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Quark star phenomenology

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We analyze models of quark matter appropriate to the description of matter at densities found within neutron stars. We consider in detail two models: quantum chromodynamics with a bag constant $B = 56$ MeV/fm^3, and quantum chromodynamics with $B = 0$. Both models are consistent with the strength of the quark-quark force found in high-energy particle phenomenology. The pressure, energy density per baryon, and baryon density are calculated. An interesting prediction of our models is that a quark star should have strangeness comparable to its baryon number.

I. INTRODUCTION

Recent work has explored the conjecture that there is a phase transition between nuclear matter and quark matter at high densities.[1-8] Such a phase transition might occur in the core of a pulsar, or during a heavy-ion collision.

Theoretical work in this area is most useful if it is developed so as to allow testing of theoretical models against empirical observation. Quark matter should have observable characteristics which distinguish it from nuclear matter. For example, theoretical predictions of upper mass limits, moments of inertia, and luminosities of neutron stars should differ between quark-matter and nuclear-matter models.

We would expect that a phase transition to quark matter occurs when nuclear matter is compressed so tightly that the hadronic constituents of nuclear matter overlap. In such circumstances, quarks in different hadrons may be freely interchanged, and the degrees of freedom of the matter are those of quarks. The matter is thus more directly described by the quark constituents of hadrons than by individual hadrons.

The density of matter at which hadrons overlap is given by the density of matter inside a hadron, $\mathfrak{N}_H \sim 1/\frac{4}{3}\pi r_H^3$. A typical hadronic radius is that of a proton, $r_H \sim 1$ fm, giving hadronic matter density as $\mathfrak{N}_H \sim \frac{1}{4}$ baryons/fm^3. Since the density of matter in a heavy pulsar may be as high as $\mathfrak{N}_p \sim 1-2$ baryons-fm^3, we should expect that quark matter may be found in neutron stars.

Of course, a phase transition would occur at densities higher than those of conventional nuclei. In conventional nuclei, quarks are bound together as neutrons and protons. Nuclear-matter density, $\mathfrak{N}_{\rm NM} \sim 0.16$ baryons/fm^3, thus provides a lower bound on the phase-transition density. It should be emphasized that nuclear-matter density is only slightly lower than that of hadronic matter. Hadrons in nuclei are almost overlapping, and if only slightly compressed should make a transition to the quark phase. We might expect, therefore, that unconventional quark nuclei could be produced in a heavy-ion collision.[9]

Logically, to determine whether a phase transition takes place between nuclear matter and quark matter, the energy per baryon of quark matter should be determined and compared to that of nuclear matter. The density at which the energy per baryon of quark matter becomes less than that of nuclear matter would be the phase-transition density.

Unfortunately, this estimate of the phase-transition density is very sensitive to a number of small uncertainties in hypothetical energy-density relationships of different nuclear-matter and quark-matter models.[6-8]

At the phase-transition density, the extended structure of hadrons is important, and nuclear-matter models do not take this extended structure into account. Moreover, at densities greater than nuclear-matter densities, models based on potential theory and those based on variational methods give different energy-density relationships.[10-13] Only at nuclear-matter densities may different models be directly compared with observation. At higher densities, the disagreement between the relationships given by different models is not surprising.

Quark-matter calculations have uncertainties arising from imprecise knowledge of the strength of the quark-quark interaction. Further, quark-matter calculations are perturbative in the strength of the quark-quark interaction, and many effects associated with nonlinearities of the interaction are ignored in this method of calculation. Since the quark-quark force increases with the distance between quarks, these nonlinearities are most important at the lowest density at which matter may exist in the quark phase, that is, at the phase-transition density.

At low densities, $\mathfrak{N}_B \lesssim \frac{1}{4}$ baryons/fm^3, nuclear-matter calculations are intrinsically more accurate than quark-matter calculations. The largest contributions to the energy per baryon of nuclear matter are proton and neutron rest masses, $m \sim 940$ MeV. Kinetic and interaction energies are small and are measured in tens of MeV's. Even large errors in nuclear-matter calculations of this small energy can yield only errors of the order of tens of MeV's for $\mathcal{E}/\mathfrak{N}$.

On the other hand, fits to hadron spectroscopy suggest small up- and down-quark masses, perhaps as small as 50 MeV.[14] Almost all the energy of quark matter resides in quark kinetic and interaction energies. Small errors, perhaps of 10%, result in errors of the order of 100 MeV in the determination of $\mathcal{E}/\mathfrak{N} \sim 1$ GeV.

We should thus be cautious about drawing any conclusions from a comparison of quark-matter and nuclear-matter calculations. In this paper, we shall not attempt such a comparison, nor shall we try to prove or disprove the existence of a phase transition between nuclear matter and quark matter. Instead, we shall assume the existence of a phase transition, and show that using a reasonable choice of the strength of the quark-quark force, the calculated energy per baryon of quark matter may be matched on to the $\mathcal{E}/\mathfrak{N}$ of nuclear matter at hadronic-matter densities.

More refined calculations than those presented in this paper may, of course, lead to different estimates of the strength of the quark-quark force. We do not expect, however, that more refined calculations will significantly alter the equation of state calculated for quark matter. The equation of state is almost entirely determined by specification of its asymptotic form at high densities, where interactions are unimportant, and by the constraint that the energy per baryon of quark matter match on to that of nuclear matter at hadronic-matter densities. At very high densities, the dynamics of quark matter are those of an ideal, relativistic gas, and $P \sim \frac{1}{3}\mathcal{E}$. At hadronic-matter densities, the equation of state must be softened so as to connect with nuclear matter. We find that at all densities of interest, the quark-matter equation of state is well approximated by $P \sim \frac{1}{3}\mathcal{E} - \mathcal{E}_0$. This phenomenological form for the equation of state is the simplest functional form that extrapolates between hadronic-matter densities and very high densities.

This simple form for the equation of state, given by the MIT bag model when quark interactions are ignored, has been used by Brecher and Caporaso to determine the upper mass limit of a quark star.[4,15] The constant $\mathcal{E}_0$ is given in the bag model as $\frac{4}{3}B$, where B is the bag constant.

Our calculations include interactions and produce equations of state similar in form to that used by Brecher and Caporaso over a variety of values for the bag constant and for the strength of the quark-quark force. In particular, we present two models, one with $B = 56$ MeV/fm^3 and the other with $B = 0$, which produce almost identical equations of state. These two models differ in the assumed strength of quark-quark force; the model using $B = 56$ MeV/fm^3 is assumed to have weaker quark interactions than that using $B = 0$.

We have calculated the thermodynamic properties of quark matter by using quantum chromodynamics. This theory provides a description of different flavored and colored quarks interacting by the exchange of massless vector gluons. In the following analysis, we shall assume three flavors and three colors of quarks. The quark flavors necessary for the description of quark matter in pulsars are up, down, and strange.

Heavy quarks of exotic flavors, such as the charmed quark, may be ignored for dynamical reasons. Heavy quark masses are $m_H \gtrsim 2$ GeV. At the densities of interest for quark stars, $\mathfrak{N}_B \lesssim 1-2$ baryons/fm^3. The quark Fermi momenta are $k_f \lesssim \pi^{2/3}\mathfrak{N}^{1/3} \sim 500$ MeV. More energy would be needed for the production of heavy quarks. Moreover, interactions with vacuum pairs of heavy quarks are important only for $k_F^2/4m_H^2 \gtrsim 1$, a condition involving higher Fermi momenta than would be found in quark stars.

The parameters which specify quantum chromodynamics are quark masses and the quark-gluon coupling constant. In this paper, we shall take the up- and down-quark masses as zero and take the strange-quark mass as $m_s = 280$ MeV. These values are given by the MIT bag model fit to the spectroscopy of the light hadrons.[14]

The remaining parameter—the quark-gluon coupling constant, g—is not known precisely. For the MIT bag model, the chromodynamic structure constant, $\alpha_c \equiv g^2/16\pi$, is found to be $\alpha_c \sim \frac{1}{2}$. However, in the bag model fit to the light-hadron spectrum, radiative corrections to lowest-order quark interactions were not determined. In order to calculate these corrections, the quark-gluon vertex must be specified at an off-mass-shell Euclidean subtraction point, μ_0. The corrections can be small only if the subtraction point, μ_0, is chosen to be of the order of a typical bag energy scale. If the natural scale, $B^{1/4}$, is chosen, then $\mu_0 \sim B^{1/4} \sim 150$ MeV. If the average kinetic energy of quarks inside the bag is used, $\mu_0 \sim E \sim 400$ MeV. These different choices of the subtraction point imply very different values of the strength of the quark-quark interaction.

Given these uncertainties, we have considered various values of the strength of the quark-quark force in our analysis. If the strength chosen were too large, a phase transition would be found at densities far above that of hadronic matter. If the strength chosen were too small, the phase transition would be found below nuclear-matter density. These relationships follow from the repulsive character of quark interactions at high densities. Increasing the hypothetical strength of the quark-quark force increases the energy per baryon, and yields a higher phase-transition density.

We shall present in this paper two models giving a phase transition between quark matter and nuclear matter at hadronic densities. We will argue that the values for the strength of the quark-quark force used in these models are not inconsistent with the values found in high-energy particle phenomenology.[14,16-19]

The organization of this paper will be as follows:

(1) In the first section, the result of a fourth-order (g^4) evaluation of the quark thermodynamic potential appropriate for massless quarks is extended to include the effects of finite strange-quark mass. These effects are included only to second order in g. Contributions of higher order in g are unimportant at higher densities, where interactions are weak. At lower densities, we argue that the contribution of strange quarks to the thermodynamic potential is kinematically suppressed.

The renormalization group is used to express the thermodynamic potential in terms of an average screened charge.[3,8,20-24] This charge explicitly depends on the quark chemical potentials, and approaches zero at high densities. The effects of finite strange-quark mass are also included in the screened charge.[17,20]

(2) In the second section, the equations for the quark number densities, energy density, and pressure are discussed. The constraints of beta-decay equilibrium are implemented, allowing for the production of an equilibrium distribution of electrons and strange quarks.

The inclusion of a bag constant, B, in the thermodynamic potential is discussed. We show that B appears only as a positive constant added to the thermodynamic potential and does not enter into the renormalization-group equations.

Finally, we discuss the equations for the quark number densities, energy density, and pressure in the limits of large and small baryon number density. We conclude that a phase transition takes place between quark matter and nuclear matter. At densities greater than the phase-transition density, a consistent evaluation of all order-α_c^2 effects demonstrates that $\alpha \lesssim \frac{1}{4}$.

(3) In the final section, we present two models of quark matter, both of which undergo phase transitions to nuclear matter at hadronic-matter densities. The first model takes the bag constant as $B=56$ MeV/fm^3, and assumes $\alpha_c=1$ at a subtraction point of $\mu_0=100$ MeV. In this model, the phase transition occurs at $\mathfrak{N}=0.34$ baryons/fm^3. The second model assumes $B=0$ and $\alpha_c=1$ at $\mu_0=275$ MeV. The phase transition then occurs at $\mathfrak{N}=0.28$ baryons/fm^3. We show that these two different models give almost identical equations of state and energy-density–density relations.

The nuclear-matter model used for a comparison with quark matter is the BJVH model.[12] This model is based on fits of potential theory to nucleon-nucleon scattering, and allows for the production of an equilibrium distribution of hyperons. The BJVH model is representative of a large class of potential theory models.

An interesting prediction of both our models is that quark matter is strange. At densities found within quark stars, the strangeness would be of the order of the baryon number. At the phase transition from nuclear matter to quark matter, there might be a large discontinuity in strangeness associated with a large transmutation of down quarks into strange quarks.

I. THE THERMODYNAMIC POTENTIAL

In Ref. 21, the thermodynamic potential of a massless quark gas of N_c colors and N_f flavors was evaluated to fourth order in the quark-gluon coupling, g. The result of this evaluation was

$$\Omega=-\frac{1}{\pi^2}\frac{1}{4}\left\{\sum_{i=1}^{N_f}\mu_i^4\left[\frac{N_c}{3}-\frac{\alpha_c(\tilde{\mu}_0)}{\pi}N_g+\left(\frac{\alpha_c(\tilde{\mu}_0)}{\pi}\right)^2\ln\frac{\mu_i^2}{\tilde{\mu}_0^2}\left(\frac{11N_c-2N_f}{3}\right)N_g\right.\right.$$
$$\left.+\left(\frac{\alpha_c(\tilde{\mu}_0)}{\pi}\right)^2N_g[-2.250N_c+0.409N_f-3.697-(4.24\pm0.12)/N_c]\right]$$
$$\left.-(\vec{\mu}^{\,2})^2\left(\frac{\alpha_c(\tilde{\mu}_0)}{\pi}\right)^2N_g\left[2\ln\left(\frac{\alpha_c(\tilde{\mu}_0)}{\pi}\right)-0.476\right]-\left(\frac{\alpha_c(\tilde{\mu}_0)}{\pi}\right)^2N_gF(\mu)\right\}, \tag{1.1}$$

where

$$F(\mu)=-2\sum_{i=1}^{N_f}\vec{\mu}^2\mu_i^2\ln\frac{\mu_i^2}{\vec{\mu}^2}+\sum_{i>j}\left[\tfrac{2}{3}(\mu_i-\mu_j)^4\ln\frac{|\mu_i^2-\mu_j^2|}{\mu_i\mu_j}+\tfrac{8}{3}\mu_i\mu_j(\mu_i^2+\mu_j^2)\ln\frac{(\mu_i+\mu_j)^2}{\mu_i\mu_j}-\tfrac{2}{3}(\mu_i^4-\mu_j^4)\ln\frac{\mu_i}{\mu_j}\right]. \qquad (1.2)$$

In Eqs. (1.1) and (1.2), the number of gluons is $N_g=N_c^2-1$, the chromodynamic structure constant is defined as $\alpha_c\equiv g^2/16\pi$, $\vec{\mu}^2\equiv\sum_{i=1}\mu_i^2$, and the quark flavor chemical potentials are μ_i.

The parameter $\bar{\mu}_0$ which appears in Eq. (1.1) is the Euclidean subtraction point at which the charge is defined. Later, we shall make a judicious choice of value for this parameter, a value which minimizes radiative corrections appearing in the perturbative evaluation of the thermodynamic potential. This choice requires that $\bar{\mu}_0$ be a function of the quark flavor chemical potentials. The explicit dependence of α_c on $\bar{\mu}_0$ may be obtained by solving the Gell-Mann–Low equation.[25-27]

It should be noted that the coefficient of terms proportional to $\alpha_c^2\sum_i\mu_i^4$ varies with the subtraction procedure used to define a renormalized charge. In Ref. (21), a conventional prescription was used which is analogous to the prescription of QED. However, other prescriptions could be used with different results.[24] However, to all orders in perturbation theory, these different prescriptions would lead to identical results, and differ only in the definition of the coupling g. To finite order in perturbation theory, the results can be different.

The virtue of our renormalization procedure, and of the renormalization-group analysis which follows, is that at all densities of interest the interactions are sufficiently weak so that perturbation theory may be valid. The difference between our results and increasingly accurate evaluations resides in contributions of order α_c^3 and higher, and in our analysis α_c is small. We therefore believe our analysis to be self-consistent.

At the densities of interest for quark-matter calculations, a zero-strange-quark-mass approximation is not appropriate, and Eq. (1.1) must be modified to include the effects of finite strange-quark mass. In lowest order in perturbation theory, strange quarks give an ideal gas contribution to the thermodynamic potential. This contribution is shown in Fig. 1 and may be evaluated with the result

$$\Omega^s_{(0)}=-\frac{1}{\pi^2}\frac{N_c}{3}\left[\tfrac{1}{4}\mu_s(\mu_s^2-m_s^2)^{1/2}(\mu_s^2-\tfrac{5}{2}m_s^2)+\tfrac{3}{8}m_s^4\ln\frac{\mu_s+(\mu_s^2-m_s^2)^{1/2}}{m_s}\right]. \qquad (1.3)$$

In this equation, the strange-quark mass is m_s.

In second order in perturbation theory, the strange-quark contribution to the thermodynamic potential is given by the exchange energy, shown in Fig. 2. The strange-quark exchange energy is[28,29]

$$\Omega^s_{(2)}=\frac{1}{\pi^2}\frac{\alpha_c(\bar{\mu}_0)}{\pi}N_g\left\{\tfrac{3}{4}\left[\mu_s(\mu_s^2-m_s^2)^{1/2}-m_s^2\ln\frac{\mu_s+(\mu_s^2-m_s^2)^{1/2}}{m_s}\right]^2-\tfrac{1}{2}(\mu_s^2-m_s^2)^2\right\}. \qquad (1.4)$$

As $\mu_s\to m_s$—that is, near the threshold for production of strange quarks—the ideal gas contribution becomes

$$\Omega^s_{(0)}=-\frac{1}{\pi^2}\frac{N_c}{3}\tfrac{1}{5}(\mu_s^2-m_s^2)^{5/2}/m_s\,, \qquad (1.5)$$

and the exchange energy becomes

$$\Omega^s_{(2)}=-\frac{1}{\pi^2}\frac{\alpha_c(\bar{\mu}_0)}{\pi}N_g\tfrac{1}{2}(\mu_s^2-m_s^2)^2\,. \qquad (1.6)$$

Thus, near threshold, both the strange-quark contributions to the thermodynamic potential vanish. On the other hand, up- and down-quark contributions are nonvanishing, and therefore dominate over the strange-quark contributions.

This is a fortunate circumstance, as many high-

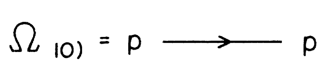

FIG. 1. The ideal gas contribution to the thermodynamic potential.

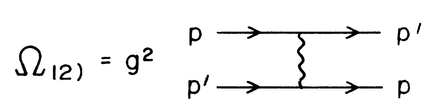

FIG. 2. The exchange energy.

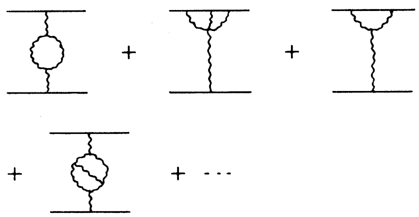

FIG. 3. Diagrams which yield threshold enhancements of the exchange energy.

order processes modify the strange-quark contributions near threshold. Processes such as those shown in Fig. 3 give logarithmic threshold enhancements. However, these enhanced diagrams vanish at finite order in perturbation theory as

$$(\mu_s^2 - m_s^2)^2(\alpha_c/\pi)^n \ln^{n-m} \frac{\mu_s^2 - m_s^2}{m_s^2},$$

where $n > m > 0$.

In all orders of perturbation theory, on the other hand, these enhanced diagrams might sum up to powers, or inverse powers, of $(\mu^2 - m_s^2)$. On physical grounds, nevertheless, we believe that the sum of all diagrams will not yield a singular contribution to the thermodynamic potential near the strange-quark threshold. Any singular behavior near threshold would have to arise from the long-distance singular behavior of the quark-quark force. The plasmon effect, however, appears in higher orders of perturbation theory and cuts off the electrostatic contribution to the quark-quark force at distances greater than the inverse plasma frequency.[3] Since the strange quarks are nonrelativistic at threshold, the magnetic contributions to the quark-quark force may be ignored, and, therefore, the quark-quark force is shielded at long distances.

Besides the plasmon effect, there are additional significant charge screening contributions associated with multiparticle scattering processes. These processes are important for the cancellation of infrared divergences in the perturbative evaluation of the thermodynamic potential. The long-distance singular behavior of the quark-quark force should also be cut off by these processes.

In addition to the threshold suppression of strange quarks mentioned earlier, another effect suppresses the strange-quark exchange energy at intermediate densities. At low densities, the exchange energy is generated by chromoelectric forces and decreases the energy per strange quark. At higher densities, chromomagnetic forces and retardation effects become important, and increase the energy per strange quark. The exchange energy at high densities is

$$\Omega^s_{(2)} = \frac{1}{\pi^2}\frac{\alpha_c(\bar{\mu})}{\pi}\frac{1}{4}\mu_s^4, \tag{1.7}$$

and its sign is the opposite of that of the exchange energy near threshold [Eq. (1.6)].

The exchange energy passes through zero at

$$\mu_s \simeq 2.72 m_s = 760 \text{ MeV}, \tag{1.8}$$

if $m_s = 280$ MeV. In most regions of interest $\mu_s \lesssim 500$ MeV, the strange-quark exchange energy thus is small, and of opposite sign to the up- and down-quark contributions.

Since contributions of strange quarks are dynamically suppressed in order α_c, we may hope that contributions of order α_c^2 may also be suppressed. At low densities, threshold effects strongly suppress strange-quark contributions. At intermediate densities, order-α_c^2 contributions may be kinematically suppressed since these contributions involve iterations and radiative corrections of the kinematically suppressed exchange energy. At higher densities, where strange-quark interactions are not kinematically suppressed, the screened charge, $\alpha_c(\bar{\mu}_0)$, is small. Ignoring the contributions arising from strange-quark interactions in order α_c^2 and higher may be justified.

Strange quarks do, however, make a significant contribution in some fourth-order processes. These processes comprise strange-quark vacuum polarization corrections to the up- and down-quark exchange energy. Such vacuum polarization corrections need only be calculated approximately since they are relatively minor fourth-order processes. Insertions of massless quark and gluon loops for example give larger contributions, since there are more ways of inserting these loops into the exchange energy. Moreover, the vacuum polarization correction from strange quarks is kinematically suppressed if $\mu^2/4m_s^2 \lesssim 1$. At the highest densities of interest, $\mu^2/4m_s^2 \sim 1$.

An approximation for strange-quark vacuum polarization may be taken into account by including the effects of strange-quark mass in the lowest-order contribution to the Gell-Mann–Low equation for the screened charge. In the higher-order contributions to the Gell-Mann–Low equation, strange quarks may be ignored. These higher-order contributions are important only at low densities where the coupling is large, and vacuum pairs of massive strange quarks are unimportant.

The Gell-Mann–Low equation for the screened charge, to order α_c^3, including the strange quarks in order α_c^2, is [15,18,30,31]

$$\mu\frac{d\alpha_c(\mu)}{d\mu}=\left[c_0-8\pi\mu\frac{d\pi_s(\mu)}{d\mu}\right]\alpha_c^2(\mu)+c_1\alpha_c^3(\mu). \tag{1.9}$$

In this equation, the parameters c_0 and c_1 are constants which are, for three colors and two flavors of massless quarks,

$$c_0=-58/3\pi, \tag{1.10}$$

$$c_1=-460/3\pi^2. \tag{1.11}$$

The vacuum polarization tensor for strange quarks, $\pi_s(\mu)$, is given by

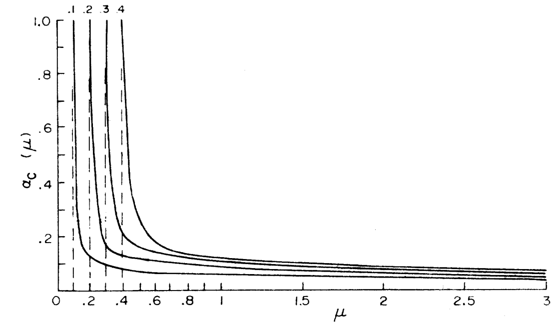

FIG. 4. The screened charge, $\alpha_c(\mu/\mu_0)$ for $\alpha_c(1)=1$ at $\mu_0=0.1$, 0.2, 0.3, and 0.4 GeV.

$$\pi_s(\mu)=\frac{1}{4\pi^2}\left[\tfrac{5}{9}-\tfrac{4}{3}m_s^2/\mu^2-\tfrac{2}{3}(1-2m_s^2/\mu^2)(1+4m_s^2/\mu^2)^{1/2}\operatorname{arctanh}(1+4m_s^2/\mu^2)^{-1/2}\right]. \tag{1.12}$$

Ignoring effects of strange-quark mass of order α_c^3 in Eq. (1.9), we find that $\alpha_c(\mu)$ is given as the solution of the equation

$$-c_0\ln\frac{\mu}{\mu_0}+8\pi[\pi^s(\mu)-\pi^s(\mu_0)]=\frac{1}{\alpha_c(\mu/\mu_0)}-\frac{1}{\alpha_c(1)}+\frac{c_1}{c_0}\ln\frac{1+c_0/c_1\,\alpha_c(1)}{1+c_0/c_1\,\alpha_c(\mu/\mu_0)}. \tag{1.13}$$

In this equation, $\alpha_c(1)$ is the value of α_c at the subtraction point μ_0. A plot of $\alpha_c(\mu/\mu_0)$ is shown in Fig. 4 for $\alpha_c(1)=1$ at the subtraction points $\mu_0=0.1$, 0.2, 0.3, and 0.4 GeV. The value of m_s used in this analysis is $m_s=280$ MeV.

Several features of Fig. 4 are noteworthy. The screened charge decreases very rapidly from $\alpha_c\sim1$. All the values of μ_0 considered are consistent with a value of $\alpha_c\sim\frac{1}{2}$ at a subtraction point μ: $0.1\lesssim\mu\lesssim0.4$. These values of μ are consistent with the MIT bag model fit to the spectroscopy of the light hadrons.[14] At $\mu=3$ GeV, the screened charge is $0.04\lesssim\alpha_c\lesssim0.06$. This range of values for α_c, while slightly lower than the values of $0.05\lesssim\alpha_c\lesssim0.10$ found by Poggio, Quinn, and Weinberg in their fit to e^+e^- annihilation data, is not inconsistent with these values.[15] If we had included charmed quarks in our analysis, we would have found slightly higher values of α_c. Our values are also close to the value found by Appelquist and Politzer for charmonium, $\alpha_c=0.65$ at $\mu=3$ GeV.[16]

The parameter $\bar{\mu}_0$ which appears as the argument of the screened charge in Eqs. (1.1) and (1.4) has not yet been specified. We shall choose a value for $\bar{\mu}_0$ which makes higher-order radiative corrections small. These radiative corrections appear as powers of $\ln(\mu_i^2/\bar{\mu}_0^2)$. Thus, if we choose $\bar{\mu}_0^2\sim\mu_i^2$, these corrections will be small. At low densities, where strange quarks are unimportant, a good choice is

$$\bar{\mu}_0^2=\tfrac{1}{2}(\mu_u^2+\mu_d^2) \tag{1.14}$$

since $\mu_u^2\sim\mu_d^2$. This choice is also good for densities at which strange quarks are important, since $\mu_s^2\sim\mu_d^2\sim\bar{\mu}_0^2$. (A small change in the choice of $\bar{\mu}_0^2$, moreover, yields only a very small change in the thermodynamic potential, $\delta\Omega\propto\delta\mu_0^2\alpha_c^3$.)

Using Eq. (1.14) in Eqs. (1.1), (1.3), and (1.4) for the thermodynamic potential, we find

$$\Omega=\Omega_u+\Omega_d+\Omega_s+\Omega_{\rm int}, \tag{1.15}$$

where

$$\Omega_u=-(1/\pi^2)\tfrac{1}{4}\mu_u^4[1-8\alpha_c/\pi-16(\alpha_c/\pi)^2\ln(\alpha_c/\pi)-31.1(\alpha_c/\pi)^2], \tag{1.16}$$

$$\Omega_d=-(1/\pi^2)\tfrac{1}{4}\mu_d^4[1-8\alpha_c/\pi-16(\alpha_c/\pi)^2\ln(\alpha_c/\pi)-31.1(\alpha_c/\pi)^2], \tag{1.17}$$

$$\Omega_s=\frac{-1}{\pi^2}\frac{1}{4}\Bigg(\mu_s(\mu_s^2-m_s^2)^{1/2}(\mu_s^2-\tfrac{5}{2}m_s^2)+\tfrac{3}{2}m_s^4\ln\left(\frac{\mu_s+(\mu_s^2-m_s^2)^{1/2}}{m_s}\right)-\frac{8\alpha_c}{\pi}\left\{3\left[\mu_s(\mu_s^2-m_s^2)^{1/2}-m_s^2\ln\left(\frac{\mu_s+(\mu_s^2-m_s^2)^{1/2}}{m_s}\right)\right]^2-2(\mu_s^2-m_s^2)^2\right\}\Bigg), \tag{1.18}$$

$$\Omega_{\rm int} = \frac{1}{\pi^2}\left(\frac{\alpha_c}{\pi}\right)^2\Bigg[8\mu_u^2\mu_d^2\ln\frac{\alpha_c}{\pi} - 1.9\mu_u^2\mu_d^2 - 19.3\left(\mu_u^4\ln\frac{\mu_u^2}{\mu_u^2+\mu_d^2} + \mu_d^4\ln\frac{\mu_d^2}{\mu_u^2+\mu_d^2}\right)$$
$$- 4(\mu_u^2+\mu_d^2)\left(\mu_u^2\ln\frac{\mu_u^2}{\mu_u^2+\mu_d^2} + \mu_d^2\ln\frac{\mu_d^2}{\mu_u^2+\mu_d^2}\right)$$
$$+ \tfrac{4}{3}(\mu_u-\mu_d)^4\ln\frac{|\mu_u^2-\mu_d^2|}{\mu_u\mu_d} + \tfrac{16}{3}\mu_u\mu_d(\mu_u^2+\mu_d^2)\ln\frac{(\mu_u+\mu_d)^2}{\mu_u\mu_d} - \tfrac{4}{3}(\mu_u^4-\mu_d^4)\ln\frac{\mu_u}{\mu_d}\Bigg]. \tag{1.19}$$

In these equations, Ω_u, Ω_d, and Ω_s represent the contributions of up, down, and strange quarks to the thermodynamic potential. The contribution associated with the interference between up and down quarks is $\Omega_{\rm int}$. The chromodynamic structure constant, $\alpha_c = \alpha_c(\bar{\mu}_0/\mu_0)$, is given by the solution of Eq. (1.13).

II. THERMODYNAMIC PARAMETERS

In this section, we shall derive formulas appropriate for the description of quark matter in a quark star. We shall also discuss the inclusion of a bag constant, B, in the calculation of the thermodynamic potential. Finally, we shall discuss a constraint which limits the magnitude of the average screened charge.

In our analysis, we allow for the production of an equilibrium distribution of strange quarks and electrons by beta decays,

$$d \rightleftarrows u + e^- + \bar{\nu} \tag{2.1}$$

and

$$s \rightleftarrows u + e^- + \bar{\nu}. \tag{2.2}$$

Since the electrons produced by the beta decays are relativistic, and since the electromagnetic interactions of the electrons are small, the electron contribution to the thermodynamic potential may be approximated by that of an ideal, relativistic electron gas,

$$\Omega_e = \frac{1}{3\pi^2}\tfrac{1}{4}\mu_e^4. \tag{2.3}$$

The implication of beta-decay equilibrium is that the energy density,

$$\mathcal{E} = \Omega + \vec{\mu}\cdot\vec{\mathfrak{N}}, \tag{2.4}$$

is at a minimum with respect to the density of strange quarks and electrons. Minimizing $\mathcal{E}$ with respect to $\mathfrak{N}_e$ and $\mathfrak{N}_s$ gives the equations of chemical equilibrium

$$\mu_e = \mu_d - \mu_u \tag{2.5}$$

and

$$\mu_s = \mu_d. \tag{2.6}$$

At high densities, $\mu_d \sim \mu_u \sim \mu_s$, and electrons give a very small contribution to the thermodynamic potential. Our numerical analysis showed that electrons may be ignored at all densities. However, for completeness, we shall demonstrate how to include their effect in quark-matter calculations.

In addition to the constraint of beta-decay equilibrium, the quark and electron number densities must satisfy the constraint of electrical neutrality,

$$Q = e(\tfrac{2}{3}\mathfrak{N}_u - \tfrac{1}{3}\mathfrak{N}_d - \tfrac{1}{3}\mathfrak{N}_s - \mathfrak{N}_e) = 0. \tag{2.7}$$

Since the chemical potentials are related to the number densities by

$$\mathfrak{N}_i = -\frac{\partial}{\partial\mu_i}\Omega, \tag{2.8}$$

Eqs. (2.5)–(2.8) allow for elimination of all chemical potentials in favor of the baryon number density.

The baryon number density is related to the quark number densities as

$$\mathfrak{N}_B = \tfrac{1}{3}(\mathfrak{N}_u + \mathfrak{N}_d + \mathfrak{N}_s). \tag{2.9}$$

Using the constraint of electric neutrality, Eq. (2.7), we see that

$$\mathfrak{N}_B = \mathfrak{N}_u - \mathfrak{N}_e. \tag{2.10}$$

Since at all densities $|\mathfrak{N}_e/\mathfrak{N}_u| \ll 1$, we see that the baryon number density is approximately equal to the up-quark number density.

All thermodynamic quantities may be written in terms of the baryon number density. In particular, the energy, pressure, and up, down, strange, and electron number densities are functions only of $\mathfrak{N}_B$. The constraint of beta-decay equilibrium, unfortunately, complicates the expression of these quantities in terms of $\mathfrak{N}_B$. We have been able to determine thermodynamic quantities as a function of baryon number only by performing a computer analysis.

For performing a numerical analysis, it is useful to introduce the spherical variables

$$\mu = (\mu_u^2 + \mu_d^2)^{1/2} \tag{2.11}$$

and

$$\tan\theta = \mu_u/\mu_d \ . \qquad (2.12)$$

Using Eqs. (2.5) and (2.6) to impose beta-decay equilibrium, and defining

$$\lambda = m_s/\mu\cos\theta \ , \qquad (2.13)$$

Eqs. (1.16)–(1.19) for the thermodynamic potential become

$$\Omega_u = -(1/\pi^2)\tfrac{1}{4}\mu^4\sin^4\theta[1 - 8\alpha_c/\pi - 16(\alpha_c/\pi)^2\ln(\alpha_c/\pi) - 31.1(\alpha_c/\pi)^2] \ , \qquad (2.14)$$

$$\Omega_d = -(1/\pi^2)\tfrac{1}{4}\mu^4\cos^4\theta[1 - 8\alpha_c/\pi - 16(\alpha_c/\pi)^2\ln(\alpha_c/\pi) - 31.1(\alpha_c/\pi)^2] \ , \qquad (2.15)$$

$$\begin{aligned}\Omega_{\text{int}} = (1/\pi^2)(\alpha_c/\pi)^2\sin^2\theta\cos^2\theta\{&8\ln(\alpha_c/\pi) - 1.9 - 38.6[\tan^2\theta\ln(\sin\theta) + \cot^2\theta\ln(\cos\theta)]\\ &- 8[\sec^2\theta\ln(\sin\theta) + \csc^2\theta\ln(\cos\theta)] + \tfrac{4}{3}(\tan^{1/2}\theta - \cot^{1/2}\theta)^4\ln|\tan\theta - \cot\theta|\\ &+ \tfrac{32}{3}\sec\theta\csc\theta\ln(\tan^{1/2}\theta + \cot^{1/2}\theta) - \tfrac{4}{3}(\tan^2\theta - \cot^2\theta)\ln(\tan\theta)\} \ , \end{aligned} \qquad (2.16)$$

and

$$\begin{aligned}\Omega_s = -\frac{1}{\pi^2}\tfrac{1}{4}\mu^4\cos^4\theta\Bigg(&(1-\lambda^2)^{1/2}(1-\tfrac{5}{2}\lambda^2) + \tfrac{3}{2}\lambda^4\ln\left(\frac{1+(1-\lambda^2)^{1/2}}{\lambda}\right)\\ &- \frac{8\alpha_c}{\pi}\left\{3\left[(1-\lambda^2)^{1/2} - \lambda^2\ln\left(\frac{1+(1-\lambda^2)^{1/2}}{\lambda}\right)\right]^2 - 2(1-\lambda^2)^2\right\}\Bigg). \end{aligned} \qquad (2.17)$$

In these equations, the average screened charge depends only on μ as

$$\alpha_c = \alpha_c(\mu^2/2\mu_0^2) \ . \qquad (2.18)$$

The electron contribution to the thermodynamic potential is

$$\Omega_e = -\frac{1}{3\pi^2}\tfrac{1}{4}\mu^4(\cos\theta - \sin\theta)^4 \ . \qquad (2.19)$$

The number densities may be obtained from Eqs. (1.16)–(1.19) and Eq. (2.3), by differentiating with respect to the chemical potentials and then imposing the constraints of beta-decay equilibrium. In the differentiation, the average screened charge must be differentiated so as to produce terms of order α_c^2 and α_c^3. Since in evaluating the thermodynamic potential we have ignored terms of order α_c^3, and those terms of order α_c^2 involving the strange-quarks mass, it is not consistent to maintain these terms here. We believe that the thermodynamic potential, energy density, and number densities must all be evaluated consistently in a perturbative expansion in the average screened charge. Performing the evaluation of the number densities, we find

$$\begin{aligned}\mathfrak{N}_u = (1/\pi^2)\mu^3\sin^3\theta\big(&1 - 8\alpha_c/\pi - 16(\alpha_c/\pi)^2\ln(\alpha_c/\pi)\csc^2\theta\\ &+ (\alpha_c/\pi)^2\{31.6 + 11.8\cot^2\theta - \tfrac{8}{3}\csc^2\theta(\sin\theta + \cos\theta)^{-1}[(\sin\theta - \cos\theta)^3 + 4\cos\theta]\\ &+ (170.4 + 16\csc^2\theta)\ln(\sin\theta) + 16\cot^2\theta\ln(\cos\theta) - \tfrac{16}{3}(1-\cot\theta)^3\ln|\tan\theta - \cot\theta|\\ &- \tfrac{32}{3}\cot\theta(3 + \cot\theta^2)\ln(\tan^{1/2}\theta + \cot^{1/2}\theta) + \tfrac{16}{3}\ln(\tan\theta)\}\big) , \end{aligned} \qquad (2.20)$$

$$\mathfrak{N}_s = \frac{1}{\pi^2}\mu^3\cos^3\theta\left\{(1-\lambda^2)^{3/2} - \frac{8\alpha_c}{\pi}\left[1 - \lambda^2 - 3\lambda^2(1-\lambda^2)^{1/2}\ln\left(\frac{1+(1-\lambda^2)^{1/2}}{\lambda}\right)\right]\right\}, \qquad (2.21)$$

and

$$\mathfrak{N}_e = (1/3\pi^2)\mu^3(\cos\theta - \sin\theta)^3 . \qquad (2.22)$$

The down-quark number density is given by Eq. (2.20) with $\cos\theta \leftrightarrow \sin\theta$.

Ignoring terms of order α_c^3 (and terms of order α_c^2 involving the strange-quark mass) in the derivation of Eq. (2.20) and (2.21) for the quark number densities introduces a difficulty in the calculation of the pressure

$$P = -dV\mathcal{E}/dV = -\Omega \, . \qquad (2.23)$$

Here, V is the volume and $\mathcal{E}$ is the energy density. If these terms were included in the evaluation of the number densities, it would be simple to verify that

$$\frac{d}{dV} V\mathcal{E} = \Omega \, , \qquad (2.24)$$

where $\mathcal{E}$ is given by Eq. (2.4) and Ω by Eqs. (2.14)–(2.17). If, however, these terms are not included then this equation will be invalidated by terms of the same order as those which were ignored.

This difficulty can be avoided either by including these terms in the number densities, or by modifying Eqs. (2.14)–(2.17) for Ω by including a correction, δP, which is of order α_c^3 (and order α_c^2 for those terms involving the strange-quark mass). We have chosen the latter course to modify Eqs. (2.14)–(2.17).

In either case, the magnitude of the modifications must be small at all densities where the perturbative evaluation is valid, since the modifications are of the same order as terms which have not been calculated for Ω. In the next section, we shall see that $|\delta P/\mathcal{E}| \ll 1$ at all densities greater than the phase-transition density. Nevertheless, δP is numerically significant at densities near the phase-transition density. At these densities, the pressure becomes small, and high-order terms in the perturbative evaluation become important. By including δP, we have kept only those higher-order terms in the evaluation of the pressure which maintain the validity of Eq. (2.23), when the energy density is evaluated to order α_c^2 in a chemical-potential-dependent charge.

In Ref. 23, an evaluation of the energy density of a relativistic quark gas was performed to order α_c in a density-dependent screened charge. Terms of order α_c^2 were retained when the energy was differentiated with respect to the density to obtain the pressure. Terms of order $\alpha_c^2 \ln\alpha_c$, and terms of order α_c^2 which were not generated by differentiation, were ignored. Since the conclusions of Ref. 23 depended on the terms of order α_c^2 which was generated by differentiation, we believe that the analysis of Ref. 23 was inconclusive.

If we let

$$P = -\Omega_0 + \delta P \, , \qquad (2.25)$$

where Ω_0 is given by Eqs. (2.14)–(2.17), we may use Eqns. (2.23), (2.24), and (2.4) to obtain

$$\begin{aligned} \delta P = \mathfrak{N}_u \frac{1}{\alpha_c} \frac{d\alpha_c}{d\mathfrak{N}_u} \Bigg[\frac{1}{\pi^2} \tfrac{1}{4} \mu^4 \Bigg(\frac{8\alpha_c}{\pi} \cos^4\theta \Bigg\{ 3\Bigg[(1-\lambda^2)^{1/2} - \lambda^2 \ln\Bigg(\frac{1+(1-\lambda^2)^{1/2}}{\lambda} \Bigg) \Bigg]^2 - 2(1-\lambda^2)^2 \Bigg\} \\ + \left(\frac{\alpha_c}{\pi}\right)^2 \sin^2\theta \cos^2\theta \left[32 + (\tan^2\theta + \cot^2\theta)\left(78.2 + 32 \ln \frac{\alpha_c}{\pi} \right) \right] \Bigg) + 2\Omega_{\text{int}} \Bigg] \\ + \mathfrak{N}_u \frac{1}{\mu} \frac{d\mu}{d\mathfrak{N}_u} \left\{ -\frac{16}{\pi^2} \mu^4 \left(\frac{\alpha_c}{\pi}\right)^2 (\cos^4\theta + \sin^4\theta) \left[\pi^2 \mu \frac{d\pi_s}{d\mu} + \frac{115}{6} \left(\frac{\alpha_c}{\pi}\right) \right] \right\} . \end{aligned} \qquad (2.26)$$

We did not evaluate analytically the derivative of α_c and μ with respect to the up-quark number density, but have instead evaluated it numerically.

Up to this point, we have not discussed the inclusion of a bag constant in the description of the dynamics of quark matter. In the MIT bag model, a positive constant, B, is added to the quark-gluon Hamiltonian. The quark-gluon fields are required to be enclosed within a finite volume, and to satisfy boundary conditions at the surface of the volume. The field equations, together with the boundary conditions, determine the shape of the volume.

For a thermodynamic system, boundary conditions are unimportant. Different choices of boundary conditions give differences in the thermodynamic potential which vanish as $V^{-1/3}$ for large V. The thermodynamic potential becomes modified by the bag dynamics only in that

$$\Omega \to \Omega + B \, . \qquad (2.27)$$

This modification can in no way change the distribution of quark number densities, since number densities are given by derivatives of Ω. The pressure and energy density, however, are modified as

$$\mathcal{E} \to \mathcal{E} + B \, , \qquad (2.28)$$

$$P \to P - B \, . \qquad (2.29)$$

For a thermodynamic system, the bag constant has a simple interpretation as the thermodynamic potential of the vacuum. When the thermodynamic potential is calculated from a relativistic field theory, the divergent zero-point energy of the vacuum must be subtracted from the unrenormalized thermodynamic potential to define a renormalized thermodynamic potential,[21]

$$\Omega^{\text{ren}}(\beta, \mu) = \Omega(\beta, \mu) - \lim_{\beta\to\infty;\, \mu\to 0} \Omega(\beta, \mu) . \qquad (2.30)$$

In this subtraction procedure, the value of the thermodynamic potential of the vacuum is implicitly set equal to zero. In a unified theory of particle dynamics, this requirement has no physical implications, since only energy and pressure differences would be measured experimentally. However, we wish to compare our calculations of the energy and pressure of quark-matter to nuclear-matter calculations, calculations which are based on potential theory models of nuclear interactions, and not derived from underlying quark dynamics. In such a comparison, the thermodynamic potential of the vacuum, B, may be regarded as a phenomenological parameter.

Since the bag constant is a parameter independent of density, it does not enter into the renormalization-group equations. Nevertheless, the bag constant, like the chromodynamic structure constant, is a phenomenological parameter determined from a comparison of theoretical calculations and experimental data. Increasingly accurate calculations in the bag model may lead to different values for the bag constant and the chromodynamic strructure constant. When effects of order α_c are calculated, the MIT bag model fit to the spectroscopy of the light hadrons gives $B = 56$ MeV/fm^3 and $\alpha_c = 0.55$.

In our analysis, calculations are carried out to order α_c^2 in the screened charge. We should not expect that the same parameters which give the best fit to light-hadron spectroscopy in order α_c will be those which give the best fit in order α_c^2. In fact, our calculations show that, for a thermodynamic system, baglike effects appear in order α_c^2. The appearance of these effects suggests that higher-order calculations in the bag model itself might decrease the values found for B and α_c.

In the next section, two models are presented. One uses $B = 56$ MeV/fm^3 and $\alpha_c = 1$ at $\mu_0 = 100$ MeV; the other uses $B = 0$ and $\alpha_c = 1$ at $\mu_0 = 275$ MeV. The equations of state, and energy densities as a function of density, produced by these two models are for all practical purposes identical. In addition to the calculations presented in the next section, we have considered still other models, using various values for B and μ_0. In these calculations, we found that the equation of state and energy density were in sensitive to variation in B and μ_0, so long as B and μ_0 were constrained to give a phase transition between nuclear matter and quark matter at hadronic-matter densities.

Before proceeding with a detailed analysis of specific quark-matter models, we shall first attempt to offer some insight into the structure of the relations between pressure, energy density, and density. We begin by observing that at very high densities, the screened charge, α_c, is sufficiently small so that interactions may be ignored. In addition, at very high densities the quark Fermi energies are much greater than the strange-quark mass, and a good approximation is permitted by taking the strange-quark mass to zero. In this approximation, the up, down, and strange quarks appear symmetrically in the thermodynamic potential with

$$\mu_u = \mu_d = \mu_s \qquad (2.31)$$

and

$$\mathfrak{N}_u = \mathfrak{N}_d = \mathfrak{N}_s . \qquad (2.32)$$

The equations for the pressure, energy density, and number densities at high densities are

$$P = \tfrac{1}{3}\mathcal{E} , \qquad (2.33)$$

$$\mathcal{E} = \tfrac{9}{4}\pi^{2/3}\,\mathfrak{N}_u^{4/3} , \qquad (2.34)$$

and

$$\mathfrak{N}_i = (1/\pi^2)\mu_i^3 . \qquad (2.35)$$

At low densities, on the other hand, a fair approximation is found by ignoring strange quarks and taking $\mu_u \sim \mu_d$. In such an approximation, Eqs. (2.14)–(2.22) become

$$\mathfrak{N}_i = (1/\pi^2)\mu_i^3(1 - 2.55\alpha_c - 3.24\alpha_c^2 \ln\alpha_c - 5.74\alpha_c^2) , \qquad (2.36)$$

$$\mathcal{E} = \tfrac{3}{4}\sum_i \mu_i \mathfrak{N}_i + 0.204\alpha_c^2 \sum_i \mu_i^4 , \qquad (2.37)$$

and the pressure is given by

$$P = \mathfrak{N}_u^2 \frac{d}{d\mathfrak{N}_u}(\mathcal{E}/\mathfrak{N}_u) . \qquad (2.38)$$

A constraint on the screened charge follows from Eq. (2.36). In order to maintain positivity in the relationship between the quark number densities and chemical potentials, the screened charge must satisfy the condition

$$2.55\alpha_c + 3.24\alpha_c^2 \ln\alpha_c + 5.74\alpha_c^2 < 1 . \qquad (2.39)$$

This condition is satisfied only if $\alpha < 0.32$. At $\alpha_c = 0.32$, the quark number densities vanish. The screened charge approaches this upper bound at a finite value of μ_i, corresponding to the solution of

$$\alpha_c(\mu_i/\mu_0) = 0.32 \ . \tag{2.40}$$

If μ_0 is taken as the subtraction point at which $\alpha_c = 1$, then $\mu_i \gtrsim \mu_0$.

Before zero density is reached, however, a phase transition between quark matter and nuclear matter will occur. The occurrence of this phase transition is implied in Eq. (2.37) for the energy density. As the number density approaches zero, the quark energy density approaches a finite limit,

$$\mathcal{E}(0) = 0.204\alpha_c^2 \sum_i \mu_i^4 > 0 \ . \tag{2.41}$$

Since $\mathcal{E}/\mathfrak{N} \sim \mathfrak{N}^{1/3}$ at high densities and $\mathcal{E} \sim \mathcal{E}(0)/\mathfrak{N}$ at low densities, the energy per quark must have a minimum at some critical density. At this density, the pressure would vanish and the quark matter becomes unstable.

A phase transition between nuclear matter and quark matter would occur at a density higher than that at which the pressure vanishes. When the quark energy density and pressure become small, quark matter becomes unstable, forming droplets of hadronic matter. The occurrence of a minimum for the energy per quark as a function of density signals a phase transition.

The use of Eqs. (2.36) and (2.37) is, of course, not valid for densities below or near the critical density at which the pressure vanishes. Near the critical density, the perturbative evaluation of the pressure breaks down. As the pressure approaches zero, cancellations occur between all the terms which result from this evaluation.

We can conclude, however, that the equation of state for quark matter is softer than that of an ideal relativistic gas at all densities for which the perturbative evaluation of the pressure is valid. At the phase-transition density, moreover, the value of the screened charge need not be large. The numerical analysis, which we shall present in the next section, provides hope that perturbation theory may be valid above and near the phase-transition density. At all densities greater than the phase-transition density, we find the screened charge is small $\alpha_c \lesssim \frac{1}{4}$.

III. QUARK-MATTER MODELS

In this section, we shall present the results of our numerical analysis of two quark-matter models. Model I is a bag model, $B = 56$ MeV/fm^3, with a screened charge parametrized so that $\alpha_c = 1$ at $\mu_0 = 100$ MeV. One virtue of this model is that quark confinement is incorporated in the quark dynamics through the presence of B. In addition, the screened charge is small at all densities of interest, so that effects of interactions are likewise small.

The second model, model II, uses $B = 0$ and a screened charge parametrized so that $\alpha_c = 1$ at $\mu_0 = 275$ MeV. Calculations using this model demonstrate that there is little difference between the descriptions of quark matter provided by quantum chromodynamics (QCD) with $B \neq 0$ and by QCD with $B = 0$. In fact, we have considered various models with $0 < B < 56$ MeV/fm^3 and 100 MeV $< \mu_0 < 275$ MeV. When the parameters were chosen so that a phase transition between quark matter and nuclear matter occurred at hadronic-matter densities, the differences in the energy density, pressure, and density relations of those models were small.

We have included the effects of an equilibrium distribution of electrons, as discussed in Sec. II. At all densities greater than hadronic-matter densities, electrons give only negligible contamination of quark matter. We shall thus ignore electrons in the results we present.

The density dependence of the screened charge is shown in Fig. 5. The screened charge approaches $\alpha_c \sim 0.32-0.34$ at small densities, and approaches zero at very high densities in both models I and II. At hadronic-matter densities ($\mathfrak{N}_B \sim \frac{1}{4}$ baryons/fm^3), the screened charge is $\alpha_c \sim 0.24$ for model II and $\alpha_c \sim 0.1$ for model I. At the highest densities found within pulsars, $\mathfrak{N}_B \sim 1-2$ baryons/fm^3, the screened charge is $\alpha_c \sim 0.1$ in both models.

The density dependence of the energy per baryon is shown in Fig. 6. The quark-matter curves have been smoothly matched to the BJVH nuclear-matter model at hadronic-matter densities. The phase-transition density is $\mathfrak{N}_B = 0.34$ baryons/fm^3 for model I, and $\mathfrak{N}_B = 0.28$ baryons/fm^3 for model II.

In this smooth matching, the phase transition is

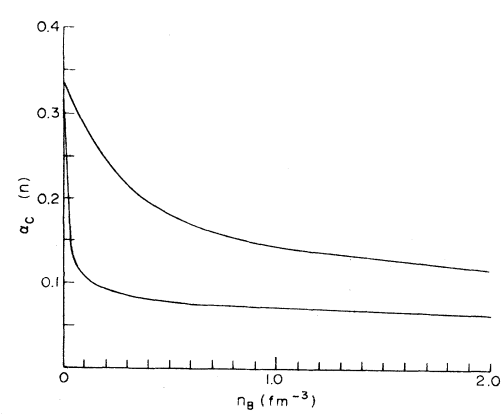

FIG. 5. The density dependence of the screened charge for models I and II.

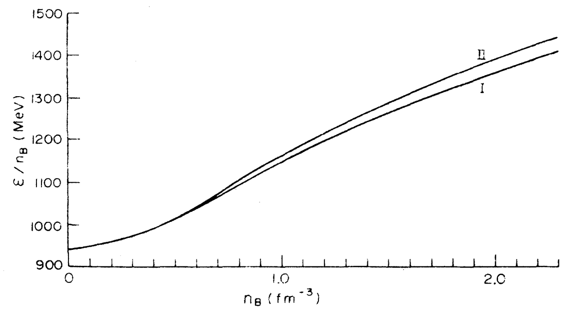

FIG. 6. The energy per baryon as a function of density.

assumed to be of second order and both $\mathcal{E}/\mathcal{N}$ and the first derivative of $\mathcal{E}/\mathcal{N}$ are required to be continuous at the phase-transition density. The possibility of a first-order phase transition was considered in models not presented here. We found little qualitative difference between the equations of state resulting from assuming either a first-order or a second-order phase transition.

The density dependence of $\mathcal{E}/\mathcal{N}$ is remarkably similar in both models. In the density region 0.25 baryons/fm^3 < $\mathcal{N}_B$ < 2 baryons/fm^3, the energy is well approximated by

$$\mathcal{E}=A\mathcal{N}^{4/3}+B\ . \tag{3.1}$$

This functional form is consistent with the ideal relativistic gas behavior at high densities, $\mathcal{E}\sim\mathcal{N}^{4/3}$, and with an increase in the energy per baryon at lower densities.

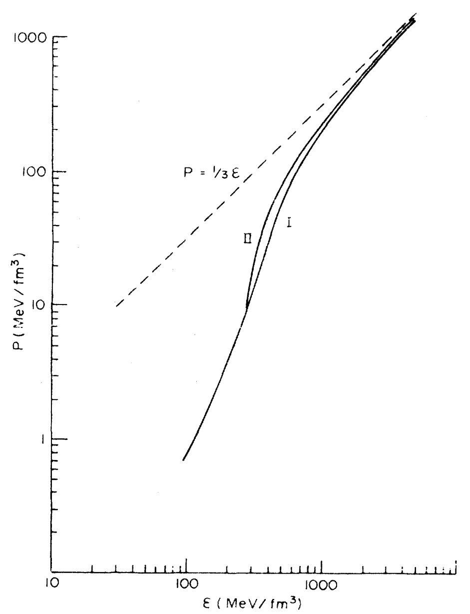

FIG. 7. The equation of state. The equation of state of an ideal relativistic gas, $P=\frac{1}{3}\epsilon$, is denoted by the dashed line.

The equation of state is shown in Fig. 7. The equation of state in both models is well approximated by

$$P=\tfrac{1}{3}\mathcal{E}-\tfrac{4}{3}B\ , \tag{3.2}$$

where B is given by Eq. (3.1). This form is completely determined by Eq. (3.1) for the energy density. The pressure approaches that of an ideal relativistic gas, $P=\frac{1}{3}\mathcal{E}$, at high densities. At lower densities, the equation of state is "softer" than $P=\frac{1}{3}\mathcal{E}$, since it must match to a "soft" nuclear-matter equation of state.

As was discussed in Sec. II, a small correction term, δP, must be added to the thermodynamic potential to define properly the pressure. This correction term in the screened charge is of higher order than the order in which $\mathcal{E}$ and $\mathcal{N}_i$ have been calculated. The ratio $|\delta P/\mathcal{E}|$ must be small in order that the perturbative evaluations of P, $\mathcal{E}$, and $\mathcal{N}_i$ may be consistent. This ratio is shown in Fig. 8. This correction term is not plotted for model I, as $|\delta P/\mathcal{E}|>0.005$ at all densities $\mathcal{N}_B>0.34$. For model II, this ratio is less than 0.1 at densities $\mathcal{N}_B>0.28$.

The fact that this correction is less than 10% suggests that a perturbative evaluation of P, $\mathcal{E}$, and $\mathcal{N}_i$ may be valid. However, such a small uncertainty in $\mathcal{E}$ introduces a large uncertainty in the calculation of the density for a phase transition between quark matter and nuclear matter. Since we assume that a phase transition exists, we have matched quark matter to nuclear matter by adjusting the parameters of our various quark-matter models. The uncertainty in our evaluation of these models leads to a corresponding uncertainty in the parameters which give a phase transition at hadronic-matter densities. This uncertainty is largest in the model with $B=0$, and we estimate that the 10% uncertainty in $\mathcal{E}$ would give an un-

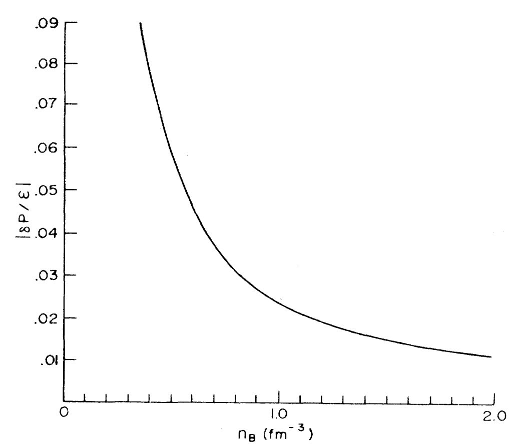

FIG. 8. The ratio of $|\delta P/\epsilon|$ in model II.

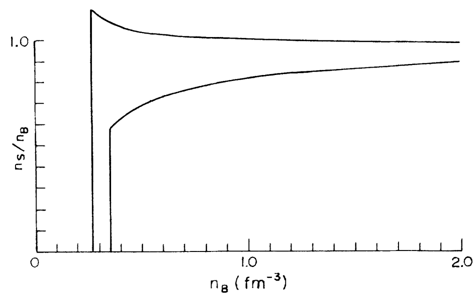

FIG. 9. The ratio of strangeness to baryon number as a function of density.

certainty in μ_0 of order 10%.

The ratio of strangeness to baryon number, $\mathfrak{N}_s/\mathfrak{N}_B$, is shown in Fig. 9. For the quark phases of both models, the strangeness approximates the baryon number. At high densities, where quark matter is flavor symmetric, the strangeness approaches the baryon number.

Below the phase-transition density, the strangeness is very small. The production of a large number of strange quarks in the phase transition to quark matter may be attributed to two dynamical effects, the most important of which is the Fermi exclusion principle. At high densities, the Fermi energy of up and down quarks becomes greater than the strange-quark mass, $m_s \sim 280$ MeV. The up and down quarks can lower their Fermi energies by transforming themselves into strange quarks. A somewhat less important effect is the attractive interaction of strange quarks at low densities. As was discussed in Sec. I, the exchange energy is attractive at low densities and becomes repulsive at high densities. The interactions of strange quarks lower the energy, whereas the interactions of up and down quarks raise the energy.

TABLE I. The pressure, energy density, and number density relations for model I.

n_B (fm^{-3})	$\mathcal{E}/n_B$ (MeV)	P (MeV/fm^3)	n_s/n_B
0.100	952	0.7	
0.150	956	1.8	
0.200	961	3.7	
0.250	966	6.8	
0.300	972	11.4	
0.347	978	20.0	0.59
0.402	988	35.5	0.63
0.494	1010	63.7	0.69
0.597	1037	98.1	0.73
0.713	1070	139	0.76
0.796	1093	171	0.78
0.888	1117	207	0.80
0.985	1142	246	0.81
1.20	1194	338	0.84
1.38	1234	420	0.86
1.57	1276	514	0.87
1.79	1318	620	0.88
2.02	1362	740	0.89
2.27	1405	876	0.90
2.53	1449	1028	0.91
2.72	1479	1138	0.91
3.02	1523	1320	0.92

TABLE II. The pressure, energy density, and number density relations for model II.

n_B (fm^{-3})	$\mathcal{E}/n_B$ (MeV)	P (MeV/fm^3)	n_s/n_B
0.100	952	0.7	
0.150	956	1.8	
0.200	961	3.7	
0.250	966	6.8	
0.312	972	15.4	1.12
0.347	979	25.8	1.10
0.383	987	37.0	1.08
0.495	1018	73.8	1.05
0.615	1055	116	1.03
0.701	1081	149	1.02
0.793	1109	185	1.01
0.890	1137	225	1.00
1.05	1180	290	1.00
1.16	1209	341	1.00
1.40	1268	454	0.99
1.60	1313	554	0.99
1.82	1358	665	0.99
1.98	1388	749	0.98
2.23	1434	884	0.98
2.49	1480	1046	0.98
2.78	1526	1206	0.98
2.98	1557	1329	0.98

This second effect is most important when interactions are strong. The quark-quark force is larger for model II than for model I. The larger strange-quark content of model II is, therefore, not surprising.

The amount of strangeness predicted by quark-matter models is sensitive to changes in the parameters of different models. The strangeness is sensitive not only to the interaction strength, but also to the magnitude of the strange-quark mass. An increase in the strange-quark mass leads to a decrease in strangeness, since with increased strange-quark mass the number of up and down quarks with sufficient energy to decay into strange quarks is decreased. The energy density relations of models I and II are summarized in Tables I and II.

IV. CONCLUSIONS

We have seen that viable models of quark matter can be constructed from fundamental theories of quark dynamics. At the present level of accuracy of perturbative calculations, and given the uncertainties in the experimental knowledge of the parameters of these theories, a prediction of the precise density of a phase transition between quark matter and nuclear matter cannot be made. Measurements of the charge radius of the proton, however, indicate that the proton is an extended object of radius 1 fm. If the extended structure of the proton arises from quarks, then we expect a phase transition from nuclear matter to quark matter at the density of matter inside of a proton.

Using a density of this order as the phase-transition density, we have seen that different quark-matter models give remarkably similar energy density, pressure, and density relations. This similarity is not unexpected, since asymptotic freedom and the value of phase-transition density strongly constrain the high- and low-density limits of the quark-matter models. The extrapolation of the energy density and pressure as functions of density between these limits is insensitive to the changes in parameters which characterize different models.

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