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Dynamical symmetry breakdown in the supersymmetric nonlinear σ model

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The supersymmetric nonlinear σ model is studied to leading order in $1/N$. The spontaneous breakdown of a discrete chiral symmetry leads to mass generation, and the appearance of a supersymmetric pair of bound states.

I. INTRODUCTION

Several years ago, Gross and Neveu[1] discovered that dynamical symmetry breakdown was possible in asymptotically free field theories. The Gross-Neveu model consists of N fermions governed by the Lagrangian

$$\mathcal{L}=\bar{\psi}i\partial\!\!\!/\psi+\tfrac{1}{2}g^2(\bar{\psi}\psi)^2 .$$

This Lagrangian possesses a discrete chiral symmetry $\psi\to\gamma_5\psi$ which prevents the appearance of masses to any finite order in perturbation theory in g^2. To obtain nonperturbative results they study the behavior of the model as $N\to\infty$ with $\lambda=g^2N$ fixed. They obtain an expansion in powers of $1/N$ and nonperturbative in g^2. The model is asymptotically free and to leading order the chiral symmetry breaks down spontaneously. This leads to a massive fermion and to a $\bar{\psi}\psi$ bound state at threshold.

Polyakov[2] has discovered that the O(N) nonlinear σ model is asymptotically free and that the fundamental particle acquires a mass for $N>2$.

Recently, Di Vecchia and Ferrara,[3] and Witten have constructed a supersymmetric version of the two-dimensional O(N) nonlinear σ model. This model is a hybridization of the nonlinear σ model and the Majorana version of the Gross-Neveu model. There are several questions which materialize: Do the boson-fermion interactions destroy asymptotic freedom and mass generation? What happens to the Gross-Neveu bound state? How much are the individual characters of the Gross-Neveu and the σ model preserved?

The object of this paper is to provide a partial answer to these questions. We find that in the large-N limit asymptotic freedom and mass generation remain. There appears a fermion-boson bound state in addition to the fermion-fermion bound state. As $N\to\infty$, purely bosonic (fermionic) amplitudes are dominated by the nonlinear σ-model (Gross-Neveu) sector, but amplitudes coupling the two systems are of the same order. We expect the individual sectors to lose their character as N decreases.

This paper is organized as follows: In Sec. II, the model is reviewed. Section III derives the main results. In Sec. IV, additional topics are discussed. In the Appendix the δ function for fermions is defined.

II. REVIEW

The Lagrangian for the supersymmetric O(N) nonlinear σ model is given by[5]

$$\mathcal{L}_F=\frac{1}{2g^2}(-n^a\partial\!\!\!/^2n^a+\bar{\psi}i\partial\psi^a+F^aF^a)\,, \tag{2.1}$$

where n^a, ψ^a, and F^a transform under the vector representation of O(N). The fields n and F are real, and ψ is a Majorana spinor. The fields n, ψ, and F are not aribtrary and are required to satisfy the constraints

$$n^an^a-1=0\,, \tag{2.2a}$$

$$\psi^an^a=0\,, \tag{2.2b}$$

$$n^aF^a-\tfrac{1}{2}\bar{\psi}^a\psi^a=0\,. \tag{2.2c}$$

The action defined by (2.1) and constraints (2.2) is invariant under the supersymmetry transformations

$$\delta n^a=\bar{\epsilon}\psi^a\,, \tag{2.3a}$$

$$\delta\psi^a=-i\gamma^\mu\epsilon\partial_\mu n^a+F^a\epsilon\,, \tag{2.3b}$$

$$\delta F^a=-i\bar{\epsilon}\partial\!\!\!/\psi^a\,, \tag{2.3c}$$

where ϵ is a constant anticommuting Majorana spinor.

Since F enters the Lagrangian algebraically, it may be eliminated by using its equations of motion. After a rescaling of the fields, the new Lagrangian is given by

$$\mathcal{L}=\tfrac{1}{2}[-n^a\partial^2n^a+\bar{\psi}i\partial\!\!\!/\psi+\tfrac{1}{4}g^2(\bar{\psi}\psi)^2]\,, \tag{2.4}$$

with the constraints

$$g^2n^2-1=0\,, \tag{2.5a}$$

$$n\cdot\psi=0\,. \tag{2.5b}$$

The supersymmetry transformations are given by

$$\delta n^a = \bar{\epsilon}\psi^a , \quad (2.6a)$$

$$\delta\psi^a = -i\gamma^\mu \epsilon \partial_\mu n^a + \tfrac{1}{2} g^2 n^a (\bar\psi\psi)\epsilon . \quad (2.6b)$$

III. THE QUANTUM THEORY

The $1/N$ expansion is a nonperturbative way of studying certain field theories. The expansion is generated by keeping $\lambda = g^2 N$ fixed as $N \to \infty$. The results are nonperturbative in λ and perturbative to the order of $1/N$ desired. Lagrangian (2.4) is not the most efficient way of generating the $1/N$ expansion; it does, however, provide a systematic way of doing perturbation theory in $g^2 = \lambda/N$. In Fig. 1 we have two graphs which are of order g^4, but of different order in the $1/N$ expansion. The remedy is to replace Lagrangian (2.4) by an effective Lagrangian that takes into account the dominance of loops involving isospin traces over nontrace ones.[1,6] The perturbation expansion of the effective Lagrangian will be in powers of $1/N$.

The generating functional for Green's functions is given by

$$Z[J,\xi] = \eta \int [dn][d\psi] \delta(g^2 n^2 - 1)\delta(n\cdot\psi) \times e^{i(\mathcal{L} + J\cdot n + \bar\xi\psi)} , \quad (3.1)$$

where η is a normalization factor.[7] The normalization factor will change from line to line, but it shall always be denoted by η. The δ function for fermions is defined in the Appendix, and may be represented by

$$\delta(n\cdot\psi) = \int [d\beta]\, e^{i\bar\beta\psi^a n^a} , \quad (3.2)$$

where β is an anticommuting Majorana fermion. The generating functional Z may be written as

$$Z[J,\xi] = \eta \int [dn][d\psi][d\alpha][d\phi] \times \exp\{i[\mathcal{L} - \tfrac{1}{2} i\alpha(g^2 n^2 - 1) + i\beta\bar\psi\cdot n - \tfrac{1}{2}(\phi + \tfrac{1}{2} g\bar\psi\psi)^2 + J\cdot n + \bar\xi\psi]\} . \quad (3.3)$$

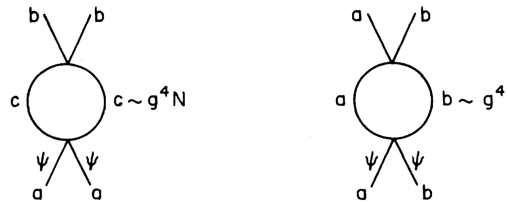

FIG. 1. Two diagrams which are of order g^4 but of different order in $1/N$.

The integral over ϕ is a constant that may be absorbed into η.[1,6] The theory defined by (3.3) is equivalent to the one defined by (3.1). Equation (3.3) may be rewritten as

$$Z[J,\xi] = \eta \int [dn][d\psi][d\alpha][d\beta][d\phi] e^{i(\mathcal{L}' + J\cdot n + \bar\xi\psi)} , \quad (3.4)$$

where

$$\mathcal{L}' = \tfrac{1}{2} n(-\partial^2 - g^2\alpha)n + \tfrac{1}{2}\bar\psi(i\partial\!\!\!/ - g\phi)\psi + \tfrac{1}{2}\alpha - \tfrac{1}{2}\phi^2 + g\bar\beta\psi^a n^a . \quad (3.5)$$

Since the α, β, and ϕ fields are not coupled to external sources, they represent internal effects in the Feynman diagram. We shall borrow the language of gauge theories and call these ghost particles. At this stage, the α and β ghost particles do not propagate, but later we shall see how to define an effective propagator.

Lagrangian (3.5) is quadratic in n and ψ; therefore it is possible to do the n and ψ functional integrations. The n and ψ integrations may be done with sources, but it is more convenient to perform all manipulations of the functional integral without sources. At the end, Lagrangian (4.5) will be used to determine the effect of the sources. Doing the ψ integration we obtain

$$Z = \eta \int [dn][d\alpha][d\beta][d\phi] \exp\{i[\tfrac{1}{2} n(-\partial^2 - g^2\alpha)n - \tfrac{1}{2}\phi^2 + \tfrac{1}{2}\alpha]\}[\det(i\partial\!\!\!/ - g\phi)]^{1/2} \exp\left(-\tfrac{1}{2} g^2 n^a \bar\beta \frac{i}{i\partial\!\!\!/ - g\phi}\beta n^a\right). \quad (3.6)$$

We only get a $(\det)^{1/2}$ since the fermions are Majorana, i.e., real. This expression is still quadratic in n; therefore, we can do the n integration:

$$Z = \eta \int [d\alpha][d\beta][d\phi] \exp\{i[-\tfrac{1}{2}\phi^2 + \tfrac{1}{2}\alpha]\}[\det(i\partial\!\!\!/ - g\phi)]^{1/2}\left[\det\left(-\partial^2 - g^2\alpha - g^2\bar\beta\frac{1}{i\partial\!\!\!/ - g\phi}\beta\right)\right]^{-1/2}$$
$$= \eta \int [d\alpha][d\beta][d\phi] e^{i\mathcal{L}_{\rm eff}} . \quad (3.7)$$

The effective Lagrangian $\mathcal{L}_{\text{eff}}$ is given by

$$\mathcal{L}_{\text{eff}}=-\tfrac{1}{2}\phi^2+\tfrac{1}{2}\alpha-\tfrac{1}{2}(iN)\,\text{Tr}\ln(i\partial\!\!\!/-g\phi)+\tfrac{1}{2}iN\,\text{Tr}\ln\left(-\partial^2-g^2\alpha-g^2\bar{\beta}\frac{i}{i\partial\!\!\!/-g\phi}\beta\right). \tag{3.8}$$

This Lagrangian is the key to the $1/N$ expansion.

Lagrangian (3.5) seemed to tell us that in addition to the n and ψ particles there were additional particles ϕ, α, β interacting via

$$\mathcal{L}'_{\text{int}}=-\tfrac{1}{2}g^2\alpha n^2-\tfrac{1}{2}g\phi\bar{\psi}\psi+g\bar{\beta}\psi^a n^a, \tag{3.9}$$

except that α and β did not propagate. The correct way of interpreting this remark is to say that the particles n and ψ couple to the internal ghosts α, β, ϕ via (3.9) and these interact among themselves via (3.8). The precise way of computing a Green's function with a total of $2m$ external n and ψ lines is as follows:

(1) Draw m straight lines and make an arbitrary number of $\mathcal{L}'_{\text{int}}$ insertions on the straight lines. This will determine whether the external lines are n or ψ particles.

(2) Connect the α, β, ϕ lines together by using $\mathcal{L}_{\text{eff}}$ only. *Do not use* $\mathcal{L}'_{\text{int}}$ to generate internal nn, $\psi\psi$, or $n\psi$ virtual pairs. These are already taken into account by $\mathcal{L}_{\text{eff}}$. Notice that the vertices in $\mathcal{L}_{\text{eff}}$ are nonlocal. For simplicity, we shall draw these as if they were local.

Thus far the functional integral has been manipulated in a cavalier manner without regard to renormalization. The supersymmetric nonlinear σ model may be regulated in a supersymmetric manner by adding higher-order derivative terms which are supersymmetric. Consider Lagrangian (2.4) with constraints (2.5). Under a supersymmetric regularization, the only possible counterterm is of the form $a\mathcal{L}$; the bare Lagrangian must be given by

$$\mathcal{L}_0=(1+a)\mathcal{L}=Z_3\mathcal{L}. \tag{3.10}$$

If the bare fields are defined by $\psi_0=Z_3^{1/2}\psi$, $n_0=Z_3^{1/2}n$, and the bare coupling by $g_0^{\,2}=g^2Z_3^{-1}$, we see that (3.10) may be rewritten as

$$\mathcal{L}_0=\tfrac{1}{2}[n_0-\partial^2 n_0+\bar{\psi}_0 i\partial\!\!\!/\psi_0+\tfrac{1}{4}g_0^{\,2}(\bar{\psi}_0\psi_0)^2], \tag{3.11}$$

with the constraints

$$g_0^{\,2}n_0^{\,2}-1=0, \tag{3.12a}$$

$$n_0\cdot\psi_0=0. \tag{3.12b}$$

The action corresponding to (3.11) is invariant under

$$\delta n_0^a=\bar{\epsilon}\psi_0^a \tag{3.13a}$$

$$\delta\psi_0^a=-i\gamma^\mu\epsilon\partial_\mu n_0^a+\tfrac{1}{2}g_0^2 n_0^a(\bar{\psi}_0\psi_0)\epsilon. \tag{3.13b}$$

Notice that (3.13) is consistent with rescaling (2.6).

To leading order it will suffice to use a momentum-space cutoff. We will discover that the renormalization to lowest order is supersymmetric and consistent with the remarks made above. The actual renormalization will be performed on the effective Lagrangian and afterwards interpreted in terms of the original Lagrangian.

Before continuing with the discussion on renormalization, it will be convenient to discuss chiral symmetry. Lagrangian (2.4) and its constraints are invariant under the discrete chiral transformation $\psi\to\gamma_5\psi$. This symmetry forces the fermion to be massless to any finite order in perturbation theory. The fermion can only become massive if the chiral symmetry is spontaneously broken. The chiral symmetry manifests itself in Lagrangian (3.5) in the form $\psi\to\gamma_5\psi$, $\phi\to-\phi$, $\bar{\beta}\to\bar{\beta}\gamma_5$. After doing the functional integration over n and ψ, we obtain the effective Lagrangian which should be invariant under $\phi\to-\phi$, $\bar{\beta}\to\bar{\beta}\gamma_5$. The term in (3.8) involving β is easily seen to be chiral invariant. The term $\text{Tr}\ln(i\partial\!\!\!/-g\phi)$ is also chiral invariant. This may be seen by expanding the logarithm and noticing that terms odd in ϕ disappear since the trace of an odd number of γ matrices is zero. The chiral symmetry also forces the β particle to be massless to all orders in perturbation theory.

Chiral symmetry plays a second important role. It restricts the class of allowed counterterms. One of the advances of renormalization theory in the 1970's was the realization that symmetry breakdown does not affect renormalization. Even though chiral symmetry will be broken at the end, the renormalization will be performed in a chirally symmetric manner.

The first step in any renormalization scheme is to isolate the primitive divergences. This may be done by noticing that there is a duality between internal and external particles. Thus far we have taken the attitude that the interaction Lagrangian (3.9) connects the external particles, n and ψ, to the internal ghost particles governed by Lagrangian (3.8). We can also take the viewpoint that an internal ghost particle can become a pair of virtual external particles. There are three primitively divergent diagrams to lowest order (Fig. 2). Figure 2(c) is identically zero because the chiral-symmetric propagator is $i/k\!\!\!/$. Chiral symmetry tells us that we cannot have a counterterm linear in ϕ, i.e., Fig. 2(c). The counterterms will be of the form $\mathcal{L}_{\text{CT}}=c\phi^2\ln\Lambda+d\alpha\ln\Lambda$,

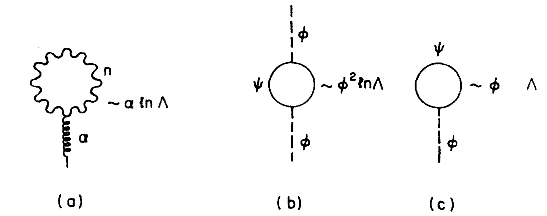

FIG. 2. Leading-order, primitively divergent diagrams, and their counterterms.

where Λ is the momentum-space cutoff. The bare effective Lagrangian is

$$\mathcal{L}_{0,\mathrm{eff}} = \mathcal{L}_{\mathrm{eff}} + \mathcal{L}_{\mathrm{CT}} . \tag{3.14}$$

To generate the $1/N$ expansion, functional integral (3.7) is expanded around the point of stationary phase. If we use angular brackets to denote the vacuum expectation of a field, then the point of stationary phase is given by

$$\left.\frac{\partial \mathcal{L}_{0,\mathrm{eff}}}{\partial \phi}\right|_{\langle\phi\rangle,\langle\alpha\rangle} = 0 , \tag{3.15a}$$

$$\left.\frac{\partial \mathcal{L}_{0,\mathrm{eff}}}{\partial \alpha}\right|_{\langle\phi\rangle,\langle\alpha\rangle} = 0 . \tag{3.15b}$$

Translational invariance demands that $\langle\phi\rangle$ and $\langle\alpha\rangle$ be constant. Differentiating the Lagrangian gives

$$\begin{aligned} 0 &= \left.\frac{\partial \mathcal{L}_{0,\mathrm{eff}}}{\partial \phi}\right|_{\langle\phi\rangle,\langle\alpha\rangle} \\ &= -\langle\phi\rangle + \frac{i}{2} gN \int \frac{d^2k}{(2\pi)^2} \mathrm{Tr} \frac{1}{\not{k} - g\langle\phi\rangle} + \left.\frac{\partial \mathcal{L}_{\mathrm{CT}}}{\partial \phi}\right|_{\langle\phi\rangle,\langle\alpha\rangle} \\ &= \left(-1 + \frac{g^2N}{4\pi} \ln \frac{\Lambda^2}{g^2\langle\phi\rangle^2}\right)\langle\phi\rangle + \left.\frac{\partial \mathcal{L}_{\mathrm{CT}}}{\partial \phi}\right|_{\langle\phi\rangle,\langle\alpha\rangle} . \end{aligned} \tag{3.16}$$

Let M be a renormalization mass. By rewriting expression (3.16) we can guess the counterterm:

$$\begin{aligned} 0 = &\left(-1 + \frac{g^2N}{4\pi} \ln \frac{M^2}{g^2\langle\phi\rangle^2}\right)\langle\phi\rangle \\ &+ \left(\frac{g^2N}{4\pi}\langle\phi\rangle \ln \frac{\Lambda^2}{M^2} + \frac{\partial \mathcal{L}_{\mathrm{CT}}}{\partial \phi}\right)_{\langle\phi\rangle,\langle\alpha\rangle} . \end{aligned} \tag{3.17}$$

The ϕ counterterm will be chosen to be

$$\mathcal{L}_{\mathrm{CT}}^{\phi} = -\left(\frac{g^2N}{4\pi} \ln \frac{\Lambda^2}{M^2}\right) \frac{\phi^2}{2} . \tag{3.18}$$

With this choice of $\mathcal{L}_{\mathrm{CT}}$, the condition for a stationary point is given by

$$0 = \left(-1 + \frac{g^2N}{4\pi} \ln \frac{M^2}{g^2\langle\phi\rangle^2}\right)\langle\phi\rangle . \tag{3.19}$$

This equation has more than one solution. To choose the correct solution, we integrate (3.16) to obtain the effective potential[7] for ϕ:

$$V_{\phi}(\phi) = \tfrac{1}{2}\phi^2 + \frac{g^2N}{8\pi}\phi^2\left(\ln \frac{g^2\phi^2}{M^2} - 1\right) . \tag{3.20}$$

The chiral-symmetric vacuum $\langle\phi\rangle = 0$ is seen to have higher energy than the asymmetric vacuums given by

$$g^2\langle\phi\rangle^2 = M^2 e^{-4\pi/g^2N} . \tag{3.21}$$

We shall break the chiral symmetry by choosing the positive solution to (3.21):

$$g\langle\phi\rangle = M e^{-2\pi/g^2N} . \tag{3.22}$$

The same result could also have been obtained by expanding $\mathrm{Tr}\ln(i\not{\partial} - g\phi)$ in powers of ϕ and following the manipulations of Coleman and Weinberg.[8] Notice that $\langle\phi\rangle$ is of order $\sqrt{N}$.

The stationary point for α is found in the same way. The counterterm is

$$\mathcal{L}_{\mathrm{CT}}^{\alpha} = \alpha \frac{g^2N}{8\pi} \ln \frac{\Lambda^2}{M^2} . \tag{3.23}$$

The condition for stationary phase is

$$\tfrac{1}{2} - \frac{g^2N}{8\pi} \ln \frac{M^2}{g^2\langle\alpha\rangle} = 0 . \tag{3.24}$$

This equation has exactly one solution,

$$g^2\langle\alpha\rangle = M^2 e^{-4\pi/g^2N} . \tag{3.25}$$

The vacuum expectation value of α is of order N.

To demonstrate the supersymmetric renormalization of the theory we write down $\mathcal{L}_{0,\mathrm{eff}}$ as follows:

$$\begin{aligned} \mathcal{L}_{0,\mathrm{eff}} = &-\tfrac{1}{2}\left(1 + \frac{g^2N}{4\pi} \ln \frac{\Lambda^2}{M^2}\right)\phi^2 \\ &+ \tfrac{1}{2}\left(1 + \frac{g^2N}{4\pi} \ln \frac{\Lambda^2}{M^2}\right)\alpha - \frac{iN}{2} \mathrm{Tr}\ln(i\not{\partial} - g\phi) \\ &+ \frac{iN}{2} \mathrm{Tr}\ln\left(-\partial^2 - g^2\alpha - g^2\bar{\beta}\frac{1}{i\not{\partial} - g\phi}\beta\right) . \end{aligned} \tag{3.26}$$

The bare quantities are defined by

$$\begin{aligned} \phi_0 &= Z_3^{1/2}\phi , \\ \alpha_0 &= Z_3\alpha , \\ \beta_0 &= Z_3^{1/2}\beta , \\ g_0^2 &= Z_3^{-1}g^2 , \end{aligned} \tag{3.27}$$

where the field renormalization constant is given by

$$Z_3 = 1 + \frac{g^2N}{4\pi} \ln \frac{\Lambda^2}{M^2} . \tag{3.28}$$

Equation (3.26) may be rewritten as

$$\mathcal{L}_{0,\,\mathrm{eff}} = -\tfrac{1}{2}\phi_0{}^2 + \tfrac{1}{2}\alpha_0 - \tfrac{1}{2}iN\,\mathrm{Tr}\ln(i\partial\!\!\!/ - g_0\phi_0) + \tfrac{1}{2}iN\,\mathrm{Tr}\ln\left(-\partial^2 - g_0{}^2\alpha_0 - g_0^2\bar\beta_0\frac{1}{i\partial\!\!\!/ - g_0\phi_0}\right)\beta_0 . \qquad (3.29)$$

This equation has the same form as the effective Lagrangian (3.8); therefore by proceeding backwards along the steps which led to (3.8) we discover that the bare Lagrangian is given by (3.11). The renormalization is supersymmetric.

The vacuum is not chiral invariant. As a consequence, the fermions β and ψ may acquire a mass. We shall expand around the asymmetric vacuum by considering fields ϕ', α' defined by

$$\phi = \langle\phi\rangle + \phi' , \quad \alpha = \langle\alpha\rangle + \alpha' . \qquad (3.30)$$

By looking at (3.5) we learn that the masses of the n and the ψ particles are given to lowest order by

$$m_n{}^2 = g^2\langle\alpha\rangle , \qquad (3.31a)$$

$$m_\psi = g\langle\phi\rangle . \qquad (3.31b)$$

Using the expressions for the vacuum expectation values we learn that the n particles and the ψ particles have the same mass. Their common mass is given by

$$m = Me^{-2\pi/g^2N} . \qquad (3.32)$$

Supersymmetry requires the masses of the n and the ψ to be equal. This may be seen by noticing that to leading order in $1/N$, transformation laws (2.6) may be written as

$$\delta n^a = \bar\epsilon\psi^a , \qquad (3.33a)$$

$$\delta\bar\psi^a = \bar\epsilon(i\partial\!\!\!/ - m)n^a . \qquad (3.33b)$$

If the vacuum is supersymmetric then

$$\epsilon\langle n\rangle_0 = -\delta\langle\psi\rangle_0/m = 0 ,$$

and

$$\delta\langle T\{n^a(x), \bar\psi^b(0)\}\rangle_0 = 0 .$$

The last relation leads to the Ward identity

$$\langle T\{\psi^a(x), \bar\psi^b(0)\}\rangle_0 = (i\partial\!\!\!/ + m)\langle T\{n^a(x), n^b(0)\}\rangle_0 . \qquad (3.34)$$

Therefore, n and ψ must have the same mass. The supersymmetry of the vacuum will be demonstrated later. Since n is massive we do not expect long-range correlations; this is consistent with $\langle n\rangle_0 = 0$.

The ordinary nonlinear σ model has mass generation, and the mass is precisely given by (3.32). The Gross-Neveu model with Majorana fermions also has mass generation with the mass given by (3.32). The constraint $n\cdot\psi = 0$ which couples the bosonic sector to the fermionic sector does not seem to play a role in affecting the masses of the fundamental particles to lowest order.

The coupling of the external particles to the ghost particles is given by

$$\mathcal{L}'_{\mathrm{int}} = -\tfrac{1}{2}g^2\alpha' n^2 - \tfrac{1}{2}g\phi'\bar\psi\psi + g\bar\beta\psi\cdot n . \qquad (3.35)$$

The propagators for the ghost lines may be found by computing the quadratic part of Lagrangian (3.26):

$$\mathcal{L}^{(2)}_{0,\,\mathrm{eff}} = -\tfrac{1}{2}\left(1 + \frac{g^2N}{4\pi}\ln\frac{\Lambda^2}{M^2}\right)\phi'^2 - \tfrac{1}{2}iNg^2(-\tfrac{1}{2})\,\mathrm{Tr}\frac{1}{i\partial\!\!\!/ - m}\phi'\frac{1}{i\partial\!\!\!/ - m}\phi' - \tfrac{1}{2}ig^2N\,\mathrm{Tr}\frac{1}{-\partial^2 - m^2}\bar\beta\frac{1}{i\partial\!\!\!/ - m}\beta - \tfrac{1}{4}ig^4N\,\mathrm{Tr}\frac{1}{-\partial^2 - m^2}\alpha'\frac{1}{-\partial^2 - m^2}\alpha' . \qquad (3.36)$$

The propagators are given by

$$D_{\phi'}(k^2) = \frac{-4\pi i}{g^2N}I(k^2) , \qquad (3.37a)$$

$$S_\beta(k) = \frac{-4\pi i}{g^2N}(k\!\!\!/ - 2m)I(k^2) , \qquad (3.37b)$$

$$D_{\alpha'}(k^2) = \frac{4\pi iN}{(g^2N)^2}(4m^2 - k^2)I(k^2) , \qquad (3.37c)$$

where

$$I(k^2) = \left(\frac{-k^2}{4m^2 - k^2}\right)^{1/2}\left[\ln\frac{(4m^2 - k^2)^{1/2} - (-k^2)^{1/2}}{(4m^2 - k^2)^{1/2} + (-k^2)^{1/2}}\right]^{-1} . \qquad (3.38)$$

The best way of checking whether the above results are consistent is to use the Ward identities.

By using the formalism of Ref. 3 it may be shown that (2.1) belongs to a supermultiplet where $n\cdot F$, $\psi\cdot F - ni\partial\!\!\!/\psi$, and $(-n\partial^2 n + \bar\psi i\partial\!\!\!/\psi + F^2)$ are, respectively, the analogs of n, ψ, and F. The classical equations of motion obtained from Lagrangian (3.5) (Ref. 9) tell us that the multiplet may be written as $-\phi$, β, $g(\alpha - \phi^2)$, where

$$\phi = -\tfrac{1}{2}g\bar\psi\psi , \qquad (3.39a)$$

$$\beta = -n\cdot i\partial\!\!\!/\psi , \qquad (3.39b)$$

$$\alpha = (\partial_\mu n)^2 . \qquad (3.39c)$$

The Green's function satisfies the classical equations of motion, and it is legitimate to use the equations of motion in deriving the Ward identities. To leading order $g(\alpha-\phi^2)$ equals $g\alpha' - 2m\phi'$. The transformation laws for this multiplet [analog of (2.3)] are given by

$$\delta\phi = -\bar{\epsilon}\beta \,, \tag{3.40a}$$

$$\delta\bar{\beta} = -\bar{\epsilon}(i\partial\!\!\!/ + 2m)\phi' + \bar{\epsilon}g\alpha' \,, \tag{3.40b}$$

$$\delta(g\alpha' - 2m\phi') = -i\bar{\epsilon}\partial\!\!\!/\beta \,. \tag{3.40c}$$

By taking vacuum expectation values of the above we see that the vacuum is supersymmetric to leading order.

Since the vacuum is supersymmetric we have

$$\delta\langle T\{\phi(x),\bar{\beta}(0)\}\rangle_0 = 0$$

and

$$\delta\langle T\{g\alpha'(x) - 2m\phi'(x),\bar{\beta}(0)\}\rangle_0 = 0 \,.$$

From these two relationships we obtain the Ward identities:

$$S_\beta(k) - (k\!\!\!/ - 2m)D_{\phi'}(k^2) = 0 \,, \tag{3.41a}$$

$$k\!\!\!/ S_\beta(k) - g^2 D_{\alpha'}(k^2) + 2m(k\!\!\!/ - 2m)D_{\phi'}(k^2) = 0 \,. \tag{3.41b}$$

These relations are satisfied by the propagators given in (3.37).

Propagators (3.37a) and (3.37b) diverge as $k^2 \to 4m^2$. This indicates the appearance of a bound state at threshold. Normally, the signal for a bound state is the appearance of a pole in the scattering amplitudes. When the bound state appears at threshold one no longer gets a pole.[10] This is the behavior in the Gross-Neveu model.[1]

In the supersymmetric nonlinear σ model there are two bound states. The ϕ' particle corresponds to a fermion-fermion bound state created by the operator $\bar{\psi}\psi$. The β particle corresponds to a fermion-boson bound state created by the operator $n^a i\partial\!\!\!/\psi^a$. To leading order, the ϕ' particle is due entirely to the Gross-Neveu sector. The β particle is a new feature of the supersymmetric nonlinear σ model. This is the first and the most pronounced example of interaction between the ordinary σ-model sector and the Gross-Neveu sector. The amplitudes for the creation of a ϕ' particle and a β particle are both of order $1/\sqrt{N}$.

The allowed particle spectrum of the theory is determined by the symmetry group that acts on the physical Hilbert space. The vacuum to leading order in $1/N$ is found to be isospin, Poincaré, and supersymmetry invariant. The allowed particle spectrum for $N>3$ (Ref. 11) will be determined by the possible finite-dimensional unitary irreducible representations (FDUIR) of the continuous symmetry group (we shall not discuss the implications of the discrete symmetries). The FDUIR's of O(N) are well known and will not be discussed. The FDUIR's of the remaining continuous symmetries are easily constructed. The generators Q_α, $\alpha = 1, 2$, of supersymmetry and the generators of translation P_μ satisfy the algebra

$$\{Q_\alpha, Q_\beta\} = 2(\gamma^\mu\gamma^0)_{\alpha\beta}P_\mu \,, \tag{3.42a}$$

$$[P_\mu, Q_\alpha] = 0 \,. \tag{3.42b}$$

Since the little group of the two-dimensional Poincaré group consists of the identity (there is no spin in two dimensions), its FDUIR's are labeled by one invariant, the mass squared, and the different states within the representation may be taken to be momentum eigenstates. By (3.42b) the momentum eigenstates are invariant under supersymmetry. When restricted to fixed-momentum eigenstates, (3.42a) defines a Clifford algebra. The only finite-dimensional representation of the Clifford algebra is four dimensional and given by 2×2 matrices. For a fixed momentum, supersymmetry demands the existence of two possible states. We expect particles to come in multiplets which are twice the dimension of the isospin representation.

The α' propagator does not exhibit bound-state behavior. This is fortunate because the two bound states that were found form a complete representation of supersymmetry. If the α' ghost particle manifested itself as a bound state then there would have to exist another bound state.

IV. MORE ABOUT THE QUANTUM THEORY

The renormalization of the quantum theory required the introduction of an arbitrary renormalization mass, M. Standard arguments tell us that the one-particle irreducible (1PI) Green's function must satisfy a renormalization-group equation. If $\Gamma^{(2k)}$ is a 1PI Green's function consisting of a total of $2k$ external n and ψ lines, then

$$\left[M\frac{\partial}{\partial M} + \beta(g)\frac{\partial}{\partial g} - k\gamma(g)\right]\Gamma^{(2k)}(p,g,M) = 0 \,, \tag{4.1}$$

where

$$\beta(g) = M\frac{\partial g}{\partial M}\bigg|_{g_0,\Lambda\ \text{fixed}} \,, \tag{4.2}$$

$$\gamma(g) = M\frac{\partial \ln Z_3}{\partial M}\bigg|_{g_0,\Lambda\ \text{fixed}} \,. \tag{4.3}$$

The last of Eqs. (3.27) says that β and γ are related by $\gamma(g) = 2\beta(g)/g$. By using (3.28) we learn that $\beta(g)$ is given to lowest order by

$$\beta(g) = -g^3N/4\pi \,. \tag{4.4}$$

Since $\beta(g)<0$, the model is asymptotically free. It is also possible to determine the β function by realizing that the mass of the ψ or n is a physical parameter, and must satisfy a renormalization-group equation of the form

$$\left[M\frac{\partial}{\partial M}+\beta(g)\frac{\partial}{\partial g}\right]m=0 \qquad (4.5)$$

Expression (3.32) for m, in conjuction with the above renormalization-group equation, yields (4.4).

The β function to lowest order is identical to that of the ordinary nonlinear σ model or to the Majorana Gross-Neveu model. This is seen by remembering that to leading order the masses of the n and ψ particles are identical to the respective masses of the regular σ model or Gross-Neveu model. This fact and the analog of Eq. (4.5) applied to the respective model yields the desired result.

The above and several other results may also be explained by noticing that to leading order, the renormalization of the supersymmetric nonlinear σ model is identical to the renormalization of a regular nonlinear σ model and a Majorana Gross-Neveu model which are not coupled together.

We began with a formally massless theory containing one coupling constant, g^2. Ultraviolet divergences required the introduction of a renormalization procedure requiring an arbitrary mass parameter. The renormalization group tells us that the physics would remain the same if we had renormalized at a different mass, but had used a different coupling constant and a rescaling of the fields. Since the value of M is arbitrary, it will be convenient to eliminate this ambiguity by choosing $g^2N=4\pi$. This means that m and M are related by $M=e^{1/2}m$. The theory now depends on one dimensional parameter m (or M). This is the phenomenon of dimensional transmutation discovered by Coleman and Weinberg.[8]

The masses of the fundamental particles and the β function to lowest order are identical to the ones obtained if the ordinary σ-model sector and the Gross-Neveu sector were not coupled via $n\cdot\psi=0$. A generalization of this statement is possible. The 2k-particle connected Green's functions (not 1PI) for n and/or ψ particles are of order N^{1-k}. The ones containing only n (ψ) particles are described to order N^{1-k} by the σ-model (Gross-Neveu) sector of the theory. The mixing effects on the pure n (ψ) sectors from the $n\cdot\psi=0$ constraint are of order N^{-k}. This does not mean that the model is "trivial" to lowest order. The statement made above also implies that a connected Green's function containing $2j$ of the n lines and $(2k-2j)$ of the ψ lines is of order N^{1-k}. We expect that as N decreases, purely n (ψ) amplitudes will resemble less their σ-model (Gross-Neveu) counterparts.

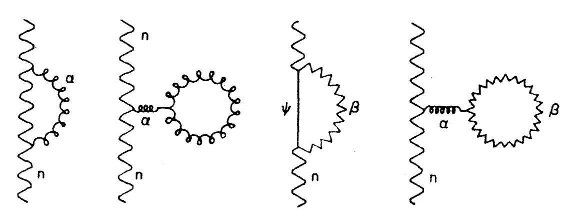

FIG. 3. Order-$1/N$ corrections to the self-energy of the n particle.

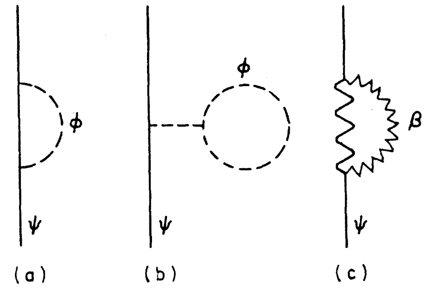

FIG. 4. Order-$1/N$ corrections to the self-energy of the ψ particle.

The most remarkable manifestation of the coupling of the two sectors of the model is the appearance of the n-ψ bound state. This bound state appears to the same order as the $\bar{\psi}\psi$ bound state which in leading order is due entirely to the Gross-Neveu sector.

The coupling of the two sectors to higher orders is best demonstrated by the $1/N$ corrections to the self-energy of the n (ψ) particle. To order N^0, we already learned that it was given by the pure σ-model (Gross-Neveu) sector. For the n (ψ) particles the relevant graphs are given in Fig. 3 (Fig. 4). Figures 3(c), 3(d), [4(c)] describe the mixing between the n and ψ sectors of the theory.

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APPENDIX

To incorporate constraint (2.2b) it is necessary to define the δ function over anticommuting numbers.[12]

Let ω be a single anticommuting real variable. The fundamental property of δ functions is that

$$\int d\omega\, \delta(\omega - \omega_0) f(\omega) = f(\omega_0)\,. \tag{A1}$$

If ω and ξ anticommute then $\int d\omega\, \omega\xi$ is ambiguous. The double valuedness of the integral in question is resolved by defining

$$\xi = \int d\omega\, \omega\xi = -\int d\omega\, \xi\omega\,. \tag{A2}$$

In (A1) it is important that $\delta(\omega - \omega_0)$ is to the left of f. Expanding δ and f in powers of ω and using (A1) and (A2) yields

$$\delta(\omega) = \omega\,. \tag{A3}$$

Having written δ to the right of f in (A1) would have led to a factor of (-1) in (A3).

For the fields n and ψ, constraint $n \cdot \psi = 0$ may be written as

$$\delta(n \cdot \psi) = \int [d\beta]\, e^{i\bar{\beta}\psi \cdot n}\,. \tag{A4}$$

As long as we are consistent, the normalization of (A4) is irrelevant.

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[1]D. J. Gross and A. Neveu, Phys. Rev. D 10, 3235 (1974).

[2]A. M. Polyakov, Phys. Lett. 59B, 79 (1975).

[3]P. Di Vecchia and S. Ferrara, ICTP Report No. IC/77/63 (unpublished).

[4]Edward Witten, Phys. Rev. D 16, 2991 (1977).

[5]We shall use the notation of Ref. 4.

[6]S. Coleman, R. Jackiw, and H. D. Politzer, Phys. Rev. D 10, 2491 (1974).

[7]For a review of functional methods: S. Coleman, in *Laws of Hadronic Matter*, proceedings of the 1973 International Summer School "Ettore Majorana," Erice, Italy, edited by A. Zichichi (Academic, New York, 1975). Integral signs in the argument of the exponential will be omitted.

[8]S. Coleman and E. Weinberg, Phys. Rev. D 7, 1888 (1973). These authors renormalize in such a way that the argument of the logarithm in (3.20) is g^2 independent. For our purposes it is convenient to consider $g^2\phi^2$ since this is a renormalization invariant (see 3.27).

[9]Notice that α and β may be interpreted as Lagrange multipliers for the corresponding constraints.

[10]I would like to thank J. Schoenfeld for having explained this to me. His Princeton Ph.D. thesis contains the details.

[11]At $N=3$, the supersymmetry algebra enlarges. See Ref. 4 for details.

[12]F. A. Berezin, *The Method of Second Quantization* (Academic, New York, 1966); S. Ferrara and O. Piguet, Nucl. Phys. B93, 261 (1975), Appendix A.