Impact-parameter representation for the forward and backward scattering of N-N with spin

B. S. Bhakar*
Theoretical Division, Los Alamos Scientific Laboratory, Los Alamos, New Mexico 87545
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In this paper, an impact-parameter representation for nucleon-nucleon scattering is developed in detail. Spins, isotopic spins, and the Pauli principle are taken into account. This representation is valid at all energies and all angles. A discussion of the unitarity requirement is also given for high energies only. A method for the dynamical calculation of the profile functions from the singularities of the invariant amplitudes present in the cross channels is discussed.

I. INTRODUCTION

In the past few years, various phenomenological investigations in the realm of medium and high energies have revealed that a representation for the scattering amplitude, known variously as the eikonal, the Glauber, the impact-parameter, or the Fourier-Bessel representation provides an effective *modus operandi* to explore particles and nuclear scatterings.[1-4] The eikonal or the Glauber representation is usually obtained by approximating the wave function of the nonrelativistic Schrödinger equation,[5,6] or by approximating the Green's function[6,7] in the integral form of the Schrödinger equation. The main advantage of this representation lies in satisfying the requirements of asymptotic unitarity, which naturally give rise to considerably better agreements between the calculated and the experimental results than is possible by performing purely Born-approximation calculations.[4]

However, most calculations have been performed so far by completely or partially ignoring the spin considerations. The reason appears to be that an inclusion of spins causes an increase in the number of independent amplitudes along with complicated couplings among them. But any study of physical processes, e.g., meson-nucleon (π-N) and nucleon-nucleon (N-N) scatterings, etc., requires a complete account of the spin dependence. Further, recent experiments[8] with polarized beams and polarized targets have revealed that spin effects are quite sizable (~30% around 1 GeV) and the polarization parameters have interesting structures. Thus one has to deal seriously with the spin complications at least in the fundamental processes (e.g., π-N and N-N) at medium and high energies.

First Glauber[6] and later Franco[9] considered the spin effects in the form of spin-orbit and spin-spin interactions. But their analysis is restricted because they assumed a local potential and neglected general spin-space couplings. A general spin dependence within the framework of the eikonal or the Glauber formalism was considered in the N-N problem first by McCauley and Brown[10] and later by Dadic and Martinis[11] and by Geicke[12] and in the π-N problem by Arnold.[13] Although these works are quite general, they are valid for relativistic energies but only in the forward direction. Further, in these works, the exchange symmetries, e.g., the Pauli principle in the case of N-N scattering, have not been incorporated.

In this paper, I discuss a Fourier-Bessel representation or an impact-parameter representation for nucleon-nucleon scattering. Although, the analysis presented here is restricted to the N-N system because of its inherent importance in particle and nuclear physics, it could easily be extended to other systems involving different spins. The representation discussed below embodies complete spin dependence, exchange forces, and effects of the Pauli principle. Further, it is valid for all angles and energies contingent upon the form of profile functions chosen and reduces to the Glauber form as given in Refs. 10–12 for small angles and high energies.

To make the procedure transparent, a spinless case is discussed in the next section. Most of the expressions presented there are quite well known. For this reason, I have omitted details of the mathematical steps and instead quote references where the details can be found. In Sec. III, the attention is focused on the problem of N-N scattering where full considerations of spin, I-spin, and the Pauli principle are discussed. The requirements of the unitarity are examined in Sec. IV. These requirements are quite complicated and the discussion is therefore restricted to high energies where the expressions are simple. Finally, I conclude with some comments.

II. SPINLESS CASE

To illustrate the methodology, consider two spinless nucleons interacting via a potential as follows:

$$V(r)=V_d(r)+V_{\rm ex}(r)P_r\,. \tag{1}$$

The operator P_r permutes the spatial coordinates of the two nucleons separated by a distance r. The interaction (1) has been chosen for two reasons: First, the treatment followed in this case suggests a possibility for generalizing semiclassical approaches in the presence of nonlocal potentials, and it also yields the Glauber type representation for both forward and backward directions. Second, the scattering by the interaction (1) satisfies the Mandelstam representation, which is also assumed to be satisfied by the relativistic scattering.

In the semiclassical approaches (the WKB, the Glauber, or the Green's-function eikonalization) the scattering amplitude is calculated by defining potentials $V^{\pm}=V_d\pm V_{\rm ex}$. The corresponding even and odd parts of the resulting amplitudes with respect to c.m. angles θ and $\pi-\theta$ are[14,15]

$$f^{\pm}(k,\theta)\approx\frac{-ik}{2}\int_0^\infty b\,db[J_0(b\sqrt{-t})\pm J_0(b\sqrt{-u})]\Gamma^{\pm}(s,b). \tag{2}$$

Here, $J_m(x)$ denotes the Bessel's function of the first kind. The profile functions $\Gamma^{\pm}(s,b)$ are given by

$$\Gamma^{\pm}(s,b)=\exp[\chi^{\pm}(s,b)]-1, \tag{3a}$$

$$\chi^{\pm}=\chi_d\pm\chi_{\rm ex}, \tag{3b}$$

and

$$\chi_{d,\rm ex}=\frac{-1}{k}\int_{-\infty}^\infty dz'V_{d,\rm ex}[(b^2+z'^2)^{1/2}]. \tag{3c}$$

The sum of the even and odd parts then gives the following scattering amplitude:

$$f(k,\theta)\approx-ik\int_0^\infty b\,dbJ_0(b\sqrt{-t})(e^{i\chi_d}\cos\chi_{\rm ex}-1)+k\int_0^\infty b\,dbJ_0(b\sqrt{-u})e^{i\chi_d}\sin\chi_{\rm ex}. \tag{4}$$

The two integrals appearing in (4) dominate in the forward and backward directions, respectively. Further, expression (4) is valid only at high energies and within a small angular range around the incident beam. This could be an undesirable feature if the interaction (1) is unknown, which is invariably the situation. Consequently, the profile functions are calculated from a knowledge of amplitudes in the physical region employing an inverse Fourier-Bessel transform. This procedure presumes not only the validity of expression (4) over the whole energy and angular ranges, but also the knowledge of the scattering amplitudes for unphysical values of momentum transfers. It is therefore appropriate to start by assuming the Mandelstam representation, the hypothesis of maximum analyticity, and crossing symmetry, which at least define an amplitude for all values of momentum transfers and energies.

Thus, following Blankenbecler and Goldberger,[16] one writes

$$f(s,t,u)=\int_{t_0}^\infty\frac{dt'}{\pi}\frac{{}_tD(s,t')}{t'-t}+\int_{u_0}^\infty\frac{du'}{\pi}\frac{{}_uD(s,u')}{u'-u}\,, \tag{5}$$

where ${}_tD$ and ${}_uD$ are discontinuities across the cuts in the t and u channels, respectively. The amplitude (5) could then be written as

$$f(s,t,u)=-ik\int_0^\infty b\,dbJ_0(b\sqrt{-t})\Gamma_t(s,b)-ik\int_0^\infty b\,dbJ_0(b\sqrt{-u})\Gamma_u(s,b) \tag{6}$$

and

$$\Gamma_{t,u}(s,b)=\int_{x_0}^\infty\frac{dx}{\pi}K_0(bx^{1/2})\,{}_{u,t}D(s,x). \tag{7}$$

The expressions (6) and (7) are defined for all angular and energy ranges. Also, the unitarity constraints take the simpler forms

$$\mathrm{Im}\Gamma_t=\Gamma_t^*\Gamma_t+\Gamma_u^*\Gamma_u\,, \tag{8a}$$

and

$$\mathrm{Im}\Gamma_u=\Gamma_t^*\Gamma_u+\Gamma_u^*\Gamma_t\,, \tag{8b}$$

only at asymptotic energies for these amplitudes. Further, defining signatured profile functions $\Gamma^{\pm}=\Gamma_t\pm\Gamma_u$, the expressions (8) reduce to the uncoupled form

$$\mathrm{Im}\Gamma^{\pm}=|\Gamma^{\pm}|^2. \tag{9}$$

The constraints (9) could now be satisfied trivially by writing

$$\Gamma^{\pm}=\exp(i\chi^{\pm})-1\,, \tag{10}$$

or alternatively writing $\Gamma^{\pm}$ in the form analogous to the K matrix.[16] If Eq. (10) is employed, expression (6) becomes the same as (4); but it is now valid for all angles at high energies. Also, one now has the possibility of performing dynamical calculations making use of various input informations in the following Born terms:

$$\chi^{\pm}(s,b)=\int_{x_0}^\infty\frac{dx}{\pi}K_0(bx^{1/2})[{}_tD(s,x)\pm{}_uD(s,x)]. \tag{11}$$

But the Mandelstam representation defines the amplitude only up to an unspecified number of arbitrary subtraction constants in each of the variables s, t, and u. Assuming the asymptotic behavior of the D functions as polynomial, Henzi[16] derived an impact-parameter representation which does not have the same simple form as that of (6), besides containing a number of arbitrary subtraction constants depending on the degree of the polynomial assumed. However, this asymptotic behavior is related to the nature of bound states and resonance poles which are contained in the theory, i.e., governed by the dynamics of the problem. The impact-parameter description should be a kinematic description in some sense like the partial-wave expansion, and therefore should not involve detailed assumptions on the analyticity and the polynomial boundedness. Therefore in an alternative procedure[17,18] one begins by assuming a partial wave-expansion of the scattering amplitude:

$$f(k,z)=(2ik)^{-1}\sum_l (2l+1)f_l(s)P_l(z)\,, \tag{12}$$

where $f_l(s)=\exp(2i\delta_l)-1$. Rewritting the sum in (12) for even and odd partial waves separately, and using (A5) given in the Appendix for $-1<z<1$, one obtains

$$f(k,z)=-\tfrac{1}{2}ik\int_0^\infty b\,db\{[J_0(b\sqrt{-t})+J_0(b\sqrt{-u})]\Gamma^+(s,b) + [J_0(b\sqrt{-t})+J_0(b\sqrt{-u})]\Gamma^-(s,b)\}\,. \tag{13}$$

Expression (13) has the same form as expressions (2) and (4). But now the profile functions are

$$\Gamma^\pm(s,b)=(kb)^{-1}\Sigma_{l\pm}(2l_\pm+1)f_{l\pm}(s)J_{2l_\pm+1}(2kb), \tag{14}$$

where $l\pm$ denotes the even- or odd-integer values of l. The expressions (13) and (14) are valid for all angles and energies. At high energies one could replace $J_{2l+1}(2kb)\sim\delta(2kb-2l-1)$, which reduces expression (14) to expression (3a) and simultaneously provides an insight into the notion of a peripheral interaction due to an enhancement of partial waves around $2kb\sim 2l+1$. However, the expressions (13) and (14) are not *unique*. This nonuniqueness appears because the summation and the integration have been interchanged in deriving expression (13) from (12), without defining the summation in (12) for unphysical values of angle angles.[17,19] Recently, Islam[19] made expression (13) unique by specifying the summation in (12) outside the physical range of z by means of the Sommerfeld-Watson transformation. The same procedure could be carried out here for the signatured amplitudes. Now the profile functions are given by

$$\Gamma^\pm=\Gamma_1^\pm+\Gamma_2^\pm\,, \tag{15}$$

$$\Gamma_1^\pm=(kb)^{-1}\Sigma_{l\pm}(2l_\pm+1)f_{l\pm}(s)J_{2l_\pm+1}(2kb), \tag{16}$$

and

$$\Gamma_2^\pm=\tfrac{1}{2}i\oint_C dl\,f^\pm(l,s)(2l+1) \times\frac{1}{\sin\pi l\cos\pi l}\left[\Delta(l,\beta)-\frac{J_{2l+1}(\beta)}{\beta}\right]. \tag{17}$$

Here $f^\pm(l,s)=f_{l\pm}(s)$ for $l=l_\pm$, $\beta=2kb$, and the contour C encloses the real axis in the complex l plane in a clockwise sense. The distribution Δ is given by

$$\Delta(l,\beta)=\lim_{\epsilon\to 0}\beta^{2l+1}J_{-2l-1}(\beta) \times\left[\frac{\Gamma(-l)}{2\pi i}\int_\infty^{0+}\frac{d\lambda}{\lambda^2}(-\lambda)^{-l}e^{-\beta^2/(\lambda^2+\epsilon)}\right]. \tag{18}$$

Finally, I wish to emphasize the close similarity between the above discussion and the Reggeization procedure. In both cases, one must use the signatured amplitudes in the presence of exchange forces to avoid the trouble coming from the $(-)^l$ factor. Further, when the angular momentum is made continuous, the replacement $P_l(z)\approx\rangle \to\frac{1}{2}[J_0(b\sqrt{-t})\pm J_0(b\sqrt{-u})]$ takes care of the mixing between even and odd l values. This point has been exploited to incorporate the Pauli principle in the following analysis.

III. INCLUSION OF SPIN

When the scattering particles have spins, the discussion of Sec. II must be generalized. For this purpose, I shall consider the nucleon-nucleon scattering as a concrete example. Further, an imposition of the Pauli principle gives rise to an exchange symmetry between t and u variables in the representation, and consequently in both forward and backward scatterings. Later, a similar representation is obtained by starting from the Mandelstam representation, which again has both forward and backward scatterings because of the exchange forces, even when the particles are not identical. I begin with a discussion of a spin- and isotopic-spin-dependent central interaction between the two nucleons, eventually leading to the most general case. Finally, the impact-parameter representation for the helicity amplitude is obtained.

A. Central interaction with spin and *I*-spin

Consider two nucleons (e.g., 1 and 2) interacting via a central potential

$$V(\vec{r}) = U_{33}(r)P_\sigma^+P_\tau^+ + U_{31}(r)P_\sigma^+P_\tau^- + U_{13}(r)P_\sigma^-P_\tau^+ + U_{11}(r)P_\sigma^-P_\tau^- , \quad (19)$$

where $r = |\vec{r}_1 - \vec{r}_2|$. Here, $P^\pm_{\sigma,\tau}$ are spin, I-spin projection operators for the states characterized by eigenvalues $(2S+1, 2I+1)$. These projection operators are given by $P^\pm_{\sigma,\tau} = \frac{1}{2}[1 \pm (12)_{\sigma,\tau}]$, where the permutation operator (12) exchanges the coordinates of the nucleons 1 and 2. As the wave function for the two nucleons should be totally antisymmetric, it implies that the $V(\vec{r})$ could be partitioned into two parts:

$$V_e = U_{31}P_\sigma^+P_\tau^- + U_{13}P_\sigma^-P_\tau^+ , \quad (20a)$$

$$V_0 = U_{33}P_\sigma^+P_\tau^+ + U_{11}P_\sigma^-P_\tau^- , \quad (20b)$$

which give scatterings in the even and odd partial waves, respectively. The projection operators $P^\pm$ satisfy $[P^+, P^-] = 0$, therefore the states characterized by different values of $(2S+1, 2I+1)$ are orthogonal. One then calculates partial-wave amplitudes using the interactions given by (20) and follows similar steps as employed in deriving Eqs. (12)–(15). The resulting signatured amplitudes are

$$f^\pm(k,z) = -ik\int_0^\infty b\,db\,\tfrac{1}{2}[J_0(b\sqrt{-t}) \pm J_0(b\sqrt{-u})]\Gamma^\pm(s,b), \quad (21)$$

where

$$\Gamma^+(s,b) = \Gamma_{31}(s,b)P_\sigma^+P_\tau^- + \Gamma_{13}(s,b)P_\sigma^-P_\tau^+ , \quad (22a)$$

and

$$\Gamma^-(s,b) = \Gamma_{33}(s,b)P_\sigma^+P_\tau^+ + \Gamma_{11}(s,b)P_\sigma^-P_\tau^- . \quad (22b)$$

The profile functions for the states characterized by $i = (2S+1)$, and $j = (2I+1)$ are given by

$$\Gamma_{ij} = 2\beta^{-1}\Sigma_{l_{ij}}(2l_{ij}+1)f(l_{ij},s)J_{2l_{ij}+1}(\beta) + \frac{i}{2}\oint_C dl\, f_{ij}(l,s)\,\frac{2l+1}{\sin\pi l\cos\pi l}\,[\Delta(l,\beta) - \beta^{-1}J_{2l+1}(\beta)] \quad (23)$$

and consequently the expression (21) is valid for all angles and energies. The functions $f(l_{ij}, s)$ are defined as partial-wave amplitudes for integer values of l_{ij}, and $f_{ij}(l,s) = f(l_{ij},s)$ for $l = l_{ij}$. Note that there are no couplings between the various partial-wave amplitudes as J, L, S, and I are all good quantum numbers.

B. Allowing spin-space coupling

In addition to the above example, the N-N interaction also involves coupling between spin and space variables. Consequently, one does not have the orbital angular momentum as a conserved quantity. This fact could be reflected in the above analysis by adding extra terms in (19). But to keep the analysis model independent as far as possible and also to have the possibility of a relativistic generalization, I shall start with the N-N amplitude written for a particular isospin state as a matrix in the spin space[20]:

$$f^I(k,z) = B^I P_\sigma^- + \{C^I(\vec{\sigma}^1+\vec{\sigma}^2)\cdot\hat{n} + N^I(\vec{\sigma}^1\cdot\hat{n})(\vec{\sigma}^2\cdot\hat{n}) + \tfrac{1}{2}G^I[(\vec{\sigma}^1\cdot\hat{\Delta})(\vec{\sigma}^2\cdot\hat{\Delta}) + (\vec{\sigma}^1\cdot\hat{p})(\vec{\sigma}^2\cdot\hat{p})] + \tfrac{1}{2}H^I[(\vec{\sigma}^1\cdot\hat{\Delta})(\vec{\sigma}^2\cdot\hat{\Delta}) - (\vec{\sigma}^1\cdot\hat{p})(\vec{\sigma}^2\cdot\hat{p})]\}P_\sigma^+ . \quad (24)$$

The unit vectors $(2k\cos\theta/2)\hat{P} = \vec{k}_i + \vec{k}_f$, $(2k\sin\theta/2)\hat{\Delta} = \vec{k}_f - \vec{k}_i$, and $(k^2\sin\theta)\hat{n} = \vec{k}_f \times \mathrm{k}_i$ form a Cartesian coordinate system, and $\vec{k}_i$ and $\vec{k}_f$ are, respectively, the initial and final c.m. momenta. The invariance with respect to space rotation, reflection, and time reversal implies that the coefficients B, C, N, G, and H are invariant functions of k and z. These coefficients (known as Wolfenstein amplitudes) have also a simple behavior under the $\theta \to \pi - \theta$ transformation, viz.,

$$\begin{aligned} B^I(\pi-\theta) &= (-1)^{I+1}B^I(\theta), \\ C^I(\pi-\theta) &= (-1)^{I+1}C^I(\theta), \\ N^I(\pi-\theta) &= (-1)^{I}N^I(\theta), \\ H^I(\pi-\theta) &= (-1)^{I+1}H^I(\theta), \\ G^I(\pi-\theta) &= (-1)^{I}G^I(\theta). \end{aligned} \quad (25)$$

Now, starting from the partial-wave expansion[20] of (24), one follows exactly the analogous procedure as that of Islam.[19] The validity of his procedure for the present case is proved in the Appendix. The straightforward application of it yields the Fourier-Bessel representation for (24) given by

$$\begin{aligned} B^I(k,z) &= I_B^{0,I}(s,z), \\ C^I(k,z) &= I_C^{1,I}(s,z), \\ N^I(k,z) &= I_N^{0,I+1}(s,z) + I_N^{2,I+1}(s,z), \\ G^I(k,z) &= I_G^{0,I+1}(s,z) + I_G^{2,I+1}(s,z), \\ \sin\theta H^I(k,z) &= I_H^{1,I}(s,z); \end{aligned} \quad (26)$$

$$I_A^{m,I} = \frac{-ik}{2}\int_0^\infty b\,db\left[(-)^m\frac{J_m(b\sqrt{-t})}{(\sqrt{-u})^m} - (-)^I\frac{J_m(b\sqrt{-u})}{(\sqrt{-t})^m}\right]\Gamma_A^{m,I}(s,b); \quad (27)$$

$$\Gamma_A^{m,I}(s,b) = \Gamma_{A,1}^{m,I}(s,b) + \Gamma_{A,2}^{m,I}(s,b); \quad (28)$$

$$\Gamma_{A,1}^{m,I}(s,b) = i(2^2k)^{m+1}\sum_j\frac{\Gamma(1+j+m)}{\Gamma(1+j-m)} \times f_{A,j}^{m,I}(s)\frac{J_{2j+1}(\beta)}{\beta^{m+1}}\,, \quad (29)$$

and

$$\Gamma^{m,I}_{A,2}(s,b)=\frac{(-i)^m 2^{2m}k^{m+1}}{\pi}\times\oint d\nu f^{m,I}_{A}(\nu,\beta)\frac{\Gamma(1+m+\nu)\Gamma(m-\nu)}{\cos\pi\nu}\times\beta^{-M}\mathfrak{D}(\nu,\beta);\qquad(30)$$

$$\mathfrak{D}(\nu,\beta)=\Delta(\nu,\beta)-\beta^{-1}J_{2\nu+1}(\beta).\qquad(31)$$

Here $f^{m,I}_{A,j}(s)$, used for compactness, denotes a combination of partial-wave amplitudes for a state characterized by total angular momentum j and isotopic-spin-I. The explicit form of $f^{m,I}_{A,j}(s)$ could be identified from Table II in Ref. 20, e.g., $f^{0,I}_{B,j}=(2ik)^{-1}(2l+1)R_l$. The representation given by expressions (26)–(31) is valid for all angles and all energies. However, it should be noted that in deriving these expressions, restrictions due to the Pauli principle are built in quite transparently as forward and backward exchange symmetries.

Finally, if the particles are not identical, it is easier to define exchange terms starting from the Mandelstam representation. For this purpose, it is appropriate to start with helicity amplitudes whose kinematical singularities are known explicitly.[21] The kinematical-singularity–free amplitudes defined from the helicity amplitude, have only dynamical singularities, and by the hypothesis of maximum analyticity they satisfy the Mandelstam representation. Therefore in the following I shall discuss the Fourier-Bessel representation for the helicity amplitudes.

C. Helicity amplitudes

As in writing down the Regge representation, one uses the helicity amplitudes defined by Goldberger *et al.*[21] (GGMW); it is also appropriate to employ them here. Therefore I use the following five independent amplitudes in the s channel for an isotopic-spin state I to describe the nucleon-nucleon scattering:

$$\begin{aligned}&\phi_1^I(k,z)\equiv\langle ++|++\rangle,\quad \phi_2^I(k,z)=\langle ++|--\rangle\\&\phi_3^I(k,z)\equiv\langle +-|+-\rangle,\quad \phi_4^I(k,z)=\langle +-|-+\rangle\\&\phi_5^I(k,z)=\langle ++|+-\rangle.\end{aligned}\qquad(32)$$

These amplitudes are normalized such that

$$\frac{d\sigma^I}{dr}=|\phi_1^I|^2+|\phi_2^I|^2+|\phi_3^I|^2+|\phi_4^I|^2+4|\phi_5^I|^2,$$

and

$$\sigma^I_{\text{tot}}=\frac{4\pi}{k}\,\text{Im}(\phi_1^I+\phi_3^I)_{t=0}.$$

Further, the requirements of the Pauli principles are satisfied by demanding that

$$\begin{aligned}&\phi_1^I(\pi-\theta)=(-1)^{I+1}\phi_1^I(\theta),\quad \phi_2^I(\pi-\theta)=(-1)^{I+1}\phi_2^I(\theta)\\&\phi_3^I(\pi-\theta)=(-1)^{I}\phi_4^I(\theta),\quad \phi_5^I(\pi-\theta)=(-1)^{I}\phi_5^I(\theta).\end{aligned}\qquad(33)$$

Further, following Wang,[21] the kinematical-singularity–free amplitudes are defined by

$$E\phi_1^I,\quad E\phi_2^I,\quad E\phi_3^I/(1+z),\quad E\phi_4^I/(1-z),$$

and $\qquad(34)$

$$m\phi_5^I/(1-z^2)^{1/2}.$$

By the assumption of maximal analyticity, these amplitudes satisfy the Mandelslam representation:

$$\tilde{\Phi}_i^I(k,z)=\int_{t_0}^{\infty}\frac{dt'}{\pi}\frac{t^{\tilde{D}_i^I(s,t')}}{t'-t}+\int_{u_0}^{\infty}\frac{du'}{\pi}\frac{u^{\tilde{D}_i^I(s,u')}}{u'-u}.\qquad(35)$$

Since the parity and the total spin are good quantum numbers, the following linear combinations[22] of the amplitudes (34) respecting them could be defined along with their partial-wave representations as follows:

$$\begin{aligned}&E(\phi_1^I-\phi_2^I)=\sum_j(2j+1)[h_0^+(j_+,s)\rho_j^+\delta_{I,1}+h_0^-(j_-,s)\rho_j^-\delta_{I,0}]d^j_{00}(\theta),\\&E(\phi_1^I+\phi_2^I)=\sum_j(2j+1)[h_{11}^+(j_+,s)\rho_j^+\delta_{I,1}+h_{11}^-(j_-,s)\rho_j^-\delta_{I,0}]d^j_{00}(\theta),\\&E\phi_3^I=\tfrac{1}{2}\sum_j(2j+1)\{[h_{22}^+(j_+,s)\delta_{I,1}+h_1^+(j_+,s)\delta_{I,0}]\rho_j^++[h_1^-(j_-,s)\delta_{I,1}+h_{22}^-(j_-,s)\delta_{I,0}]\rho_j^-\}d^j_{11}(\theta),\\&E\phi_4^I=\tfrac{1}{2}\sum_j(2j+1)\{[h_{22}^+(j_+,s)\delta_{I,1}-h_1^+(j_+,s)\delta_{I,0}]\rho_j^++[h_{22}^-(j_-,s)\delta_{I,0}-h_1^-(j_-,s)\delta_{I,1}]\rho_j^-\}d^j_{-11}(\theta),\end{aligned}\qquad(36)$$

and

$$m\phi_5^I=\tfrac{1}{2}\sum_j(2j+1)[h_{12}^-(j_-,s)\delta_{I,1}\rho_j^-+h_{12}^+(j_+,s)\delta_{I-0}\rho_j^+]d^j_{10}(\theta),$$

where $\delta_{I,1}$, $\delta_{I,0}$ are Kronecker δ, and $\rho_j^{\pm}=\frac{1}{2}[1\pm(-)^j]$ are the projection operators for even and odd integer values of j. Here $j_\pm$ denotes even and odd values of j. Further, the rotation functions $d^j_{\lambda\mu}(\theta)$ used here are the same as the ones in GGMW.

To derive the impact parameter starting from the partial-wave expansion, one could follow the procedure as discussed in the later part of the Appendix, which is similar to the one used in Sec. II. Thus one obtains the following impact-parameter representation for the helicity amplitudes:

$$E[\phi_1^I(k,z)-\phi_2^I(k,z)]=-ik\int_0^\infty b\,db[J_0(b\sqrt{-t})+(-1)^{I+1}J_0(b\sqrt{-u})]\Gamma_0^I,$$

$$E[\phi_1^I(k,z)+\phi_2^I(k,z)]=-ik\int_0^\infty b\,db[J_0(b\sqrt{-t})+(-1)^{I+1}J_0(b\sqrt{-u})]\Gamma_{11}^I,$$

$$E\phi_3^I(k,z)=\frac{-ik}{2}\int_0^\infty b\,db\{[\cos^2\theta/2\,J_0(b\sqrt{-t})\delta_{\lambda,0}+(-1)^M J_2(b\sqrt{-u})\delta_{\lambda,2}]\,[{}_\lambda\Gamma_{22}^I\delta_{M,I}+{}_\lambda\Gamma_1^I\delta_{M,I+1}]\}, \tag{37}$$

$$E\phi_4^I(k,z)=\frac{-ik}{2}\int_0^\infty b\,db\{[J_2(b\sqrt{-t})\delta_{\lambda,2}+(-1)^M\sin^2\theta/2\,J_0(b\sqrt{-u})\delta_{\lambda,0}]\,[{}_\lambda\Gamma_{22}^I\delta_{M,I}-{}_\lambda\Gamma_1^I\delta_{M,I+1}]\},$$

$$m\phi_5^I(k,z)=\frac{-ik}{2}\int_0^\infty b\,db\{[\cos\theta/2\,J_1(b\sqrt{-t})-(-1)^I\sin\theta/2\,J_1(b\sqrt{-u})]\Gamma_{12}^I\}.$$

The corresponding profile functions are given by

$$\Gamma_N^I=\Gamma_{1,N}^I+\Gamma_{2,N}^I,\quad \text{for } N=0,11,12 \tag{38a}$$

and

$${}_\lambda\Gamma_N^I={}_\lambda\Gamma_{1,N}^I+{}_\lambda\Gamma_{2,N}^I,\quad \text{for } N=1,22 \tag{38b}$$

where

$$\Gamma_{i,N}^I=\Gamma_{i,N}^+\delta_{I,1}+\Gamma_{i,N}^-\delta_{I,0},\quad \text{for } N=0,11 \tag{39a}$$

$$\Gamma_{i,N}^I=\Gamma_{i,N}^-\delta_{I,1}+\Gamma_{i,N}^+\delta_{I,0},\quad \text{for } N=12 \tag{39b}$$

and

$${}_\lambda\Gamma_{i,N}^I={}_\lambda\Gamma_{i,N}^+(\delta_{N,22}\delta_{I,1}+\delta_{N,1}\delta_{I,0})+{}_\lambda\Gamma_{i,N}^-(\delta_{N,22}\delta_{I,0}+\delta_{N,1}\delta_{I,1}). \tag{39d}$$

The explicit expressions of the profile functions in terms of the partial wave are obtained as discussed in the Appendix. These expressions are given as follows: For $|Z|<1$, we obtain

$$\Gamma_{1,N}^\pm(k,b)=2ik\sum_j(2j+1)h_N^\pm(j,s)\frac{J_{2j+1}(\beta)}{\beta},\quad \text{for } N=0,11 \tag{40a}$$

$$\Gamma_{1,N}^\pm(k,b)=-2ik\sum_j(2j+1)[j(j+1)]^{-1/2}h_N^\pm(j,s)J_{2j+1}(\beta),\quad \text{for } N=12 \tag{40b}$$

and

$${}_\lambda\Gamma_{1,N}^\pm(k,b)=\frac{ik}{2}\sum_j(2j+1)h_N^\pm(j,s)\left(\frac{\beta}{[j(j+1)]^{1/2}}\delta_{\lambda,0}+\frac{4}{\beta}\delta_{\lambda,2}\right)J_{2j+1}(\beta),\quad \text{for } N=22,1; \tag{40c}$$

for $|z|>1$, we obtain

$$\Gamma_{2,N}^\pm(k,b)=\frac{k}{2}\oint d\nu\,\frac{(2\nu+1)h_N^\pm(\nu,s)\mathfrak{D}(\nu,\beta)}{\sin\pi\nu\cos\pi\nu},\quad \text{for } N=0,11 \tag{41a}$$

$$\Gamma_{2,N}^\pm(k,b)=\frac{-k\beta}{2}\oint\frac{(2\nu+1)}{[\nu(\nu+1)]^{1/2}}\,\frac{h_N^\pm(\nu,s)\mathfrak{D}(\nu,\beta)}{\sin\pi\nu\cos\pi(\nu-1)},\quad \text{for } N=12 \tag{41b}$$

$${}_\lambda\Gamma_{2,N}^\pm(k,b)=\frac{1}{8k}\oint d\nu\,\frac{(2\nu+1)h_N^\pm(\nu,s)\mathfrak{D}(\nu,\beta)}{\sin\pi(\nu-1)\cos\pi(\nu-1)}\left[\frac{\beta^2}{\nu(\nu+1)}\delta_{\lambda,0}+4\delta_{\lambda,2}\right] \tag{41c}$$

In expressions (40) and (41), I have used signature partial-wave amplitudes defined, respectively, as

$$\begin{aligned} h_N^{\pm}(j,s) &= h_N(j,s) \text{ for even and odd values of } j, \\ h_N^{\pm}(j,s) &= 0 \text{ for odd and even values of } j. \end{aligned} \tag{42}$$

Note that the kinematical singularities appear explicitly in some of the terms of (37). In others it is in the Bessel functions and could be made explicit by using polynomial expansions for them. Thus the amplitudes satisfy $\phi_3 \to 0$ at 180°, $\phi_4 \to 0$ at 0°, and $\phi_5 \to 0$ at 0° and 180°. This behavior is not quite transparent in (27), e.g., C^I should vanish at 0° and 180° which is not obvious by merely observing (28), but could be checked using (A5) for x and $y \to 1$. This explicit kinematical behavior makes representation (37) quite useful for phenomenological analysis.

Before leaving the present section, I would like to discuss a couple of important points. First, I am not presenting the proof of the uniqueness of the impact-parameter representation derived for the various cases. The proof of the uniqueness can be trivially carried out with the help of the expressions given in the Appendix in exactly the same manner as given in Ref. 19. Second, the above expressions are exact, therefore one could develop various approximation schemes to calculate the profile functions depending on the choices of angular and energy regions. These are discussed in the concluding section.

Finally, the forms of the profile functions as given by (40),(41) could further be chosen so that the unitarity is automatically satisfied. In general, the unitarity requirements yields very complicated expressions which reduces to simpler forms for the asymptotic energies. Therefore, I shall restrict myself to the discussion of it for the helicity amplitudes at high energies.

IV. UNITARITY

In order to satisfy the unitarity requirements, the following two observations are quite helpful. First, the signatured amplitudes satisfy the unitarity independently as could be noted from the discussion of the spinless case in Sec. II and also emphasized by Froissart.[23] Second, the total spin S, the total angular momentum J, and parity are good quantum numbers in the present case. That is, the states with different spin and parity do not mix. This fact allows one to satisfy the unitarity separately for the singlet $S=0$ and triplet $S=1$ states. Further, since the states with $J=L$ and $J=L\pm 1$ for $S=1$ have opposite parities, the corresponding amplitudes again satisfy the unitarity separately. Thus the appropriate amplitudes for this purpose are the ones given in (36),[22] where the first one is a spin-singlet amplitude, the third corresponds to the triplet amplitude for $J=L$ state, and remaining amplitudes form a 2×2 matrix for $J=L\pm 1, S=1$. The corresponding impact-parameter S matrix is written following Blankenbecler and Goldberger[16] as

$$S^I(k,b) - 1 = 2i\Gamma^I(s,b) \tag{43}$$

at high energies, where the profile function Γ^I is a matrix:

$$\Gamma^I(s,b) = \begin{bmatrix} \Gamma_0^I & 0 & 0 & 0 \\ 0 & \Gamma_1^I & 0 & 0 \\ 0 & 0 & \Gamma_{11}^I & \Gamma_{12}^I \\ 0 & 0 & \Gamma_{12}^I & \Gamma_{22}^I \end{bmatrix}$$

$$S=0,\ J=L$$

$$S=1,\ \begin{cases} J=L\,, \\ J=L+1, \\ J=L-1. \end{cases} \tag{44}$$

To satisfy the unitarity automatically, the impact-parameter S matrix could be defined as

$$S^I(k,b) = \exp[i\chi^I(k,b)]. \tag{45}$$

But the structure of Γ^I suggests that χ should also have the similar form

$$\chi(k,b) = \begin{bmatrix} \chi_0(k,b) & 0 & 0 & 0 \\ 0 & \chi_1(kb) & 0 & 0 \\ 0 & 0 & \chi_{11}(k,b) & \chi_{12}(k,b) \\ 0 & 0 & \chi_{12}(k,b) & \chi_{22}(k,b). \end{bmatrix} \tag{46}$$

Using the usual trick of define the mixing angle to diagonalize the phase matrix, one writes

$$\begin{pmatrix} \chi_{11}^I & \chi_{12}^I \\ \chi_{12}^I & \chi_{22}^I \end{pmatrix} \to U \begin{pmatrix} \chi_{11}^I & 0 \\ 0 & \chi_{22}^I \end{pmatrix} U^+, \tag{47}$$

where the unitary matrix U is defined as

$$U = \begin{pmatrix} \cos\epsilon^I(k,b) & -\sin\epsilon^I(k,b) \\ \sin\epsilon^I(k,b) & \cos\epsilon^I(k,b) \end{pmatrix}, \tag{48}$$

with $\epsilon(k,b)$ as the mixing angle. Then by expanding the exponential in (45) and using (43) one obtains

$$2i\Gamma_0^I = \exp[i\chi_0^I(k,b)] - 1,$$
$$2i\Gamma_1^I = \exp[i\chi_1^I(k,b)] - 1,$$
$$2i\Gamma_{11}^I = \cos^2\epsilon^I(k,b)\exp[i\chi_{11}^I(k,b)] + \sin^2\epsilon^I(k,b)\exp[i\chi_{22}^I(k,b)] - 1, \quad (49)$$
$$2i\Gamma_{22}^I = \cos^2\epsilon^I(k,b)\exp[i\chi_{22}^I(k,b)] + \sin^2\epsilon^I(k,b)\exp[i\chi_{11}^I(k,b)] - 1,$$

and

$$2i\Gamma_{12}^I = \cos\epsilon^I(k,b)\sin\epsilon^I(k,b)[\exp(i\chi_{11}^I) - \exp(i\chi_{22}^I)].$$

Note that these expressions become the same as those of Geicke's[12] for forward amplitudes by defining as follows:

$$\cos^2\chi_{11} + \sin^2\chi_{22} \equiv \epsilon_0 + \epsilon_3,$$
$$\sin\epsilon\cos\epsilon(\chi_{11} - \chi_{22}) \equiv \epsilon_1,$$

and

$$\sin^2\epsilon\chi_{11} + \cos^2\epsilon\chi_{22} \equiv \epsilon_0 - \epsilon_3,$$

where ϵ_i forms a four-vector whose ϵ_2 component is zero. Further, the inelasticity could easily be taken into account by merely making the χ's complex or by making use of the overlap functions defined by $S^+S = 1 - 2F$. Finally, to perform dynamical calculations, one may use[11]

$$\chi = \tfrac{1}{2}B - \tfrac{1}{4}(i)\ln(1-2F) + \cdots, \quad (50)$$

where B is a Born amplitude, and F represents an inelastic amplitude in the S channel. In this model, the imaginary part of the phase function is completely determined by F because B is usually a real function.

V. CONCLUDING REMARKS

In this paper, I have extended the impact-parameter representation by explicitly taking into account spins and the Pauli principle. The details of the procedure have been outlined here using the nucleon-nucleon problem. But it is not difficult to apply the technique to other problems involving different spins. The nonconservation of the orbital angular momentum and the consequential mixing of partial waves present no difficulties. This representation has the potential of synthesizing the scattering phenomenona at low, medium, and high energies. For example, the nucleon-nucleon scattering using polarized beams and polarized targets could be analyzed at the phenomenological level by parameterizing the profile functions, or at dynamical level using one-boson exchange models or Regge poles and cuts exchange models.

Since the expressions obtained in Sec. III are exact, one could develop an approximation procedure depending on the energy and angular region of interest. Thus for high energies and small angular range near $\theta \sim 0°$ and 180°, one could substitute as follows:

$$J_{2j+1}(\beta) \sim \delta(\beta - 2j - 1), \quad (51a)$$
$$\mathfrak{D}(j,\beta) \sim 0. \quad (51b)$$

The substitution of (51) into expression (23) yields the profile functions for the states characterized by $i = 2S+1$ and $j = 2I+1$:

$$\Gamma_{ij} = e^{i\chi_{ij}} - 1, \quad (52a)$$

where

$$\chi_{ij}(s,b) = \frac{-1}{k}\int_{-\infty}^{\infty} dz' U_{ij}[(b^2 + z'^2)^{1/2}]. \quad (52b)$$

The expressions (52) are the same as one would obtain following Glauber's procedure discussed in Ref. 14. Similarly, substituting (51) into (29) and (30), one obtains

$$\Gamma_f^{m,I} \approx i2^{m+1}k^{2m}b^{m-1}f_{f,j}^{m,I}\big|_{kb=j+1/2}. \quad (53)$$

Substituting (53) into (27) gives the eikonel representation of (24), which is similar to the ones obtained in Refs. 8 and 10 starting from a spin-spin and a spin-orbit interaction. But in the present case, the effects of the Pauli principle are present even after making the approximation (53) besides the spin-space coupling, i.e., an additional exchange symmetry $t \rightleftarrows u$ is present. It would not be out of place to mention an alternative procedure of Ref. 11, where Eq. (24) is written in the form

$$f^I(k,z) = \frac{-ik}{2\pi}\int d^2\vec{b}\, e^{i\vec{\Delta}\cdot\vec{b}}[e^{i\chi(s,b)} - 1]. \quad (54)$$

Then expression (24) for the amplitude $f^I(k,z)$ suggests that the eikonal phase function should satisfy the same invariance requirements as imposed on the amplitude. Thus one writes[11]

$$\chi^I(s,b) = \chi_B^I P_\sigma^- + \{\chi_c^I(\vec{\sigma}^1 + \vec{\sigma}^2)\cdot\hat{b}\times\hat{P} + \chi_N^I(\vec{\sigma}^1\cdot\hat{b}\times\hat{P})(\vec{\sigma}^2\cdot\hat{b}\times\hat{P}) + \tfrac{1}{2}\chi_G^I[(\vec{\sigma}^1\cdot\hat{b})(\vec{\sigma}^2\cdot\hat{b}) + (\vec{\sigma}^2\cdot\hat{P})] + \tfrac{1}{2}\chi_H^I[(\vec{\sigma}^1\cdot\hat{b})(\vec{\sigma}^2\cdot\hat{b}) - (\vec{\sigma}^1\cdot\hat{P})(\vec{\sigma}^2\cdot\hat{P})]\}P_\sigma^+. \quad (55)$$

Now, substituting expression (55) into (54) and expanding the exponential in the power series, which is nontrivial because of the presence of noncommuting operators but straightforward, one obtains equivalent expressions to those obtained from (53), (26), and (27). However, note that (55) has been written exploiting the fact that momentum transfer $\vec{\Delta}$ and impact parameter $\vec{b}$ are canonically conjugate variables and $\hat{P}$ is orthogonal to them. This makes it difficult to incorporate the effects of the Pauli principle which requires that the $\Delta \rightleftarrows \vec{P}$ exchange symmetry be built into (54) and (55) from the very beginning.

Finally, in the case of the helicity representation, one can perform dynamical calculations of the profile functions as defined in expressions (40) and (41), by taking the Froissart-Gribov projection of expression (35) which yields the signatured partial-wave amplitudes (GGMW notation):

$$T^{J\pm}_{\lambda\lambda'}(s) = \frac{1}{\pi}\int_{Z_0}^{\infty} dZ_s \left\{ (-)^{\lambda-\lambda'}{}_tD_{\lambda\lambda'}(s,t)e^{j}_{\lambda\lambda'}(\theta_s)\left(\frac{1-Z_s}{2}\right)^{\lambda-\lambda'/2}\left(\frac{1+Z_s}{2}\right)^{\lambda+\lambda'/2} \pm(\cos\pi\lambda-\sin\pi\lambda)(-)^{\lambda+\lambda'}{}_uD_{\lambda\lambda'}(s,u)e^{j}_{\lambda-\lambda'}(\theta_s)\left(\frac{1+Z_s}{2}\right)^{\lambda+\lambda'/2}\left(\frac{1-Z_s}{2}\right)^{\lambda-\lambda'/2}\right\} \tag{56}$$

Alternatively, one could pursue a phenomenological approach for writing the profile functions, assuming Regge-pole dominance from the expressions which are an analog of expressions (4.5) and (4.7) of Ref. 11. An application of this formulation, the nucleon-nucleon charge-exchange scattering has been investigated using a one-boson exchange model in the medium-energy region. This study will be submitted for publication soon.

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APPENDIX

The procedure of Ref. 19 is extended below for problems involving spins. The scattering amplitudes for these problems are expressed in the helicity representation involving rotation matrices $d^{J}_{\lambda'\lambda}(\theta)$ or in the representation involving associated Legendre polynomials $P^m_l(z)$. The later representation involving associated Legendre polynomials $P^m_l(z)$. The later representation will be considered first, where azimuthal angle dependence will be ignored without loss of generality. Then, a physical amplitude may be written as

$$M(k,z) = \sum_j f_j(s)P^M_j(z)\,. \tag{A1}$$

Here, j is the total angular momentum, $f_j(s)$ is the partial-wave amplitude, and $z=\cos\theta$, $s=$ square of the c.m. energy. Expression (A1) defines the amplitude for $-1\le z\le 1$. To derive a unique Fourier-Bessel representation for $M(k,z)$, one needs to specify the amplitude $M(k,z)$ for $\cos\theta > |1|$. Following analogies with the Regge theory, and using the Watson-Sommerfeld transform, one writes

$$M(k,z) = (-)^M \frac{i}{2}\oint_C dj\, \frac{P^M_j(-z)}{\sin\pi j} f(j,s)\,, \tag{A2}$$

which could be analytically continued to specify the amplitude for the unphysical angles. The contour C goes around the real axis clockwise, enclosing all the poles at integral values of j ($j=0, 1, 2\ldots$). The amplitude $f(j,s)$ represents the analytically continued partial-wave amplitude coinciding with the physical amplitude $f_j(s)$ for the integral values of j. For the physical scattering region (s is always considered above threshold) $-1\le(z=1-2y^2)\le 1$, the contour C is collapsed on the real axis and (A2) reduces to

$$M(k,z) = \frac{i(-)^M}{2}\int_{j_0}^{\infty} dj f(j,s)P^M_j(-z) \times\left(\frac{1}{\sin\pi j_+}-\frac{1}{\sin\pi j_-}\right), \tag{A3}$$

where $j_\pm = j\pm i\epsilon$ and $0>j_0>-1$. The distribution in (A3) could now be rewritten as

$$-(-)^M P^M_j(-z)\left(\frac{1}{\sin\pi j_+}-\frac{1}{\sin\pi j_-}\right) = 2\pi i\sum_{n=0}^{\infty} P^M_j(z)\delta(j-n)\,. \tag{A4}$$

Now, using the Weber-Schafheitlin discontinuous integral [Eq. (29) in Sec. 7.7.4 of HTF[24] and Eq. (2) in Sec. 3.6.1 of HTF[24]], one writes

$$\theta(1-y)P^M_j(z) = (-)^M\frac{\Gamma(j+M+1)}{\Gamma(j-M+1)}2^M\left(\frac{1+z}{2}\right)^{-M/2} \times\int_0^{\infty} d\beta\,\beta^{-M}J_M(\beta y)J_{2j+1}(\beta)\,, \tag{A5}$$

$$= (-)^j\frac{\Gamma(j+M+1)}{\Gamma(j-M+1)}2^M\left(\frac{1-z}{2}\right)^{-M/2} \times\int_0^{\infty} d\beta\,\beta^{-M}J_M(\beta x)J_{2j+1}(\beta)\,, \tag{A5$'$}$$

where $z=1-2y^2=1+2x^2$. The vanishing of (A5) for x or $y=1$ could be checked using Eq. (30) in

Sec. 7.7.4 of HTF. Further, for large j and β one obtains the MacDonald's small-angle approximation for $P_j^M(\cos\theta)$ [see Eq. (10) in Sec. 3.5 of HTF] by substituting $J_{2j+1}(\beta)\sim\delta(\beta-2j-1)$. Finally, one gets the Fourier-Bessel (FB) representation for $-1\leqslant z\leqslant 1$ from (A3), (A4), and (A5):

$$M(k,z)=(-ik)\left(\frac{1+z}{2}\right)^{-M/2}\times\int_0^\infty d\beta\,\beta J_M(\beta\sin(\tfrac{1}{2}\theta))\,\Gamma_1(s,\beta)\,, \qquad (A6)$$

and

$$\Gamma_1(s,\beta)=\frac{i(-2)^M}{k}\sum_j\frac{\Gamma(j+M+1)}{\Gamma(j-M+1)}f_j(s)\frac{J_{2j+1}(\beta)}{\beta^{M+1}}\,. \qquad (A7)$$

To specify the amplitude in the unphysical region $\cos\theta>|1|$, one writes the FB transform of $P_l^m(2y^2-1)\equiv P_l^m(z)$ for $y>1$ with the help of expressions (3), (36), and (37) from Sec. (3.3.1) and Sec. 3.2, respectively, of HTF and the Weber-Schafheitlin integral:

$$P_j^M(z)=\frac{(-)^M}{\pi}\frac{1}{\cos\pi j}[\sin\pi(j+M)Q_j^M(z)-\sin\pi(j-M)Q_{-j-1}^M(z)]\,, \qquad (A8)$$

$$P_j^M(2y^2-1)=\frac{2^{M-1}}{\cos\pi j}\left(\frac{z-1}{2}\right)^{-M/2}\int_0^\infty\beta d\beta\,\beta^{-M}J_M(\beta y)\;\frac{\Gamma(M-j)\Gamma(j+M+1)}{\Gamma(M+j)\Gamma(1-j+M)}\left(\frac{J_{2j+1}^{(\beta)}}{\beta}-\frac{J_{-2j-1}^{(\beta)}}{\beta}\right). \qquad (A8')$$

Expression (A8), however, cannot be extended to $\mathrm{Re}\,j>0$, because the singular behavior of $J_{-2j-1}(\beta)$ at the origin ($\sim\beta^{-2j-1}$) prevents the existence of the integral. Following Islam,[19] one could again smooth out the singular behavior so that the FB transform is defined and the asymptotic behavior of P_j^M is reproduced. This is achieved by replacing $J_{-2j-1}(\beta)/\beta$ in (A8) by

$$\Delta(j,\beta^2)=\lim_{\epsilon\to 0}\beta^{2j+1}J_{-2j-1}(\beta)\left[\frac{\Gamma(-j)}{2\pi i}\int_\infty^{0+}\frac{d\lambda}{\lambda^2}(-\lambda)^{-j}e^{-\beta^2/(\lambda+\epsilon)}\right]. \qquad (A9)$$

The limit $\epsilon\to 0$ is to be taken after the FB integration has been carried out. To see that this replacement in (A8′) gives back the left-hand side, one has to evaluate

$$I(y)=\frac{\Gamma(M-j)\Gamma(j+M+1)}{\Gamma(j-M)\Gamma(1-j+M)}\frac{2^{M-1}}{\cos\pi j}\left(\frac{z-1}{2}\right)^{-M/2}\int_0^\infty d\beta\,\beta^{1-M}J_M(\beta y)\Delta(j,\beta^2) \qquad (A10)$$

for $y>1$ and also for $y<1$. First consider $y>1$, then using Eq. (2) in Sec. 7.2.1 of HTF, and denoting the integral in (A10) as $I_1(y)$, one has

$$I_1(y)=\lim_{\epsilon\to 0}\frac{2^{2j+1}}{y^{2-M}}\sum_{n=0}^\infty\frac{(-)^n(2y)^{-2n}}{n!\,\Gamma(n-2j)}\int_0^\infty d\beta\,\beta^{1-M}J_M(\beta y)\left[\frac{\Gamma(-j)}{2\pi i}\int_\infty^{0+}\frac{d\lambda}{\lambda^2}(-\lambda)^{-j}\right]\beta^{2n}e^{-\beta^2/(\lambda y^2+\epsilon)}$$

$$=\frac{2^{2j+1}}{y^{2-M}}\sum_{n=0}^\infty\frac{(-)^n(2y)^{-2n}}{\Gamma(n-2j)}\left[\frac{\Gamma(-j)}{2\pi i}\int_\infty^{0+}\frac{d\lambda}{\lambda^2}(-\lambda)^{-j}\right]\frac{(\lambda y)^{n+1}}{2^{1+M}\Gamma(1+M)}e^{-\lambda y^2/4}{}_1F_1\left(M-n,1+M,\frac{\lambda y^2}{4}\right)$$

[using Eq. (22) in Sec. 7.7.3 of HTF]

$$=\frac{y^{2j+M}}{2^M\Gamma(1+j)}\sum_{n=0}^\infty\frac{(-)^ny^{-2n}}{\Gamma(n-2j)\Gamma(M-n)}\sum_{r=0}^\infty\frac{\Gamma(M-n+r)(n+r-j)}{r!\,\Gamma(M+r+1)}$$

[using Eq. (1) in Sec. 6.1 and Eq. (1) in Sec. 1.6 of HTF]

$$=\frac{y^{2j+M}}{2^M\Gamma(1+j)\Gamma(1+M)}\sum_{n=0}^\infty\frac{(-)^n\Gamma(n-j)}{\Gamma(n-2j)}y^{-2n}{}_2F_1(m-n,n-j,M+1;1)$$

[using Eq. (1) in Sec. 2.8 of HFT]

$$=\frac{y^{2j+M}}{2^M\Gamma(M+j+1)}\sum_{n=0}^\infty\frac{\Gamma(n-j)\Gamma(1-M-j)}{n!\,\Gamma(n-j)\Gamma(-M-j)}y^{-2n}$$

[using Eq. (46) in Sec. 2.8 of HFT]

$$=\frac{y^{2j+M}}{2^M\Gamma(j+M+1)}\frac{\Gamma(-j)}{\Gamma(-2j)}{}_2F_1(-j,-M-j;-2j;1/y^2)$$

$$=\frac{(-)^M2^{1-M}}{\Gamma(j+M+1)\Gamma(M-j)}\left(\frac{z-1}{2}\right)^{M/2}Q_{-j-1}^M(z)$$

[using Eq. (36) in Sec. 3.2 of HTF].

The above expression when substituted in (A10) gives

$$I(y)=\frac{(-)^M}{\cos\pi j}Q^M_{-j-1}(z)/[\Gamma(j-M)\Gamma(1-j+M)], \tag{A11}$$

as required by the second term in (A8).

Further, for $y<1$, the expression for $I_1(y)$ becomes

$$I_1(y)=\lim_{\epsilon\to 0}\left(\frac{y}{2}\right)^M\sum_{n=0}^{\infty}\frac{(-)^n}{n!\,\Gamma(M+n+1)}\left(\frac{y}{2}\right)^n\left[\frac{\Gamma(-j)}{2\pi i}\int_{\infty}^{0^+}\frac{d\lambda}{\lambda^2}(-\lambda)^{-j}\int_0^{\infty}d\beta\,\beta^{2n+2j+2}J^{(\beta)}_{-2j-1}e^{-\beta^2/(\lambda+\epsilon)}\right.$$

[Using Eq. (2) in Sec. 7.2.1 of HTF]

$$=\left(\frac{y}{2}\right)^M\sum_{n=0}^{\infty}\frac{2^{2j}(-1)^n}{\Gamma(-2j)\Gamma(M+n+1)}\left(\frac{y}{2}\right)^n\frac{\Gamma(-j)}{2\pi i}\int_{\infty}^{0^+}\frac{d\lambda}{\lambda^2}(-\lambda)^{-j}\Big]\lambda^{n+1}e^{-\lambda/4}{}_1F_1(-2j-n-1,-2j,\lambda/4)$$

[Using Eq. (22) in Sec. 7.7.3 of HTF]

$$=\left(\frac{y}{2}\right)^M\sum_{n=0}^{\infty}\frac{(-)^n y^{2n}}{\Gamma(j+1)\Gamma(M+n+1)}\,\frac{\Gamma(n-2j)}{\Gamma(-2j)}\,{}_2F_1(-2j-n-1;n-j;-2j;1)$$

[Using Eq. (1) in Sec. 6.1 and Eq. (1) in Sec. 1.6 of HTF]

$$=\left(\frac{y}{2}\right)^M\frac{1}{\Gamma(1+M)}\,{}_2F_1(-j,j+1;1+M;y^2)$$

[Using Eq. (1) and Eq. (46) in Sec. 2.8 of HFT].

Therefore the expression (A10) for $y<1$ becomes

$$I(y)=\frac{(-)^{1+M/2}}{2}P^M_j(z). \tag{A12}$$

But the first term in (A8) gives the same result, i.e. $\frac{1}{2}[(-)^{M/2+1}]P^M_j(z)$ as can be seen by using expressions (A5). Thus the expression

$$P^M_j(2y^2-1)=\frac{2^{M-1}}{\cos\pi j}\,\frac{\Gamma(M-j)\Gamma(j+M+1)}{\Gamma(j-M)\Gamma(M-j+1)}\left(\frac{z-1}{2}\right)^{-M/2}\int_0^{\infty}d\beta\,\beta^{1-M}J_M(\beta y)\left[\frac{J_{2j+1}(\beta)}{\beta}-\Delta(j,\beta^2)\right] \tag{A13}$$

vanishes for $y<1$, but gives $P^M_j(-z)$ for $y>1$. Similarly, one could write the analog of (A5) for $x>1$.

Finally, the substitution of (A5) and (A13) in expression (A2) gives a unique FB representation for the amplitude:

$$M(k,z)=-ik\left(\frac{1+z}{2}\right)^{-M/2}\int_0^{\infty}\beta\,d\beta\,J_M(\beta y)\Gamma(s,\beta), \tag{A14}$$

$$\Gamma(s,b)=\Gamma_1(s,b)+\Gamma_2(s,b), \tag{A15}$$

where $\Gamma_1(s,b)$ is given by (A7) and

$$\Gamma_2(s,b)=\frac{+i}{k}(-)^{M/2}2^{M-2}\oint dj\,\frac{f(j,s)}{\sin\pi j\cos\pi j}\,\frac{\Gamma(M-j)\Gamma(M+j+1)}{\Gamma(j-M)\Gamma(M-j+1)}\left[\frac{J_{2j+1}(\beta)}{\beta^{M+1}}-\frac{\Delta(j,\beta^2)}{\beta^M}\right]. \tag{A16}$$

A similar expression could also be written to reflect forward and backward symmetries.

Now, in the case of the helicity representation, an amplitude could be written as follows:

$$M_{\lambda',\lambda}(k,z)=\sum_{j=\lambda_{Mx}}^{\infty}A_j(s)d^j_{\lambda'\lambda}(z), \tag{A17}$$

where $\lambda_{Mx}=\mathrm{Max}(|\lambda'|,|\lambda|)$, and $\lambda_{MN}=\mathrm{Min}(|\lambda'|,|\lambda|)$. As shown by Charap and Squire,[25] Eq. (A17) allows the Sommerfeld-Watson transformation, viz.,

$$M_{\lambda'\lambda}(k,z)=\frac{(-)i}{2}\oint_c dj\,\frac{A(j,s)d^j_{\lambda'\lambda}(-z)}{\sin\pi(j-\lambda_{Mx})}. \tag{A18}$$

Again for the physical values of z, one has, following Ref. 18,

$$d^j_{\lambda'\lambda}(z) = (-)^{\lambda'-\lambda}\left[\frac{(j-\lambda')!\,(j-\lambda)!}{(j+\lambda')!\,(j+\lambda)!}\right]^{1/2} 2^{-(\lambda'+\lambda)}\left(\frac{1+z}{2}\right)^{\lambda'+\lambda/2}\int_0^\infty d\beta\,\beta^{\lambda'+\lambda}J_{\lambda'-\lambda}(\beta y)J_{2j+1}(\beta) \tag{A19a}$$

$$= (-)^{j+\lambda}\left[\frac{(j-\lambda')!\,(j+\lambda)!}{(j+\lambda')!\,(j-\lambda)!}\right]^{1/2} 2^{-(\lambda'-\lambda)}\left(\frac{1-z}{2}\right)^{\lambda'-\lambda/2}\int_0^\infty d\beta\,\beta^{\lambda'-\lambda}J_{\lambda'+\lambda}(\beta x)J_{2j+1}(\beta)\,. \tag{A19b}$$

These expressions have been derived assuming $\mathrm{Re}(\lambda'+\lambda)<1$ for (A19a) and $\mathrm{Re}(\lambda'-\lambda)<1$ for (A19b), but these restrictions could be relaxed as will be shown later. Substituting (A19a) into (A18) and collapsing the contour C to real axis one obtains

$$M_{\lambda'\lambda}(k,z) = -ik\left(\frac{1+z}{2}\right)^{|\lambda'+\lambda|/2}\int_0^\infty d\beta\,\beta J_{\lambda'-\lambda}(\beta y)\Gamma_1(s,\beta)\,, \tag{A20}$$

and

$$\Gamma_1(s,\beta) = \frac{i(-)^{\lambda'-\lambda}}{k2^{\lambda'+\lambda}}\sum_{j=\lambda_{Mx}}^{\infty}\left[\frac{(j-\lambda')!\,(j-\lambda)!}{(j+\lambda')!\,(j+\lambda)!}\right]^{1/2} A_j(s)\beta^{\lambda'+\lambda-1}J_{2j+1}(\beta)\,. \tag{A21}$$

Although expressions (A20) and (A21) are derived under the assumption that $\mathrm{Re}(\lambda'+\lambda)<1$, the restriction could now be relaxed because for large β and j; the approximation $J_{2j+1}(\beta)\sim\delta(\beta-2j-1)$, yields

$$\Gamma_1 \sim \frac{i(-)^{\lambda'-\lambda}}{\beta}A(\beta,s)2^{\lambda'+\lambda}\,, \tag{A22}$$

which maintains the convergence of (A20) for all values of $\lambda+\lambda'$. Note that Eqs. (A19) have the correct kinematical singularities.

For z, in the unphysical region, one replaces[26]

$$d^j_{\lambda'\lambda}(z) = \frac{\sin\pi(j-\lambda_{Mx})}{\pi}\left[\frac{e^j_{\lambda'\lambda}(z)}{\cos\pi(j-\lambda_{Mx})} - \frac{e^{-j-1}_{\lambda'\lambda}(z)}{\cos\pi(j-\lambda_{Mx})}\right], \tag{A23}$$

which is an analog of the expression (A8). The functions $e^j_{\lambda'\lambda}$ are related to $d^j_{\lambda'\lambda}(z)$ in the similar way as the functions $Q_l(z)$ are related to $P_l(z)$. Again following the procedure similar to one discussed in the case of P^M_j, one arrives at the form for $z>|1|$:

$$d^j_{\lambda'\lambda}(z) = (-)^{\lambda'-\lambda}2^{\lambda'+\lambda-1}\left[\frac{(j+\lambda')!\,(j+\lambda)!}{(j-\lambda')!\,(j-\lambda)!}\right]^{1/2}\left(\frac{1+z}{2}\right)^{-(\lambda'+\lambda)/2}\frac{\sin\pi(j-\lambda_{Mx})}{\sin\pi(j-\lambda_{MN})\cos\pi(j-\lambda_{Mx})}$$

$$\times\int_0^\infty d\beta\,\beta^{1-(\lambda'+\lambda)}J_{\lambda'-\lambda}(\beta y)\left[\frac{J_{2j+1}(\beta)}{\beta} - \Delta(j,\beta^2)\right]. \tag{A24}$$

Note that in the derivation of (A24) we have not assumed that $\mathrm{Re}(\lambda'+\lambda)<1$, as in derivation of (A19). If the same restriction is imposed, we get the matching expressions. Now, one follows a similar procedure as employed in deriving Eqs. (A14), (A15) and (A16), and thus obtains an analogous unique representation.

Most of the results in the case of helicity amplitudes could also be derived using operators I^μ_z and K^μ_z of Riemann-Lionville and Weyl fractional integration, respectively. These are defined by equations (see Secs. 13.1 and 13.2 of TIT[27])

$$I^\mu_x[f(x)] = \frac{1}{\Gamma(\mu)}Pf\int_1^x f(y)\,(x-y)^{\mu-1}dx$$

$$\mu\neq 0,-1,-2,\ldots,$$

$$K^\mu_x[f(x)] = \frac{1}{\Gamma(\mu)}Pf\int_x^\infty f(y)\,(y-x)^{\mu-1}dx$$

$$\mu\neq 0,-1,-2,\ldots,$$

and

$$(-)^n K^{-n}_x[f(x)] = I^{-n}_x[f(x)] = \left(\frac{d}{dx}\right)^n f(x)$$

$$n = 0,1,2,\ldots\,.$$

As shown by Andrews and Gunson,[26] all the properties of $d^j_{\lambda'\lambda}$ and $e^j_{\lambda'\lambda}$ could be derived from those of $P_j(z)$ and $Q_j(z)$ by defining operators $I^{\lambda',\lambda}$ and $K^{\lambda',\lambda}$. Without going into details, all the expressions derived above, could then be obtained with the help of Eqs. (63) and (94) in Sec. 13.1, and Eqs. (34), (59), and (77) in Sec. 13.2 of TIT.[27]

*Permanent address: Dept. of Physics, University of Manitoba, Winnipeg, Canada.

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