice, 1974 (unpublished). For a review on attempts to correlate Gaussian distributions with observed form factors for nucleons see Y. S. Kim and M. E. Noz, Phys. Rev. D 8, 3521 (1973).

[12]I. M. Gel'fand and G. E. Shilov, *Generalized Functions* (Academic, New York, 1964), Vol. I.

[13]See, e.g., J. Kogut and L. Susskind, Phys. Rev. 12, 2821 (1975).

[14]This Bargmann potential is in fact exactly soluble for S-wave states. We are indebted to Dr. Sucher for pointing out the usefulness of this potential.

[15]For an early reference on this subject see G. H. Hardy, J. E. L. Littlewood, and G. Polya, *Inequalities* (Cambridge Univ. Press, Cambridge, England, 1964), Chap. X.

[16]H. J. Brascamp, E. H. Lieb, and J. M. Luttinger, J. Funct. Anal. 17, 227 (1974).