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Polynomial conservation laws in (1 + 1)-dimensional classical and quantum field theory

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The quantum-mechanical implications of the inverse-scattering-transform method and its relationship to the structure of Bethe's ansatz are discussed in the context of the nonlinear Schrödinger equation (many-body problem) associated with the classical (quantum) field theory $\mathcal{L} = (i/2)\phi^*\overleftrightarrow{\partial}_0\phi - |\partial_1\phi|^2 - c|\phi|^4$. We review the transformation of the classical problem to action and angle variables and the derivation of an infinite number of polynomial conservation laws. The values of the conserved constants are given by the moments of the classical action variable. It is suggested that there exists a corresponding set of conserved polynomial operators in the quantum field theory and that they reflect the conservation of velocity content which characterizes the solution of the many-body scattering problem (Bethe's ansatz). This implies that the quantized action variable is just the occupation-number density operator in the asymptotic momentum-(velocity-) parameter space of Bethe's ansatz, and that Bethe's wave functions are eigenstates of all conserved operators with eigenvalues given by the moments of the N-particle distribution in asymptotic momentum space. These statements are verified for the first four operators, including one which has not previously been studied.

I. INTRODUCTION

The study of exactly solvable models has long been a valuable aid to mathematical and physical intuition about nonlinear systems. In this paper we discuss a connection between two powerful techniques which have been central to the developments in different areas of exact $(1+1)$-dimensional physics. The method known as Bethe's ansatz[1] has among its applications a variety of one-dimensional spin chains,[2,3] certain two-dimensional lattice models,[4,5] and a many-body problem with particles in one space dimension interacting via a δ function two-body potential.[6-10] It also provided the initial impetus in the study of the Thirring model.[11] The inverse-scattering-transform method of Gardner, Green, Kruskal, and Miura[12] was discovered as a means of solving the Korteweg-DeVries equation. It was subsequently shown that a generalized inverse scattering transform[13,14] provided an exact treatment of a number of nonlinear partial differential equations with one space and one time coordinate [$(1+1)$-dimensional classical field theories]. This method has been particularly valuable in exposing the profound role and remarkable properties of solitons in these theories.[15] Experience with both of these methods suggests that they, and the phenomena which they expose, are intimately related in a way which may be surmised intuitively but remains mathematically obscure.

A natural laboratory is available for studying this question: The Galilean-invariant theory of a complex scalar field ϕ with quartic self-interaction, described by the Lagrangian

$$\mathcal{L} = \tfrac{1}{2} i\phi^*\overleftrightarrow{\partial}_0\phi - |\partial_1\phi|^2 - c|\phi|^4 . \tag{1.1}$$

As a classical field theory, the equation of motion obtained from (1.1) is the "nonlinear Schrödinger equation"

$$i\partial_0\phi = -\partial_1{}^2\phi + 2c|\phi|^2\phi . \tag{1.2}$$

This equation may be exactly treated via the inverse problem of Zakharov and Shabat.[13] On the other hand, we may consider the same Lagrangian for a quantum field Φ,

$$\mathcal{L} = \tfrac{1}{2} i\Phi^*\overleftrightarrow{\partial}_0\Phi - (\partial_1\Phi^*)(\partial_1\Phi) - c\Phi^*\Phi^*\Phi\Phi , \tag{1.3}$$

which has canonical commutation relations

$$[\Phi(x,t), \Phi^*(y,t)] = \delta(x-y) . \tag{1.4}$$

It is easy to show[16] that (1.3) is the field-theoretic formulation of a many-body problem with the last term corresponding to a δ-function two-body potential.[17] In this form the quantized theory can be treated exactly by Bethe's ansatz.

The intuitive connection between the motion of particles embodied in Bethe's ansatz and the behavior of fields which emerges from the inverse method can be illustrated by considering the case $c<0$ (attractive coupling) and comparing the solitons of (1.1) with the many-particle bound states of (1.3). Using Bethe's ansatz it can be shown[9] that a collision of two bound states never results in a redistribution or breakup of the particles, e.g., if a three particle bound state with velocity v_1 collides with a five-particle bound state with velocity v_2, they always emerge as a three particle bound state with velocity v_1 and a five-particle bound state with velocity v_2, having suffered only a phase shift or time delay. This can clearly be identified with the remarkable properties of colliding solitons in the classical field theory.[21]

More generally, the form of Bethe's ansatz can be understood by simply imagining a system of colliding billiard balls moving in one spatial dimension.[19] Common experience tells us that such a system exhibits a profound simplicity: The distribution of velocities is preserved in time. This conservation of velocity content is also a property of the quantum-mechanical many-body system (1.3) and is the essence of Bethe's ansatz. We will find that this is closely related to the infinite number of conservation laws which are obtained in the classical field theory as a by-product of the inverse method.[22]

For simplicity we will restrict our discussion to the case $c>0$ (repulsive coupling) and $|\phi|\to 0$ as $x\to\pm\infty$. Thus we consider classical systems consisting of pure "radiation" (no solitons) and quantum systems composed entirely of elementary quanta. For $c<0$, similar considerations can be applied to classical (quantum) systems containing solitons (bound states), but we will not do so here. The work of Zakharov and Manakov[23] has shown that the inverse method for (1.1) can be formulated as a canonical transformation from the field variables $\phi(x)$ and $\phi^*(x)$ to a set of action and angle variables $P(\xi)$ and $Q(\xi)$. Here ξ parametrizes the scattering data for an eigenvalue problem in which the canonical field configuration $\phi(x)$ plays the role of a scattering potential. We will find that ξ can be associated with the asymptotic momentum parameters k_i which appear in Bethe's ansatz. When written in terms of action and angle variables, the classical Hamiltonian corresponding to (1.1) is found to be independent of the angle variable $Q(\xi)$. Therefore the conjugate momenta $P(\xi)$ are conserved. The infinite number of constants of motion for the classical theory are just the moments of the time-independent function $P(\xi)$ which characterizes the system. By such considerations and by the results of a calculation described in Sec. III D, we are led to propose that the quantum-mechanical generalization of the Zakharov-Manakov action variable $P(\xi)$ is the occupation number in the space of the momentum parameters of Bethe's ansatz. This will be clarified and explored in Sec. III. As we will see, the assumption that the quantized action variables $P(\xi)$ can be identified with the distribution of particles in the momentum parameter space of Bethe's ansatz implies some remarkable properties of Bethe's wave functions which can be studied directly. If this interpretation of $P(\xi)$ is correct, it tells us that Bethe's wave functions are eigenstates not only of the Hamiltonian, but also of the entire infinite set of operators $\hat{M}_l$ ($l=0,1,2,\ldots$) obtained by writing the conserved quantities M_l of the classical theory in terms of the fields ϕ and ϕ^*, and then replacing these by the quantized fields Φ and Φ^* with an appropriate ordering prescription. We will study explicitly the first four classicallly conserved quantities M_l. The operators $\hat{M}_0$ and $\hat{M}_1$ are particle number and momentum:

$$\hat{M}_0=\hat{N}=\int\Phi^*\Phi\,dx\ , \tag{1.5}$$

$$\hat{M}_1=\hat{P}=i\int\Phi_x^*\Phi\,dx\ . \tag{1.6}$$

(Here and elsewhere a subscript x denotes differentiation on the space coordinate.) The N-particle Bethe wave-functions $|\Psi_N(k_1,\ldots,k_N)\rangle\equiv|\Psi_N(k)\rangle$ are eigenstates of both, trivially,

$$\hat{M}_0\,|\Psi_N(k)\rangle=N|\Psi_N(k)\rangle\ , \tag{1.7}$$

$$\hat{M}_1\,|\Psi_N(k)\rangle=\left(\sum_{i=1}^{N}k_i\right)|\Psi_N(k)\rangle\ . \tag{1.8}$$

The operator $\hat{M}_2$ is the Hamiltonian

$$\hat{M}_2=\hat{H}=\int(\Phi_x^*\Phi_x+c\Phi^*\Phi^*\Phi\Phi\,dx)\ . \tag{1.9}$$

In Sec. III C we review the seminal property of Bethe's wave functions:

$$\hat{M}_2\,|\Psi_N(k)\rangle=\hat{H}\,|\Psi_N(k)\rangle=\left(\sum_{i=1}^{N}k_i^2\right)|\Psi_N(k)\rangle\ . \tag{1.10}$$

Our derivation of (1.10) differs from the usual in that it is carried out in momentum space. It is inspired by the graphical approach[19] to the Lagrangian (1.3), and is more easily generalized to the higher moments $\hat{M}_l$, $l\geq 3$. Finally, in Sec. III D we consider the simplest operator moment which has not been previously studied:

$$\hat{M}_3=(i)^3\int(\Phi_{xxx}^*\Phi-3c\Phi_x^*\Phi^*\Phi\Phi\,dx)\ . \tag{1.11}$$

Using a method which resembles the derivation of (1.10), we find that

$$\hat{M}_3\,|\Psi_N(k)\rangle=\left(\sum_{i=1}^{N}k_i^3\right)|\Psi_N(k)\rangle\ . \tag{1.12}$$

The combination of the familiar properties (1.7), (1.8), and (1.10), with the result (1.12) provides considerable evidence for our interpretation of the quantized action variables and for the conjecture that Bethe's wave functions are eigenstates of all the $\hat{M}_l$'s with eigenvalues$(\sum_{i=1}^{N}k_i^l)$.

II. CONSTANTS OF MOTION IN THE CLASSICAL NONLINEAR SCHRÖDINGER EQUATION

We will first review the relevant results from the inverse-scattering-transform (IST) analysis of the classical theory (1.1). For more detailed discussion we refer to the original papers of Zakharov and co-workers.[13,23] The IST for (1.1) is based on the solutions of the Zakharov-Shabat eigenvalue problem associated with the canonical field configuration $\phi(x,t)$ (t = time plays a passive role in this discussion and will hereafter be suppressed):

$$i\left(\frac{\partial}{\partial x}+\tfrac{1}{2}\xi\right)\Psi_1=-\sqrt{c}\;\phi\Psi_2\,, \tag{2.1a}$$

$$i\left(\frac{\partial}{\partial x}-\tfrac{1}{2}\xi\right)\Psi_2=\sqrt{c}\;\phi^*\Psi_1\,. \tag{2.1b}$$

When the eigenvalue ξ is real, we can define the scattering data in terms of the solution $\Psi(x,\xi)$ with asymptotic behavior

$$\Psi(x,\xi)\underset{x\to-\infty}{\simeq} e^{i\xi x/2}\binom{1}{0}. \tag{2.2}$$

If we write

$$\Psi_1(x,\xi)=e^{i\xi x/2}A(x,\xi)\,, \tag{2.3}$$

$$\Psi_2(x,\xi)=e^{-i\xi x/2}B(x,\xi)\,, \tag{2.4}$$

then the scattering data are defined by

$$a(\xi)=\lim_{x\to\infty}A(x,\xi)\,, \tag{2.5}$$

$$b(\xi)=\lim_{x\to\infty}B(x,\xi)\,. \tag{2.6}$$

By writing (2.1) as an integral equation and iterating in powers of $\sqrt{c}$, we can obtain A and B in explicit series form. Defining, for a given ξ,

$$g(x,y)=\theta(x-y)e^{-i\xi y} \tag{2.7}$$

(where θ = step function), we get

$$A(x,\xi)=1+c\int dy_1dy_2\,g(x,y_1)\phi(y_1)g^*(y_1,y_2)\phi^*(y_2)$$
$$+c^2\int dy_1dy_2dy_3dy_4\,g(x,y_1)\phi(y_1)g^*(y_1,y_2)\phi^*(y_2)g(y_2,y_3)\phi(y_3)g^*(y_3,y_4)\phi^*(y_4)+\cdots\,, \tag{2.8}$$

$$B(x,\xi)=-i\sqrt{c}\left[\int dy_1g^*(x,y_1)\phi^*(y_1)+c\int dy_1dy_2dy_3\,g^*(x,y_1)\phi^*(y_1)g(y_1,y_2)\phi(y_2)g^*(y_2,y_3)\phi^*(y_3)+\cdots\right]. \tag{2.9}$$

The scattering data can be written in a similar form by letting $x\to\infty$:

$$a(\xi)=1+c\int dy_1dy_2\,e^{-i\xi y_1}\phi(y_1)g^*(y_1,y_2)\phi^*(y_2)$$
$$+c^2\int dy_1dy_2dy_3dy_4\,e^{-i\xi y_1}\phi(y_1)g^*(y_1,y_2)\phi^*(y_2)g(y_2,y_3)\phi(y_3)g^*(y_3,y_4)\phi^*(y_4)+\cdots\,, \tag{2.10}$$

$$b(\xi)=-i\sqrt{c}\left[\int dy_1e^{i\xi y_1}\,\phi^*(y_1)+c\int dy_1dy_2dy_3\,e^{i\xi y_1}\phi^*(y_1)g(y_1,y_2)\phi(y_2)g^*(y_2,y_3)\phi^*(y_3)+\cdots\right]. \tag{2.11}$$

Equations (2.10) and (2.11) map the fields $\phi(x)$ and $\phi^*(x)$ into a set of scattering data $a(\xi)$ and $b(\xi)$. An important aspect of the inverse method is that this mapping can be inverted. The fields $\phi(x)$ and $\phi^*(x)$ can be constructed from the solution to a Gelfand-Levitan-Marchenko integral equation whose kernel depends only on $a(\xi)$ and $b(\xi)$.[13] For our purposes, we need only note that the scattering data describe the system at a given time just as completely as the original fields. (Recall that we are only discussing the case $c>0$ with $|\phi|\to\infty$ as $x\to\pm\infty$. For $c<0$ we would need additional variables to describe the soliton sector.)

The Poisson bracket of any two functionals α and β of the fields ϕ and ϕ^* is defined by

$$\{\alpha,\beta\}=i\int dx\left[\frac{\delta\alpha}{\delta\phi(x)}\,\frac{\delta\beta}{\delta\phi^*(x)}-\frac{\delta\beta}{\delta\phi(x)}\,\frac{\delta\alpha}{\delta\phi^*(x)}\right]. \tag{2.12}$$

From this it can be shown that

$$\{a(\xi),b(\xi')\}=\left(\frac{c}{\xi-\xi'-i\epsilon}\right)a(\xi)b(\xi')\,. \tag{2.13}$$

This is most easily derived from Wronskian relations for the eigenfunctions of (2.1).[23] It is also instructive to check it for low orders directly from the series expansions (2.10) and (2.11). From (2.13) and the similar relations

$$\{a(\xi),b^*(\xi')\}=\left(\frac{-c}{\xi-\xi'-i\epsilon}\right)a(\xi)b^*(\xi')\,, \tag{2.14}$$

$$\{a^*(\xi),b(\xi')\} = \left(\frac{-c}{\xi-\xi'+i\epsilon}\right)a^*(\xi)b(\xi'), \qquad (2.15)$$

$$\{a^*(\xi),b^*(\xi')\} = \left(\frac{c}{\xi-\xi'+i\epsilon}\right)a^*(\xi)b^*(\xi'), \qquad (2.16)$$

we find that the functions

$$P(\xi)=\frac{1}{\sqrt{c}}\ \ln|a(\xi)|, \qquad (2.17)$$

$$Q(\xi)=\frac{1}{\pi\sqrt{c}}\ \arg b(\xi), \qquad (2.18)$$

have canonical Poisson brackets

$$\{P(\xi),Q(\xi')\}=\delta(\xi-\xi'). \qquad (2.19)$$

For $c>0$, the function $a(\xi)$ is analytic and nonvanishing in the lower-half ξ plane, and hence we may write

$$\ln a(\xi)=\frac{i\sqrt{c}}{\pi}\int\frac{P(\xi')d\xi'}{\xi'-\xi+i\epsilon}. \qquad (2.20)$$

The constants of motion can be obtained by comparing (2.20) with another expression for $\ln a(\xi)$. Writing

$$A(x,\xi)=e^{\lambda(x,\xi)}, \qquad (2.21)$$

we can eliminate Ψ_2 from (2.1) and obtain an equation for $\lambda'=d\lambda/dx$,

$$\phi\ \frac{d}{dx}\left(\frac{\lambda'}{\phi}\right)+\lambda'^2+i\xi\lambda'-c\,|\phi|^2=0. \qquad (2.22)$$

The solution of (2.22) can be expanded in inverse powers of ξ

$$\lambda'(x,\xi)=-ic\sum_{l=0}^{\infty}\xi^{-l-1}f_l(x). \qquad (2.23)$$

The functions f_l are given by $f_0=|\phi|^2$ and the recursion formula

$$f_{l+2}=i\phi\frac{d}{dx}\left(\frac{f_{l+1}}{\phi}\right)+c\sum_{j+k=l}f_jf_k. \qquad (2.24)$$

Integrating (2.23) and noting that $\lim_{x\to\infty}\lambda(x,\xi)=\ln a(\xi)$, we obtain expressions for the moments of $P(\xi)$ which are polynomial in ϕ and its derivatives:

$$\frac{1}{\pi\sqrt{c}}\int\xi^lP(\xi)d\xi=\int f_l(x)dx\equiv M_l. \qquad (2.25)$$

We list the first five as follows:

$$M_0=\int|\phi|^2dx, \qquad (2.26)$$

$$M_1=i\int\phi\,\phi_x^*\,dx, \qquad (2.27)$$

$$M_2=(i)^2\int(\phi\phi_{xx}^*-c\,|\phi|^4)dx, \qquad (2.28)$$

$$M_3=(i)^3\int(\phi\,\phi_{xxx}^*-3c\phi_x^*\,\phi\,|\phi|^2), \qquad (2.29)$$

$$M_4=(i)^4\int[\phi\phi_{xxxx}^*-c\phi^*\phi_{xx}\,|\phi|^2-6c\phi_{xx}^*\phi\,|\phi|^2 -5c(\phi_x^*)^2\phi^2-6c\,|\phi_x|^2|\,|\phi|^2 +2c^2\,|\phi|^6]dx. \qquad (2.30)$$

The Poisson bracket between any pair of moments vanishes

$$\{M_i,M_j\}=0\ \text{ for all } i,j, \qquad (2.31)$$

by virtue of the first expression in (2.25). In particular, since M_3 is the Hamiltonian, all of the M_l's are constants of motion.

III. THE QUANTIZED NONLINEAR SCHRÖDINGER EQUATION AS A MANY-BODY PROBLEM

A. Many-Body Wave Functions

To construct eigenstates of the quantized Hamiltonian, we divide it into free and interacting parts,

$$\hat{H}=\int dx\,(-\Phi_{xx}^*\Phi+c\Phi^*\Phi^*\Phi\Phi)\equiv\hat{H}_0+\hat{V}. \qquad (3.1)$$

At $t=0$ the momentum-space creation and annihilation operators $a_k^\dagger$ and a_k are defined in the usual way:

$$\Phi(x)=\int\frac{dk}{2\pi}\ e^{ikx}a_k, \qquad (3.2)$$

and the N-particle plane-wave eigenstates of $\hat{H}_0$ are given by

$$|k_1,k_2,\ldots,k_N\rangle\equiv a_{k_1}^\dagger a_{k_2}^\dagger\cdots a_{k_N}^\dagger|0\rangle. \qquad (3.3)$$

The states (3.3) will often be abbreviated as $|k\rangle$. Next, define the Moeller wave operator as

$$U(0,-\infty)=\lim_{t\to-\infty}e^{i\hat{H}t}e^{-i\hat{H}_0t}. \qquad (3.4)$$

The N-particle in scattering states are obtained by applying this operator to the plane wave (3.3),

$$|\Psi_N(k)\rangle\rangle=U(0,-\infty)|k\rangle. \qquad (3.5)$$

We can now summarize the relevant results of Ref. 19. By writing the perturbation series for (3.5) and summing graphs, it can be shown that the states obtained are just those of Bethe's ansatz. When computed from (3.5), the wave functions $|\Psi_N(k)\rangle\rangle$ are obtained naturally in a cluster-decomposed form of Bethe's ansatz whose terms can be associated with a set of dressed skeleton graphs. A skeleton graph contains N "quasiparticle" lines drawn vertically without intersection and with some number of "phonon" lines connecting pairs of quasiparticle lines. (An example

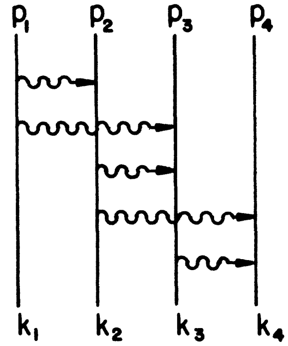

FIG. 1. A skeleton graph for the four-particle wave function.

of a four-particle graph is shown in Fig. 1.) In the billiard ball analog of the many-particle system, a quasiparticle is an entity which travels through the system at a constant velocity except for the small jumps (time advances or delays) caused by collision with other particles. The value of an undressed skeleton graph is given by the following rules:

(a) a momentum pole $(-i)(q-i\epsilon)^{-1}$ for each phonon of momentum q, (3.6a)

(b) an integration $\int dl/2\pi$ over the momentum in each closed loop. (3.6b)

We will choose a particular ordering for the asymptotic momenta of the plane wave in (3.5),

$$k_1 < k_2 < \cdots < k_N , \qquad (3.7)$$

and draw the graphs so that these momenta read from left to right across the bottom of the graph. The phonon lines will then all proceed from left to right. If we define the set of integer pairs

$$\mathcal{S}_N = \{(i,j) | \quad i < j \leq N\} , \qquad (3.8)$$

we can associate a distinct pair of quasiparticles with each member of the set $\mathcal{S}_N$. In a skeleton graph, a given pair is connected either by no phonons or by one phonon. We can therefore designate a skeleton by its "collision set" $\mathfrak{C}$ which is a subset of $\mathcal{S}_N$ containing a pair for each phonon in the graph. For example, Fig. 1 is denoted by the set $\mathfrak{C}=\{(1,2),(1,3),(2,3),(2,4),(3,4)\}$. A skeleton graph for the wave function $\langle p \,|\, \Psi_N(k)\rangle\!\rangle$ will be written $\mathcal{K}(p;\mathfrak{C})$. The dressing function of a skeleton graph consists of a factor

$$\tau_{ij} = \frac{2ic}{k_i - k_j - ic} \qquad (3.9)$$

for each pair $(i,j) \in \mathfrak{C}$, i.e., for each phonon in the graph. Finally, for notational convenience we absorb an overall $2\pi\times$(momentum-conserving δ function) in our definition of $\mathcal{K}(p;\mathfrak{C})$. Thus, in comparison with the notation of Ref. 19, we make the replacement

$$2\pi\delta\left(\sum_{i=1}^{N} p_i - \sum_{i=1}^{N} k_i\right)\mathcal{K}(p;\mathfrak{C}) \to \mathcal{K}(p;\mathfrak{C}) . \qquad (3.10)$$

With these definitions and preliminaries, we can write the many-body wave function as

$$\langle p \,|\Psi_N(k)\rangle\!\rangle = \sum_{\mathfrak{C}\subset\mathcal{P}(\mathcal{S}_N)} \left[\mathcal{K}(p;\mathfrak{C}) \prod_{(i,j)\in\mathfrak{C}} \tau_{ij}\right] . \qquad (3.11)$$

Here, $\mathcal{P}(\mathcal{S}_N)$ denotes the power set of $\mathcal{S}_N$, i.e., the set of all its subsets. $\mathcal{P}(\mathcal{S}_N)$ contains $2^{N(N-1)/2}$ elements each corresponding to a distinct skeleton graph. The connection between (3.11) and the more familiar form of Bethe's ansatz is discussed in Ref. 19.

B. Constants of Motion

Now that we have an explicit form for the N-body wave functions, we can study the operators $\hat{M}_l$ obtained by replacing the polynomial expressions of the classical theory, Eqs. (2.26)–(2.29), by their normal-ordered counterparts:

$$\hat{M}_0 = \hat{N} = \int \Phi^*\Phi\, dx , \qquad (3.12)$$

$$\hat{M}_1 = \hat{P} = i\int \Phi_x^*\Phi\, dx , \qquad (3.13)$$

$$\hat{M}_2 = \hat{H} = (i)^2\int (\Phi_{xx}^*\Phi - c\Phi^*\Phi^*\Phi\Phi)dx , \qquad (3.14)$$

$$\hat{M}_3 = (i)^3\int (\Phi_{xxx}^*\Phi - 3c\Phi_x^*\Phi^*\Phi\Phi)dx , \quad \text{etc.} \qquad (3.15)$$

Because of particle-number and total-momentum conservation, it is obvious that $|\Psi_N(k)\rangle\!\rangle$ is an eigenstate of $\hat{M}_0$ and $\hat{M}_1$ with eigenvalues equal to the zeroth and first moments of the k_i distribution:

$$\hat{M}_0\,|\Psi_N(k)\rangle\!\rangle = \omega_0(k)\,|\,\Psi_N(k)\rangle\!\rangle , \qquad (3.16)$$

$$\hat{M}_1\,|\Psi_N(k)\rangle\!\rangle = \omega_1(k)\,|\Psi_N(k)\rangle\!\rangle , \qquad (3.17)$$

where

$$\omega_l(k) \equiv \sum_{i=1}^{N} k_i^l . \qquad (3.18)$$

A well-known but less obvious property of Bethe's wave functions is that they are eigenstates of the Hamiltonian,

$$\hat{M}_2\,|\,\Psi_N(k)\rangle\!\rangle = \omega_2(k)\,|\Psi_N(k)\rangle\!\rangle \qquad (3.19)$$

Before discussing this property further, we note the following obvious conjecture based on (3.16), (3.17), and (3.19):

$$\hat{M}_l\,|\Psi_N(k)\rangle\!\rangle = \omega_l(k)\,|\Psi_N(k)\rangle\!\rangle \; . \tag{3.20}$$

To make this conjecture completely definite, we should specify the ordering prescription by which $\hat{M}_l$ is defined. We will verify (3.20) for $l=3$ and find that normal ordering is the correct prescription, as was the case for $l\leq 2$. For $l>3$ the validity of (3.20) and the correct ordering prescription for $\hat{M}_l$ remain open questions.

If we assume (3.20) for all l, it provides a basic link between the classical action variables $P(\xi)$ and the occupation-number density in the k space of Bethe's ansatz states $|\Psi_N(k)\rangle\!\rangle$. Let us define the operator

$$\hat{\rho}(k) = U a_k^\dagger a_k U^{-1}, \tag{3.21}$$

where $U=U(0,-\infty)$ is the Moeller wave operator. Then from (3.5), we obtain

$$\left[\int \xi^l\,\hat{\rho}(\xi)d\xi\right]|\Psi_N(k)\rangle\!\rangle = \omega_l(k)\,|\Psi_N(k)\rangle\!\rangle \; . \tag{3.22}$$

Thus (3.20) is just the statement that when the classical IST action variable $P(\xi)$ is quantized by replacing the polynomial (inϕ) expressions for its moments by their quantum field counterparts, one obtains $\hat{\rho}(\xi)$, up to a numerical factor $\pi\sqrt{c}$.

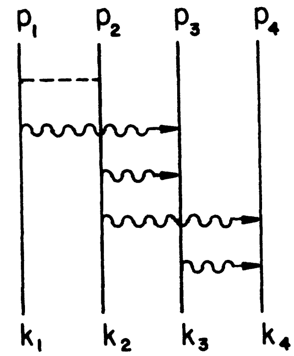

FIG. 2. A contracted skeleton graph. The dashed line represents an $|\Phi|^4$ Feynman vertex.

C. The Hamiltonian

It is instructive to verify (3.19) in momentum space, using the language of skeleton graphs. We begin with the skeleton expansion (3.11) and consider the application of the interaction Hamiltonian V to the wave function. This contracts together a pair of quasiparticle lines in a vertex which will be denoted graphically by a dashed line (see Fig. 2). Summing over pairs, we obtain an expression of the form

$$\langle p\,|\hat{V}\,|\Psi_N(k)\rangle\!\rangle = \sum_{(i,j)\in S_N}\left[\sum_{\mathfrak{C}\in\mathcal{P}(S_N)} 2c\left(\tfrac{1}{2}\right)^{\alpha_{ij}(\mathfrak{C})}\mathcal{K}_{ij}(p;\mathfrak{C}\cup\{(i,j)\})\prod_{(l,m)\in\mathfrak{C}}\tau_{lm}\right]. \tag{3.23}$$

Here, $\mathcal{K}_{ij}$ is a skeleton graph with the phonon (i,j) replaced by a dashed line, as in Fig. 2. More specifically, we can write a skeleton graph as follows:

$$\mathcal{K}(p;\mathfrak{C}) = \int\frac{dl_1}{2\pi}\cdots\frac{dl_L}{2\pi}\prod_{(l,m)\in\mathfrak{C}}\left(\frac{-i}{q_{lm}-i\epsilon}\right), \tag{3.24}$$

where L is the number of closed loops and q_{lm} is the momentum of the phonon (l,m). Then for each $(i,j)\in\mathfrak{C}$, the contracted skeleton graph $\mathcal{K}_{ij}$ is defined by

$$\mathcal{K}_{ij}(p;\mathfrak{C}) = \int\frac{dl_1}{2\pi}\cdots\frac{dl_L}{2\pi}\prod_{\substack{(l,m)\in\mathfrak{C}\\(l,m)\neq(i,j)}}\left(\frac{-i}{q_{lm}-i\epsilon}\right). \tag{3.25}$$

Also, in (3.23) we have introduced the quantity

$$\alpha_{ij}(\mathfrak{C}) = \begin{cases}1, & \text{if } (i,j)\in\mathfrak{C}\\ 0, & \text{if } (i,j)\notin\mathfrak{C}\,.\end{cases} \tag{3.26}$$

Thus an extra factor of $\frac{1}{2}$ appears in the terms of (3.23) for which the dashed-line vertex (i,j), resulting from the interaction V, contracts two lines which were already connected by a phonon. This factor emerges from the loop integration thus obtained, as discussed after Eq. (3.1) of Ref. 19.

Next we observe that in the sum over collision sets $\mathfrak{C}$ in (3.23) there are $2^{N(N-1)/2}$ terms which fall into $2^{[N(N-1)/2]-1}$ pairs. In each pair of terms, one collision set contains (i,j) and the other does not, and they are otherwise identical. Thus each pair gives a sum of the form

$$1+\tfrac{1}{2}\tau_{ij} = \left(\frac{k_i-k_j}{2ic}\right)\tau_{ij}\; . \tag{3.27}$$

The sum over such pairs can therefore be conveniently written as a sum over only those collision sets which contain (i,j), giving

$$\langle p\,|\,V\,|\Psi_N(k)\rangle\!\rangle = \sum_{(i,j)\in\mathcal{S}_N}\Bigg[\sum_{\substack{\mathfrak{C}\in\mathcal{P}(\mathcal{S}_N)\\(i,j)\in\mathfrak{C}}}(-i)(k_i-k_j)\mathcal{K}_{ij}(p;\mathfrak{C})\prod_{(l,m)\in\mathfrak{C}}\tau_{lm}\Bigg]. \tag{3.28}$$

To obtain the result in (3.19), we consider the difference between the total energy $\omega_2(k)$ and the free Hamiltonian $\hat{H}_0$ acting on the wave function

$$\langle p\,|[\omega_2(k)-\hat{H}_0]\,|\Psi_N(k)\rangle\!\rangle = \sum_{j=1}^{N}(k_j^2-p_j^2)\langle p\,|\,\Psi_N(k)\rangle\!\rangle\ . \tag{3.29}$$

Inserting the skeleton expansion (3.11) and factorizing the energy difference in (3.29), we obtain

$$\langle p\,|[\omega_2(k)-\hat{H}_0]\,|\Psi_N(k)\rangle\!\rangle = \underset{p}{\mathcal{S}}\sum_{j=1}^{N}\sum_{\mathfrak{C}\in\mathcal{P}(\mathcal{S}_N)}(k_j+p_j)(k_j-p_j)\mathcal{K}(p;\mathfrak{C})\prod_{(l,m)\in\mathfrak{C}}\tau_{lm}\ . \tag{3.30}$$

The first symbol on the right-hand side of (3.30) represents a symmetrization over the momentum variables p_i. In a skeleton graph, the momentum difference (p_j-k_j) is just the net momentum transferred to line j by all the phonons which attach to it. Using (3.24) and (3.25), this gives

$$(k_j-p_j)\mathcal{K}(p;\mathfrak{C}) = \int\frac{dl_1}{2\pi}\cdots\frac{dl_L}{2\pi}\Bigg[\sum_{\{l|\,(j,l)\in\mathfrak{C}\}}q_{jl}-\sum_{\{i|\,(i,j)\in\mathfrak{C}\}}q_{ij}\Bigg]\prod_{(m,n)\in\mathfrak{C}}\left(\frac{-i}{q_{mn}-i\epsilon}\right)$$

$$=(-i)\Bigg[\sum_{\{l|\,(j,l)\in\mathfrak{C}\}}\mathcal{K}_{jl}(p;\mathfrak{C})-\sum_{\{i|\,(i,j)\in\mathfrak{C}\}}\mathcal{K}_{ij}(p;\mathfrak{C})\Bigg]. \tag{3.31}$$

Inserting this in (3.30) and collecting terms, we obtain

$$\langle p\,|[\omega_2(k)-\hat{H}_0]|\Psi_N(k)\rangle\!\rangle = \underset{p}{\mathcal{S}}\sum_{\mathfrak{C}\in\mathcal{P}(\mathcal{S}_N)}\Bigg\{\sum_{(i,j)\in\mathfrak{C}}(-i)[k_i+p_i)-(k_j+p_j)]\mathcal{K}_{ij}(p;\mathfrak{C})\prod_{(l,m)\in\mathfrak{C}}\tau_{lm}\Bigg\}. \tag{3.32}$$

The part of (3.32) involving (p_i-p_j) vanishes because $\mathcal{K}_{ij}(p;\mathfrak{C})$ is symmetric under $p_i\leftrightarrow p_j$. Finally, we interchange the order of summation and obtain

$$\langle p\,|\,[\omega_2(k)-\hat{H}_0]\,|\Psi_N(k)\rangle\!\rangle = \sum_{(i,j)\in\mathcal{S}_N}\Bigg\{\sum_{\substack{\mathfrak{C}\in\mathcal{P}(\mathcal{S}_N)\\(i,j)\in\mathfrak{C}}}(-i)(k_i-k_j)\mathcal{K}_{ij}(p;\mathfrak{C})\prod_{(l,m)\in\mathfrak{C}}\tau_{lm}\Bigg\}. \tag{3.33}$$

This is identical to (3.28), proving (3.19).

D. The Operator $\hat{M}_3$

Using similar techniques we can now study the operator $\hat{M}_3$ which we divide into free and interacting parts,

$$\hat{M}_3 \equiv \hat{M}_3^{(1)}+\hat{M}_3^{(2)}\ , \tag{3.34}$$

where

$$\hat{M}_3^{(1)} = -\int\Phi^*_{xxx}\Phi\,dx\,, \tag{3.35}$$

and

$$\hat{M}_3^{(2)} = 3ic\int\Phi^*_x\Phi^*\Phi\Phi\,dx\ . \tag{3.36}$$

First let us consider the difference $\omega_3(k)-\hat{M}_3^{(1)}$ acting on $|\Psi_N(k)\rangle\!\rangle$,

$$\langle p\,|[\omega_3(k)-\hat{M}_3^{(1)}]\,|\Psi_N(k) = \sum_{i=1}^{N}(k_i^3-p_i^3)\langle p|\Psi_N(k)\rangle\!\rangle\ . \tag{3.37}$$

Proceeding by the same steps which led to (3.32) and using

$$(k_i^3-p_i^3)=(k_i-p_i)(k_i^2+k_ip_i+p_i^2)\,, \tag{3.38}$$

we obtain

$$\langle p\,|[\omega_3(k)-\hat{M}_3^{(1)}]\,|\Psi_N(k)\rangle = \underset{p}{\mathcal{S}} \sum_{\mathcal{C}\in\mathcal{P}(\mathcal{S}_N)} \left\{ \sum_{(i,j)\in\mathcal{C}} (-i)[(k_i^2+k_ip_i+p_i^2) - (k_j^2+k_jp_j+p_j^2)]\,\mathcal{K}_{ij}(p;\mathcal{C}) \prod_{(l,m)\in\mathcal{C}} \tau_{lm} \right\}. \tag{3.39}$$

Symmetrization in p allows us to make the replacement

$$(k_i^2+k_ip_i+p_i^2)-(k_j^2+k_jp_j+p_j^2) \to \tfrac{1}{2}(k_i-k_j)[2(k_i+k_j)+(p_i+p_j)]\,. \tag{3.40}$$

Working somewhat backwards in comparison with the previous section, we write

$$\tau_{ij} = \left(\frac{2ic}{k_i-k_j}\right)(1+\tfrac{1}{2}\tau_{ij})\,. \tag{3.41}$$

Equation (3.39) thus becomes

$$\langle p\,|[\omega_3(k)-\hat{M}_3^{(1)}]\,|\Psi_N(k)\rangle = \sum_{\mathcal{C}\in\mathcal{P}(\mathcal{S}_N)} \left\{ \sum_{(i,j)\in\mathcal{S}_N} c[2(k_i+k_j)+(p_i+p_j)](\tfrac{1}{2})^{\alpha_{ij}(\mathcal{C})} \times \mathcal{K}_{ij}(p;\mathcal{C}\cup\{(i,j)\}) \prod_{(l,m)\in\mathcal{C}} \tau_{lm} \right\}. \tag{3.42}$$

The action of the operator $M_3^{(2)}$ on the wave function is similar to that of the interaction Hamiltonian V, with an extra factor of $3(p_i+p_j)/2$ for the dashed-line vertex. This gives a result similar to (3.23):

$$\langle p\,|\hat{M}_3^{(2)}|\Psi_N(k)\rangle = \sum_{(i,j)\in\mathcal{S}_N} \left[\sum_{\mathcal{C}\in\mathcal{P}(\mathcal{S}_N)} 3c(p_i+p_j)(\tfrac{1}{2})^{\alpha_{ij}(\mathcal{C})}\mathcal{K}_{ij}(p;\mathcal{C}\cup\{(i,j)\}) \prod_{(l,m)\in\mathcal{C}} \tau_{lm} \right]. \tag{3.43}$$

Combining this with (3.41) we obtain

$$\langle p\,|[\omega_3(k)-\hat{M}_3]\,|\Psi_N(k)\rangle = \sum_{\mathcal{C}\in\mathcal{P}(\mathcal{S}_N)} \left\{ \sum_{(i,j)\in\mathcal{S}_N} 2c[(k_i+k_j)-(p_i+p_j)](\tfrac{1}{2})^{\alpha_{ij}(\mathcal{C})}\mathcal{K}_{ij}(p;\mathcal{C}\cup\{(i,j)\}) \prod_{(l,m)\in\mathcal{C}} \tau_{lm} \right\}. \tag{3.44}$$

The desired result follows from a property of skeleton graphs which is proven in the Appendix,

$$\sum_{(i,j)\in\mathcal{S}_N} [(k_i+k_j)-(p_i+p_j)](\tfrac{1}{2})^{\alpha_{ij}(\mathcal{C})}\mathcal{K}_{ij}(p;\mathcal{C}\cup\{(i,j)\}) = 0\,. \tag{3.45}$$

From this it follows that

$$\langle p\,|[\omega_3(k)-\hat{M}_3]\,|\Psi_N(k)\rangle = 0\,. \tag{3.46}$$

We have thus shown that the first four conserved quantities M_l, $l \le 3$, of the classical theory have normal-ordered operator analogs $\hat{M}_l$ in the quantum theory which are also conserved. Moreover, all four of these operators are diagonalized by Bethe's ansatz, i.e,, they are diagonal in the basis of in scattering states $|\Psi_N(k)\rangle$, with eigenvalues equal to the first four moments of the k_i distribution. A study of the higher-moment operators M_l, $l>3$ is possible using graphical techniques similar to those employed here. It seems apparent that a better understanding of the quantum-mechanical significance of the recursion formula (2.24) would be desirable. It might be hoped that an investigation along these lines would lead to a clear physical interpretation of the inverse-scattering transform in the context of quantum field theory.

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APPENDIX

To prove Eq. (3.44) we first recall the decomposition of a skeleton graph $\mathcal{K}(p;\mathcal{C})$ into "ordered plane waves" [c.f., Eq. (4.15) of Ref. 19],

$$\Phi(p;P)=N!\,\mathop{\mathrm{S}}_{p}(2\pi)\delta(\Sigma_N(p)-\Sigma_N(k))\times\prod_{l=1}^{N-1}\left[\frac{-i}{\Sigma_l(p)-\Sigma_l(Pk)-i\epsilon}\right], \quad (A1)$$

where

$$\Sigma_l(p)=\sum_{i=1}^{l}p_i\,, \quad (A2)$$

$$\Sigma_l(Pk)=\sum_{i=1}^{l}k_{P_i}\,, \quad (A3)$$

and $(P_1,P_2,\ldots,P_N)$ is some permutation of $(1,2,\ldots,N)$. A skeleton graph $\mathcal{K}(p;\mathcal{C})$ can be written as a sum of ordered plane waves $\Phi(p;P)$, where P ranges over a subset $\pi(\mathcal{C})$ of the permutation group S_N. This subset is defined by

$$\pi(i,j)=\{P\in S_N|(P^{-1})_i>(P^{-1})_j\} \quad (A4)$$

and

$$\pi(\mathcal{C})=\bigcap_{(i,j)\in\mathcal{C}}\pi(i,j)\,. \quad (A5)$$

The decomposition of a skeleton graph is

$$\mathcal{K}(p;\mathcal{C})=\sum_{P\in\pi(\mathcal{C})}\Phi(p;P)\,. \quad (A6)$$

The physical significance of this transcription is discussed in Sec. IV of Ref. 19. Here we use it to derive a similar relation for the contracted skeletons $\mathcal{K}_{ij}$ defined in (3.25). The latter can be written as a sum of contracted plane waves, which are defined by

$$\Phi_{ij}(p;P)=N!\,\mathop{\mathrm{S}}_{p}(2\pi)\delta(\Sigma_N(p)-\Sigma_N(k))\times 2\int\frac{dp_i'dp_j'}{2\pi}\,\delta(p_i'+p_j'-p_i-p_j)\times\prod_{l=1}^{N-1}\left[\frac{-i}{\Sigma_l(Pp')-\Sigma_l(Pk)-i\epsilon}\right], \quad (A7)$$

where $p_l'=p_l$ for $l\neq i,j$. The integrand in (A7) is understood to be symmetrized in p_i' and p_j' (this removes the usual spurious divergence coming from a loop integral with a single phonon[19]). By contracting both sides of (A6), we find that

$$(\tfrac{1}{2})^{\alpha_{ij}(\mathcal{C})}\mathcal{K}_{ij}(p;\mathcal{C}\cup\{(i,j)\})=\tfrac{1}{2}\sum_{P\in\pi(\mathcal{C})}\Phi_{ij}(p;P)\,. \quad (A8)$$

The result (3.44) follows from the fact that

$$\sum_{(i,j)\in\mathcal{S}_N}[(k_i+k_j)-(p_i+p_j)]\Phi_{ij}(p;P)=0\,, \quad (A9)$$

for any permutation P. We will demonstrate (A9) for $P=I=$identity. Other permutations are treated in an identical way with trivial notational changes. By contour integration of (A7), we find that $\Phi_{ij}(p;I)=0$ unless $j=i+1$, in which case

$$\Phi_{i,i+1}(p;I)\equiv\Phi_i(p;I)=N!\,\mathop{\mathrm{S}}_{p}(2\pi)\delta(\Sigma_N(p)-\Sigma_N(k))\prod_{\substack{l=1\\(l\neq i)}}^{N-1}\left[\frac{-i}{\Sigma_l(p)-\Sigma_l(k)-i\epsilon}\right]. \quad (A10)$$

Next we write

$$(k_i+k_{i+1}-p_i-p_{i+1})=[\Sigma_{i-1}(p)-\Sigma_{i-1}(k)]-[\Sigma_{i+1}(p)-\Sigma_{i+1}(k)] \quad (A11)$$

(where $\Sigma_0\equiv 0$), giving

$$(k_i+k_{i+1}-p_i-p_{i+1})\Phi_i(p;I)=N!\,\mathop{\mathrm{S}}_{p}\delta(\Sigma_N(p)-\Sigma_N(k))(i)\times\left\{\prod_{\substack{l=1\\(l\neq i,i+1)}}^{N-1}\left[\frac{-i}{\Sigma_l(p)-\Sigma_l(k)-i\epsilon}\right]-\prod_{\substack{l=1\\(l\neq i-1,i)}}^{N-1}\left[\frac{-i}{\Sigma_l(p)-\Sigma_l(k)-i\epsilon}\right]\right\}, \quad (A12)$$

for $i\neq 1$ or $N-1$, and

$$(k_1+k_2-p_1-p_2)\Phi_1(p;I)=N!\,\mathop{\mathrm{S}}_{p}(2\pi)\delta(\Sigma_N(p)-\Sigma_N(k))(i)\prod_{l=3}^{N-1}\left[\frac{-i}{\Sigma_l(p)-\Sigma_l(k)-i\epsilon}\right], \quad (A13)$$

$$(k_{N-1}+k_N-p_{N-1}-p_N)\Phi_{N-1}(p;I)=-N!\,\mathop{\mathrm{S}}_{p}(2\pi)\delta(\Sigma_N(p)-\Sigma_B(k))(i)\prod_{l=1}^{N-3}\left[\frac{-i}{\Sigma_l(p)-\Sigma_l(k)-i\epsilon}\right]. \quad (A14)$$

[Note: for $N=3$ the products in (A13) and (A14) are replaced by unity.] From (A12)–(A14) it follows that

$$\sum_{i=1}^{N-1}(k_i+k_{i+1}-p_i-p_{i+1})\Phi_i(p;I)=0\,, \quad (A15)$$

giving (A9) for the case $P=I$.

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