Quantum nondemolition measurement and coherent states

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This note investigates the possibility of quantum nondemolition measurements on an oscillator in a coherent state by means of first-order interactions with a probe particle.

Work by Braginsky and co-workers[1] has aroused interest in the possibility of experimentally measuring the state of an oscillator without altering the state of the oscillator. In a recent paper[2] I showed that the interaction must be of second order in the generalized coordinate (e.g., electric field in an electromagnetic oscillator, displacement of the ends in a bar, etc.) if such a scheme is to work. That work was done assuming that the oscillator is an energy eigenstate. Moncrief[3] has suggested that the oscillator is more likely to be in what is called a coherent state, and that a first-order interaction might work in that case. This note will examine the effect of an interaction linear in the generalized coordinate on such a coherent state.

I shall use a similar model system to that proposed in my original paper with a Lagrangian action given by

$$I=\int\left[\tfrac{1}{2}(\dot{x}^2+\dot{y}^2+\dot{z}^2)+\tfrac{1}{2}(\dot{q}^2-q^2)\right.\\ \left.-\alpha V(x,y,z)q\right]dt\,.$$

Here, x,y,z are the coordinates of the particle, and q is the generalized coordinate of the oscillator. The system can be quantized in the usual way.

The initial state of the oscillator is assumed to be the coherent state

$$\phi_c(q,t)=\exp(ce^{-it}a^\dagger)\phi_0(q,t)/e^{-|c|^2/2},$$

where ϕ_0 is the ground state of the oscillator, c is a constant, and $a^\dagger$ is the creation operator

$$a^\dagger=\frac{1}{\sqrt{2}}\left(q-\frac{d}{dq}\right)$$

(I will work throughout in the Schrödinger representation).

The wave function ϕ_c obeys the equation

$$a\phi_c=(ce^{-it})\phi_c,$$

where

$$a=\frac{1}{\sqrt{2}}\left(q+\frac{d}{dq}\right)$$

and is a solution to the Schrödinger equation for the free oscillator. Another normalized solution to the free oscillator equation is

$$\phi_p=(a^\dagger e^{-it}-c^*)\phi_c.$$

This function has unit norm and is orthogonal to ϕ_c. ϕ_p is chosen in this way because

$$q\phi_c=\frac{1}{\sqrt{2}}(ce^{-it}+c^*e^{it})\phi_c+\frac{1}{\sqrt{2}}e^{it}\phi_p.$$

Let us assume that the particle is initially in some state ψ_0 (assumed to be a normalized wave packet).

Now, because of the interaction, the particle can be scattered into some orthogonal state ψ_1 either leaving the oscillator in the state ϕ_c or by altering that state. To lowest order in α, that altered state must be ϕ_p. The amplitude for a nonperturbative scatter is therefore

$$A_n=\int_{-\infty}^{\infty}\frac{dt}{\sqrt{2}}\left[(ce^{-it}+c^*e^{it})\right.\\ \times\int dx\,dy\,dz\,\psi_1^*(x,y,z,t)V(x,y,z)\\ \left.\times\psi_0(x,y,z,t)\right].$$

The first term represents an interaction in which the particle has gained energy from the oscillator, while the second represents an energy loss to the oscillator. The perturbative scattering amplitude is given by

$$A_p=\int_{-\infty}^{\infty}\frac{dt}{\sqrt{2}}e^{it}\left[\int dx\,dy\,dz\,\psi_1^*(x,y,z,t)\right.\\ \left.\times V(x,y,z)\psi_0(x,y,z,t)\right].$$

Note that A_p is equal to $1/c^*$ times the second term in A_n. If c^* is large, the probability of a perturbative scatter is much less than that of an energy-emitting nonperturbative scatter.[4] As $|c|^2$ is just the mean number of particles in the state ϕ_c, we find that in the region of interest, i.e., small values of $|c|^2$, the ratio of perturbative to nonperturbative energy-emitting scatterings is approximately equal to one.

The energy-absorbing scatterings can only be nonperturbative. If the energy of the particle is much greater than that of the oscillator, and if $V(x, y, z)$ is reasonable, for any final state ψ_1, which corresponds to an energy-emitting interaction, there will be a similar final state ψ_1' corresponding to energy absorption. In other words, the particle will have roughly equal probability of scattering nonperturbatively through an energy emission or absorption, and for small $|c|^2$, also approximately equal probability of scattering perturbatively. Using high-energy particles to attempt to measure the state of the oscillator will result in a large probability of altering the coherent state of the system (if the detection of scattered particles is inefficient, as it will be, the probability of altering the state will increase).

If the energy of the particle is less than the energy of one quantum of the oscillator, both the energy-emitting nonperturbative scattering and the perturbative scattering have zero probability. In this case the oscillator is left in its coherent state by any scattering.

However, this does not seem too helpful for any practical scheme for measuring the state. Any mechanical oscillator has a low frequency, and even the electromagnetic oscillator used would probably have frequencies less than 10^{12} Hz which would imply electron energies of less than about 10^{-4} eV, a rather difficult requirement.

Furthermore, the above analysis is true *only* if the oscillator is in a coherent state. It is not at all clear to me that this is a reasonable expectation for a real oscillator with very few quanta. The effects of thermal fluctuations will be to destroy the coherence of the state.

One comment is appropriate here regarding the situation investigated by Moncrief. He investigated the use of electrons to study a coherence state of a free electromagnetic field. For this situation momentum conservation completely suppresses the interaction to first order in the field. The interaction is thus effectively of second order, which can allow nonperturbative measurement even of energy eigenstates.

For a unidirectional plane wave or wave packet, this does not occur, again because of momentum and energy conservation. However, the Kapitza-Dirac[5] effect can be regarded as such a measurement for two oppositely traveling plane waves.

[1]V. B. Braginsky, Yu. I. Vorontsov, and V. D. Krivchenkov, Zh. Eksp. Teor. Fiz. 68, 55 (1975) [Sov. Phys.—JETP 41, 28 (1975)].

[2]W. G. Unruh, Univ. of British Columbia report, 1977 (unpublished).

[3]V. Moncrief, Yale Univ. report, 1977 (unpublished).

[4]Energy absorbing and emitting will rèfer always to the particle.

[5]P. Kapitza and P. A. M. Dirac, Proc. Camb. Philos. Soc. 29, 297 (1933).