PHYSICAL REVIEW D VOLUME 17, NUMBER 4 15 FEBRUARY 1978

Equations of motion, variational principles, and WKB approximations in quantum mechanics and quantum field theory: Bound states

Abraham Klein and H. Arthur Weldon
Department of Physics, University of Pennsylvania, Philadelphia, Pennsylvania 19174
(Received 7 November 1977)

We describe, within the framework of the Heisenberg form of quantum mechanics, a general method for obtaining a quantization condition for bound states at the WKB level of accuracy. The method, applicable to both quantum mechanics and quantum field theory, proceeds as follows: (i) Relevant matrix elements of the equations of motions are studied in the large-quantum-number limit including first quantum corrections. (ii) These equations then are derived from several variational principles which generalize the classical versions of Hamilton's principle or the principle of least action, respectively. (iii) The quantization condition emerges in differential form from consideration of the change in either of the stationary functionals upon unit change of the quantum number of the bound state. (iv) The quantum condition in integral form thus involves an integration constant describing quantum fluctuations which is determined for every example considered by a suitable "connection formula". (v) The energy is computed in several ways, but most powerfully by employing the consequences of the quantization condition in the calculation of the expectation value of the Hamiltonian. The program outlined above is illustrated by application to one-dimensional quantum mechanics, to the nonlinear Schrödinger equation, and to the sine-Gordon model (in one space and one time dimension).

I. INTRODUCTION

Now that the initial flood of papers on quantization of solitons has passed, it is appropriate to consider what has been accomplished from a purely technical point of view[1]: In effect, a diversity of methods has been developed to carry out the equivalent of WKB approximations in field theory. To the first work involving semiclassical approximation within the path-integral formulation[2] have been added the methods of collective coordinates within the path-integral formulation,[3-5] collective coordinates and canonical quantization,[6,7] and conventional canonical quantization using Heisenberg's form of quantum mechanics.[8,9] These methods were initially applied to the quantization of topological solitons (kinks or particles) and nontopological solitons (bound states) including the first quantum correction.[10] The actual evaluation of a second quantum correction has been carried through in one instance.[11] Only within the method of collective coordinates in the path-integral formulation has the problem of scattering of solitons been carried to the first quantum correction.[12-15]

In the course of attempting to understand how the quantum corrections to scattering could be done within the Heisenberg picture, we were led to review and rethink the bound-state applications,[8-9] resulting in the present report, which includes our previous considerations but also extends them. In particular, in the previous work, we emphasized calculation of the energy (one way or another). Though such calculation is still the final goal in the present work, the most important new contribution of this paper is the development of a systematic method (in fact two such methods) for derivation of a phase-integral quantization condition. The basis for this advance was the recognition of semiclassical variational principles which represent an extension into the quantum domain of two variational principles of classical mechanics. The first of these is Hamilton's principle:

$$\delta \int_{t_1}^{t_2} dt[p\mathring{q} - H(p,q)] = 0\,,$$

where the path is varied so as to keep the time of transit constant. The second is the principle of least action:

$$\Delta \int_{t_1}^{t_2} dt\, p\mathring{q} = 0\,,$$

where the path is varied so as to keep the energy constant.

The elements of our derivation can be stated quite generally. We consider a system with a sequence of bound states labeled by an integer n. As the examples make clear, this system may be a particle system with one or many degrees of freedom, or a field theory. Let q be a fundamental operator of the system, such as a field operator or position operator: (i) We study matrix elements $(n|q|n')$ of the Heisenberg equation of motion for q. We show that for large n these can be interpreted as the Fourier components of a classical quantity $q(t)$ satisfying the classical equations of motion. A systematic expansion in orders of n^{-1} allows the first quantum correction to be included with relative ease. In principle still higher-order terms could be reached. (ii) It is not difficult to formulate a variational principle from which the equations of motion are the appropriate Euler-Lagrange equations. At the classical level, it is equivalent to Hamilton's principle. It always has the form

$$\delta_q(C_n - E_n) = 0\,, \tag{1.1}$$

where E_n is the *energy* of the state n and C_n may be viewed as a constraint on the variation of the energy. The variation is with respect to the matrix elements of q and the constraint expresses the fact that not all of these matrix elements are independent, but rather, the commutation relations limit the number of independent ones. Another variational formulation, equivalent to the principle of least action, is obtained by defining the "action" I_n by the equation

$$I_n = T_n(C_n - E_n)\,, \tag{1.2}$$

where T_n is the classical period associated with the state n [see Eq. (1.5) below]. The alternate variational principle

$$\Delta(I_n + E_n T_n) = 0 \tag{1.3}$$

replaces the fixed end-point variation associated with (1.1) by a variation of the final time, i.e., of T_n, with fixed energy E_n—precisely the variation associated with the principle of least action. (iii) A variational principle is utilized to derive a phase-integral quantization condition. Both (1.1) and (1.3) can be utilized. For introductory purposes, it is simpler to illustrate the use of (1.3). In consequences of the variational property, i.e., the equations of motion, we have

$$\frac{d}{dn}(I_n + E_n T_n) = \frac{\partial I_n}{\partial n} + \frac{dE_n}{dn} T_n\,, \tag{1.4}$$

where the partial derivative means keep q *and* T_n fixed, and it is verified that the derivative with respect to n belongs to the class of variations allowed by (1.3). In all the examples studied $(\partial I_n/\partial n) = 0$, though not always obviously so. Furthermore, the correspondence principle assures us that

$$\frac{dE_n}{dn} = \omega_n = \frac{2\pi}{T_n}\,. \tag{1.5}$$

Thus (1.4) can be integrated and yields

$$I_n + E_n T_n = T_n C_n = 2\pi n + \text{constant}\,. \tag{1.6}$$

(iv) Determination of the constant of integration is a problem special to each model. Thus in our consideration of one-dimensional quantum mechanics, we give a new derivation of the connection formula with the famous outcome, constant $= \frac{1}{2}$. For a nonrelativistic many-body model, we lean on the observation that the left-hand side of (1.6) is proportional to the number of particles, n in this case. Thus constant $= 0$. For the sine-Gordon and other relativistic field-theoretical models, the concepts of renormalization theory play an essential role in the consideration. Customarily C_n is *defined* so that constant $= 0$. (v) For the field theories (1.6) determines ω_n so that another integration is required. Alternatively we can calculate E_n directly and find that this calculation is simplified by utilization of the phase-integral condition.

In the sections which follow, this procedure is applied in whole or in part four separate times: in Sec. II to one-dimensional quantum mechanics, in Sec. III to the nonlinear Schrödinger equation as an example of a nonrelativistic field theory, in Sec. IV to the sine-Gordon model in the extreme weak-coupling limit which entails only the lowest Fourier components of the classical solution, and in Sec. V to the same model using the complete classical solution. Many of the actual details of the calculations are relegated to Appendixes A–D, associated in order with Secs. II–V.

It is almost apparent, and even true, that the methods of this paper can be extended to continuum states. For instance, once a variational principle is at hand, the variation of the stationary functional with respect to a suitable parameter such as a relative velocity yields an essential scattering relation. These developments will be presented in a subsequent work.

II. ONE-DIMENSIONAL QUANTUM MECHANICS

In this section we preface the applications to field theory by a description of how our methods work in quantum mechanics.[16]

We study the motion of a particle in one dimension described by the Hamiltonian and commutation relation ($\hbar = m = 1$)

$$H = \tfrac{1}{2}p^2 + V(x)\,, \tag{2.1}$$

$$[x, p] = i\,. \tag{2.2}$$

It is convenient to eliminate the operator p from the problem by utilizing the equation

$$p = \dot{x} = -i[x, H]\,. \tag{2.3}$$

Thus the equation of motion and commutation relation can be taken in the form

$$[[x, H], H] = \frac{dV}{dx}\,, \tag{2.4}$$

$$[[x, H], x] = 1\,, \tag{2.5}$$

and the energy can be calculated as the expectation value of

$$H = \tfrac{1}{2}[x, H][H, x] + V(x) \tag{2.6}$$

in the bound eigenstate $|n\rangle$,

$$\langle n|H|n'\rangle = E(n)\delta_{nn'} . \tag{2.7}$$

The first step in the method is to study the matrix elements of the equation of motion for large n. From (2.4) we consider

$$[E(n\pm\nu) - E(n)]^2\langle n|x|n\pm\nu\rangle = \left\langle n\left| \frac{dV}{dx}\right| n\pm\nu\right\rangle . \tag{2.8}$$

Here large n means that in any product evaluated by completeness,

$$\langle n|AB|n'\rangle = \sum_{\nu>0} (\langle n|A|n+\nu\rangle\langle n+\nu|B|n'\rangle + \langle n|A|n-\nu\rangle\langle n-\nu|B|n'\rangle), \tag{2.9}$$

upward and downward going transitions, as indicated, enter symmetrically in the sum. It is then observed that if both of Eqs. (2.8) are expanded by referring all matrix elements to a common reference matrix element, according to the formula

$$\langle n|x|n\pm\nu\rangle = \langle n-\tfrac{1}{2}\nu|x|n+\tfrac{1}{2}\nu\rangle \pm \tfrac{1}{2}\nu\partial_n\langle n-\tfrac{1}{2}\nu|x|n+\tfrac{1}{2}\nu\rangle + \cdots , \tag{2.10}$$

and introducing the definition $(\bar{n} = n + \frac{1}{2})$

$$x_\nu(\bar{n}) = x_{-\nu}(\bar{n}) \equiv \langle n-\tfrac{1}{2}\nu|x|n+\tfrac{1}{2}\nu\rangle , \tag{2.11}$$

there results a single average equation

$$[\nu\omega(\bar{n})]^2 x_\nu(\bar{n}) = \left(\frac{dV(x)}{dx}\right)_\nu , \tag{2.12}$$

where

$$\omega(\bar{n}) = \frac{dE(n)}{dn} . \tag{2.13}$$

To understand the meaning of the right-hand side of (2.12), we associate $x_\nu(\bar{n})$ with the Fourier components of a classical dynamical quantity $x(t,\bar{n})$,

$$x(t,\bar{n}) = \sum_{\nu=-\infty}^{\infty} x_\nu(\bar{n})\exp[i\nu\omega(\bar{n})t] . \tag{2.14}$$

The quantity $(dV/dx)_\nu$ is then the νth Fourier component of $[dV(x(t,\bar{n}))/dx]$, and the equation itself is then simply Newton's law in Fourier component form.

The derivation of (2.12) is given in Appendix A. The special significance of the symmetrical choice (2.11) is that only for this choice are the corrections to (2.12) of relative order n^{-2}.

The second step in the derivation of WKB is the construction of a variational principle for the equation of motion (2.12). The correct expression $L(\bar{n})$ can be written as

$$L(\bar{n}) = C(\bar{n}) - E(\bar{n}) , \tag{2.15}$$

where

$$E(\bar{n}) = \langle n|H|n\rangle = \sum_{\nu>0} [\nu\omega(\bar{n})]^2 x_{-\nu}(\bar{n})x_\nu(\bar{n}) + V(x(t,\bar{n}))_0 , \tag{2.16}$$

$$C(\bar{n}) = \langle n|\dot{x}^2|n\rangle = 2\sum_{\nu>0} [\nu\omega(\bar{n})]^2 x_{-\nu}(\bar{n})x_\nu(\bar{n}) . \tag{2.17}$$

In the present case and in many other cases, $L(\bar{n})$ has direct significance as the expectation value of the Lagrangian in the state $|n\rangle$, accurate to the first quantum correction, the same accuracy as claimed for (2.12). In (2.15) $L(\bar{n})$ has been written as a sum of two parts because of convenience for application. This separation also alludes to the quantum origin of the variational principle for the energy subject to the constraint $[x, p] = i$.[17]

The equation of motion (2.12) follows from the condition

$$\delta_{x_{-\nu}(\bar{n})} L(\bar{n}) = 0 . \tag{2.18}$$

Note that $\omega(\bar{n})$ is held fixed in this variation. This can be understood as an expression of the Rayleigh-Ritz principle, since $\omega(\bar{n})$ is defined in (2.13) as an energy difference.

The third step in the derivation of WKB is to vary $L(\bar{n})$ with respect to n. The dependence on n is both implicit (via x) and explicit (via ω). Because of (2.18), $L(\bar{n})$ is stationary under variations in x. Hence

$$\frac{dL(\bar{n})}{dn} = \frac{\partial L(\bar{n})}{\partial n} , \tag{2.19a}$$

i.e.,

$$\frac{dC(\bar{n})}{dn} - \omega(\bar{n}) = \frac{\partial C(\bar{n})}{\partial n} - \frac{\partial E(\bar{n})}{\partial n} . \tag{2.19b}$$

If $\omega(\bar{n}) \neq 0$, an elementary calculation now yields

$$1 = \frac{d}{dn} 2\sum_{\nu>0} \nu^2\omega(\bar{n})x_{-\nu}(\bar{n})x_\nu(\bar{n}) , \tag{2.20}$$

or

$$S(\bar{n}) \equiv 2\pi\sum_{\nu=-\infty}^{\infty} \nu^2\omega(n)x_{-\nu}(\bar{n})x_\nu(\bar{n}) = 2\pi(n+c) , \tag{2.21}$$

where c is a constant.

It is trivial to recognize $S(\bar{n})$ as the usual phase integral, utilizing the definition of the period,

$$\omega(\bar{n})T(\bar{n}) = 2\pi , \tag{2.22}$$

for then

$$S(\bar{n}) = T(\bar{n}) \sum_{\nu=-\infty}^{\infty} [\nu\omega(\bar{n})]^2 x_{-\nu}(\bar{n}) x_\nu(\bar{n}) \tag{2.23}$$

$$= \int_0^{T(\bar{n})} dt(\dot{x}(t,n))^2 \tag{2.24}$$

$$= \oint p\dot{x}\,dt = \oint p\,dx\,. \tag{2.25}$$

The fourth and last step in deriving the WKB approximation requires evaluation of the constant c, the famous one-half. The detailed argument for this case differs markedly from those appropriate to the field-theoretical models. Here we start by comparing the WKB case with the Bohr-Sommerfeld quantization. The latter is defined so that $S(n)$, the phase integral in (2.23), has precisely the value $2\pi n$ with no additive constant. This implies, as we verify below, a different association of matrix elements with the Fourier components of a classical dynamical variable from that taken in (2.11). In the Bohr-Sommerfeld approximation the correct association is defined by the identification

$$x_1(n) = x_1(\omega(n)) = \langle n-1|x|n\rangle\,. \tag{2.26}$$

[It is implicit in the construction of the physical Hilbert space that the matrix element occurring in (2.26) vanishes for $n=0$.] Assuming that the absolute minimum of the potential energy is at the origin of coordinates, the vanishing of the fundamental amplitude (2.26) for $n=0$ implies through the equation of motion that $x(t)=0$. This is because the equations of motion (2.12) are homogeneous in the Fourier components $x_\nu(n)$ so that it is correct to think of the "harmonics" as driven by the fundamental. The particle is thus at rest at the origin, so that $S_{\rm Bohr}(0) = E_{\rm Bohr}(0) = 0$ as required.

To complete this part of the derivation we need only note that the WKB identification (2.11) follows from the Bohr identification (2.26) by the replacement $n \to n+\frac{1}{2}$. Thus we have reached the known result.

We remark that in order to complete the derivation we have had to make a statement concerning the theory for $n=0$, where the approximation itself is not valid. This is our version of the connection formulas. Some analogous consideration will be reqiured in all other examples to be treated.

We round out the considerations of this section by showing how the WKB quantization can be obtained from a time-dependent variational principle. In this form, the considerations are more closely akin to those of the path-integral method. We define the action $I(n)$,

$$I(n) = \int_0^{T(n)} dt[\dot{x}^2(t) - H(\dot{x}(t), x(t))]$$

$$= \int_0^{T(n)} dt[\tfrac{1}{2}\dot{x}^2(t) - V(x(t))]\,, \tag{2.27}$$

where $x(t)$ is, ultimately, to be identified with (2.14). We consider a variation of $I(n)$ which is that appropriate to the principle of least action for a conservative system,[18] i.e., fixed energy but varying final time:

$$\delta x(0) = 0\,,$$
$$\delta x(T(n)) = -\dot{x}(T(n))\delta T\,. \tag{2.28}$$

Standard textbook manipulations for this variation, which we term Δ, yield what is in effect the principle of least action: The statement

$$\Delta[I(n) + E(n)T(n)] = 0 \tag{2.29}$$

implies and is implied by the classical equations of motion. If we identify $x(t)$ with $x(t,\bar{n})$ of (2.11) and (2.14), we are however, doing quantum theory to the first two orders in n.

In consequence of (2.29), however, we have

$$\frac{d}{dn}[I(\bar{n}) + E(\bar{n})T(\bar{n})] = \frac{dE(\bar{n})}{dn}T(\bar{n})$$

$$= \omega(\bar{n})T(\bar{n}) = 2\pi\,, \tag{2.30}$$

because the significance of the symbol Δ is that in carrying out the derivative, we may keep both $x(t,\bar{n})$ *and* $T(\bar{n})$ fixed. Integrating (2.30), we have

$$S(\bar{n}) = I(\bar{n}) + E(\bar{n})T(\bar{n}) = 2\pi(n+\tfrac{1}{2})\,. \tag{2.31}$$

The argument given previously leading to the known constant of integration has of course been presupposed in writing (2.31).

It can and should be checked that the derivative with respect to n is an allowed variation in the sense of (2.28). Since in this standard example (2.31) directly yields $E(n)$, it is not necessary to discuss a separate calculation for this quantity, as is either necessary or convenient in all the remaining examples of this paper. Nevertheless, it is possible to build the entire discussion about the energy, as shown in Ref. 16.

III. THE NONLINEAR SCHRÖDINGER EQUATION

As a second application of the general approach of this work, we choose the simplest of many-body problems in the guise of the nonlinear Schrödinger equation (NLSE). In this problem, the "connection formula" aspects are the simplest encountered because of the existence of a conservation law. On the other hand, the stationary functional no longer has exactly the structure of a classical Lagrangian when first quantum correlations are included. Though it would take us too far afield to present a detailed analysis of this circumstance, it can be traced to the fact that we have a complex field which is purely a lowering operator in the particle number.

Since the techniques of calculation have been illustrated in I and II, many of the details have been relegated to Appendixes.

A. WKB quantization from time-independent variational principle

From the Hamiltonian ($t=0$)

$$H=\frac{\hbar^2}{2m}\int dx\left(\frac{d}{dx}\psi^\dagger(x)\right)\left(\frac{d}{dx}\psi(x)\right) -\tfrac{1}{2}K\int dx\,\psi^\dagger(x)\psi^\dagger(x)\psi(x)\psi(x)\,, \tag{3.1}$$

and the commutation relation

$$[\psi(x),\psi^\dagger(y)]=\delta(x-y)\,, \tag{3.2}$$

we obtain the field equation ($\hbar=m=1$ henceforth)

$$i\partial_t\psi(x)=[\psi(x),H] =-\frac{1}{2}\frac{d^2}{dx^2}\psi(x)-K\psi^\dagger(x)\psi(x)\psi(x)\,. \tag{3.3}$$

From (3.3) we then deduce an equation for the matrix element

$$\Phi_{np}(x)=\frac{1}{2\pi}\int dp'\langle n-1(p')\,|\psi(x)|\,n(p)\rangle\,, \tag{3.4}$$

where

$$H\,|n(p)\rangle=\left(E_n+\frac{p^2}{2n}\right)|n(p)\rangle\,. \tag{3.5}$$

Utilizing translational and Galilean invariance, the latter in the form

$$\langle n-1(p')\,|\psi(0)|\,n(p)\rangle=\left\langle n-1\left(p'-\frac{n-1}{n}p\right)\,|\psi(0)|\,n(0)\right\rangle, \tag{3.6}$$

(3.4) can be written

$$\Phi_{np}(x)=e^{ivx}\phi_n(x),\quad v=(p/n) \tag{3.7}$$

$$\phi_n(x)=\frac{1}{2\pi}\int dp'e^{-ip'x}\langle n-1(p')\,|\psi(0)|\,n(0)\rangle\,. \tag{3.8}$$

In Appendix B, we indicate that up to the one-loop approximation $\phi_n(x)$ satisfies the equation

$$(E_n-E_{n-1})\phi_n(x)+\frac{1}{2}\left(\frac{n}{n-1}\right)\frac{d^2}{dx^2}\phi_n(x)+K\frac{(n-1)}{(n-2)}\left|\phi_{n-1}\left(\frac{n-1}{n-2}x\right)\right|^2\phi_n(x) +2K\int\frac{dk}{2\pi}|\eta_k(x)|^2\phi_n(x)+K\int\frac{dk}{2\pi}\chi_k(x)\eta_k(x)\phi_n^*(x)=0\,. \tag{3.9}$$

Here $\chi_k(x)$ and $\eta_k(x)$ are the amplitudes which describe small quantum fluctuations about the classical soliton solution (see I and Appendix B). They satisfy the equations

$$\tfrac{1}{2}k^2\chi_k(x)=-\frac{1}{2}\frac{d^2}{dx^2}\chi_k(x) -K[2\,|\phi_n(x)|^2\chi_k(x)+\phi_n{}^2(x)\eta_k^*(x)]\,, \tag{3.10}$$

$$(2\omega_n-\tfrac{1}{2}k^2)\eta_k^*(x)=-\frac{1}{2}\frac{d^2}{dx^2}\eta_k^*(x) -K\{2\,|\phi_n(x)|^2\eta_k^*(x)+[\phi_n^*(x)]^2\chi_k(x)\}\,, \tag{3.11}$$

which were derived from (3.3) in I. Here again

$$\omega_n=\frac{dE_n}{dn}\cong E_n-E_{n-1}\,. \tag{3.12}$$

For large n, (3.9) reduces in leading order to the nonlinear Schrödinger equation in the form

$$\omega_n^{(0)}\phi_n^{(0)}(x)+\frac{1}{2}\frac{d^2}{dx^2}\phi_n^{(0)}(x)+K\,|\phi_n^{(0)}(x)|^2\phi_n^{(0)}(x)=0\,, \tag{3.13}$$

for which the quantum solution is

$$\phi_n^{(0)}(x)=\frac{1}{2}\frac{K^{1/2}n}{\cosh z},\quad z=\tfrac{1}{2}Knx \tag{3.14}$$

$$\omega_n^{(0)}=-\tfrac{1}{8}K^2n^2\,. \tag{3.15}$$

The quantization "rule" which leads to (3.15) will be reviewed below.

We show next that (3.9) may, to the required approximation, be derived from a simple variational principle. This is achieved with greatest convenience if we write

$$\phi_{n-1}\left(\frac{n-1}{n-2}x\right)\cong\phi_n(x)-\frac{\partial}{\partial n}\phi_n(x) +\left[\frac{\partial}{\partial x}\phi_n(x)\right](x/n) \cong\phi_n(x)-\frac{1}{n}\phi_n(x)\,. \tag{3.16}$$

The approximation (3.16) is valid to leading order for the *correction term* since it utilizes an identity satisfied by the function (3.14). With the variational goal in mind, we rewrite (3.9) as

$$(E_n - E_{n-1})\phi_n(x) + \frac{1}{2(n-1)}\frac{d^2}{dx^2}\phi_n(x) = -\frac{1}{2}\frac{d^2}{dx^2}\phi_n(x) - K\left(1-\frac{1}{n}\right)|\phi_n(x)|^2\phi_n(x) + 2K\int\frac{dk}{2\pi}|\eta_k(x)|^2\phi_n(x) + K\int\frac{dk}{2\pi}\chi_k(x)\eta_k(x)\phi_n^*(x)\,. \tag{3.17}$$

Equation (3.17) can be then derived from a variational principle using the functional

$$L_n[\phi_n, \phi_n^*] \equiv C_n - E_n\,, \tag{3.18}$$

where E_n is the energy

$$E_n = \langle n(0)|H|n(0)\rangle$$
$$= \int dx\left\{\frac{1}{2}\left|\frac{d}{dx}\phi_n(x)\right|^2 - \tfrac{1}{2}K\left(1-\frac{1}{n}\right)[|\phi_n(x)|^2]^2\right\} + \frac{1}{2}\int\frac{dk}{2\pi}\int dx\left|\frac{d}{dx}\eta_k(x)\right|^2$$
$$- 2K\int\frac{dk}{2\pi}\int dx|\eta_k(x)|^2|\phi_n(x)|^2 - \frac{1}{2}K\int\frac{dk}{2\pi}\int dx\{\chi_k(x)\eta_k(x)[\phi_n^*(x)]^2 + \chi_k^*(x)\eta_k^*(x)[\phi_n(x)]^2\}, \tag{3.19}$$

which can be derived by the methods of Appendix B and the approximation (3.16), and C_n is the "constraint"

$$C_n = (E_n - E_{n-1})\int dx|\phi_n(x)|^2 - \frac{1}{2(n-1)}\int dx\left|\frac{d}{dx}\phi_n(x)\right|^2. \tag{3.20}$$

(Its physical meaning is that it is the tree approximation to the matrix element $\langle n(0)|\psi^\dagger(0)[\psi(0), H]|n(0)\rangle$.) In requiring

$$\delta_{\phi^*}L_n = \delta_\phi L_n = 0\,, \tag{3.21}$$

we keep n (and therefore E_n, etc.) as well as the small vibration amplitudes χ_k and η_k fixed. As shown in Appendix B, an extended variational principle, from which Eqs. (3.10) and (3.11) can also be derived, can be constructed, but it will not be needed for our purposes.

Equation (3.21) is then utilized to derive a WKB quantization rule by means of the condition

$$\frac{dL_n}{dn} = \frac{\partial L_n}{\partial n} + \left[\frac{\delta L_n}{\delta\phi(x)}\frac{\partial\phi(x)}{\partial n} + \frac{\delta L_n}{\delta\phi^*(x)}\frac{\partial\phi^*(x)}{\partial n}\right] = \frac{\partial L}{\partial n}\,, \tag{3.22}$$

which follows from (3.21). Let us apply this first in lowest order:

$$L_n^{(0)} = \omega_n^{(0)}\int dx|\phi_n^{(0)}(x)|^2 - E_n^{(0)}\,, \tag{3.23}$$

where

$$E_n - E_{n-1} = \omega_n - \frac{1}{2}\frac{d\omega_n}{dn} + \cdots \cong \omega_n^{(0)} = \frac{dE_n^{(0)}}{dn} \tag{3.24}$$

and

$$E_n^{(0)} = \int dx\left[\frac{1}{2}\left|\frac{d}{dx}\phi_n^{(0)}\right|^2 - \tfrac{1}{2}K(|\phi_n^{(0)}|^2)^2\right]. \tag{3.25}$$

We thus have

$$\frac{dL_n^{(0)}}{dn} = \frac{dC_n^{(0)}}{dn} - \omega_n^{(0)} = \frac{d}{dn}\left[\omega_n^{(0)}\int dx|\phi_n^{(0)}(x)|^2\right] - \omega_n^{(0)}\,, \tag{3.26}$$

but

$$\frac{\partial L_n^{(0)}}{\partial n} = \frac{d\omega_n^{(0)}}{dn}\int dx|\phi_n^{(0)}(x)|^2\,. \tag{3.27}$$

From (3.22), (3.26), and (3.27) we thus conclude

$$\frac{d}{dn}\int dx|\phi_n^{(0)}(x)|^2 = 1\,, \tag{3.28}$$

or with a definite assumption (to be discussed) about the constant of integration, we have the Bohr-Sommerfeld condition

$$\int dx|\phi_n^{(0)}(x)|^2 = n\,. \tag{3.29}$$

It is not surprising that this quantization condition, which was used to obtain (3.15), coincides in lowest order with the number quantization condition

$$n = \langle n(0)|\psi^\dagger(0)\psi(0)|n(0)\rangle\,. \tag{3.30}$$

We return to this point after extending the present considerations to the next order.[19]

To obtain an improved quantization condition, we utilize the full functional L_n, Eqs. (3.18), (3.19) and (3.20). From (3.22), we find immediately

$$\omega_n = \frac{dC_n}{dn} - \frac{\partial C_n}{\partial n}\bigg|_{\phi^*,\phi} + \frac{\partial E_n}{\partial n}\bigg|_{\phi^*,\phi} . \tag{3.31}$$

From (3.20), we find

$$\begin{aligned}\frac{dC_n}{dn} - \frac{\partial C_n}{\partial n} &\cong \omega_n \frac{d}{dn}\int dx\,|\phi_n(x)|^2 \\ &\quad -\frac{1}{2}\frac{d\omega_n}{dn}\frac{d}{dn}\int dx\,|\phi_n(x)|^2 \\ &\quad -\frac{1}{2n}\frac{d}{dn}\int dx\left|\frac{d\phi_n(x)}{dx}\right|^2 \\ &= \omega_n \frac{d}{dn}\int |\phi_n(x)|^2 dx ,\end{aligned} \tag{3.32}$$

since the last two terms cancel when evaluated with the help of (3.14). The last term of Eq. (3.31) is evaluated in Appendix B 2 and equals

$$\omega_n \frac{d}{dn}\int \frac{dk}{2\pi}\int dx\,|\eta_k(x)|^2 = \omega_n \frac{d}{dn}(\tfrac{1}{3}) = 0 . \tag{3.33}$$

Thus in place of (3.29), we obtain upon integration

$$\int dx\,|\phi_n(x)|^2 = n + c . \tag{3.34}$$

We can, of course, obtain the value of c by recourse to the particle number conservation equation (3.30). It is instructive for future purposes to present this argument in a disguised form. Thus the integral in (3.34) is recognized as containing the classical approximation to the expectation value

$$N_n \equiv \langle n(0)|\psi^\dagger(0)\psi(0)|n(0)\rangle = n + b , \tag{3.35}$$

i.e., the large-n approximation to N_n must go like n. In the role of a connection formula, we invoke the condition $N_0 = 0$ to discard the constant b. This is a much simpler consideration than in the preceding section where obtaining the zero-point value of the phase integral required a stratagem with a more subtle origin.

Equation (3.35) now yields the condition

$$\int dx\left[\,|\phi_n(x)|^2 + \int \frac{dk}{2\pi}\,|\eta_k(x)|^2\right] = n , \tag{3.36}$$

or using the value given in (3.33), we find

$$\int dx\,|\phi_n(x)|^2 = n - \tfrac{1}{3} . \tag{3.37}$$

The utilization of (3.37) as a quantization condition requires that we solve Eq. (3.9) to first order. This has not been done previously. It is carried out in Appendix B 3 where we then obtain from (3.37) the known result

$$\omega(n) = \omega^{(0)}(n)[1 + O(n^{-2})] . \tag{3.38}$$

In our previous work, we avoided the calculation of $\phi_n(x)$ to first order by utilizing the consequences of (3.36) to simplify the direct computation of the energy. We next give a brief review of this procedure.

B. WKB quantization from energy self-consistency

The method consists of solving the equation of motion (3.9) for $\phi_n(x, E_{n,n-1})$ where $E_{n,n-1} = E_n - E_{n-1}$. If we substitute into (3.19) and evaluate the integral over x, we obtain a difference equation

$$E_n = F_n(E_n - E_{n-1}) \cong F_n(dE_n/dn) . \tag{3.39}$$

In lowest approximation this has been shown by example in I an II to be equivalent to Bohr-Sommerfeld quantization in the form

$$n = \int \frac{dE}{(dE/dn)} = \int \frac{dE}{\omega(E)} . \tag{3.40}$$

Consistent treatment of the difference equation in the large-n approximation will then yield a consistent set of higher-order corrections.

To understand why, to first-order terms, we need only $\phi_n^{(0)}$ and χ_k, η_k, let us write (3.19) as

$$E_n = E_n^{(0)}[\phi_n] + E_n^{(1)} , \tag{3.41}$$

where $E_n^{(0)}$ is the functional of (3.25) and $E_n^{(1)}$ is explicitly a first-order correction containing only $\phi_n^{(0)}, \chi_k, \eta_k$. We now write

$$\phi_n(x) = \phi_n^{(0)} + \phi_n^{(1)} . \tag{3.42}$$

With the help of (3.13), the lowest-order equation, we find to first order

$$E_n^{(0)}[\phi_n] = E_n^{(0)}[\phi_n^{(0)}] + \omega_n^{(0)}\int dx(\phi_n^{(0)*}\phi_n^{(1)} + \phi_n^{(1)*}\phi_n^{(0)}) . \tag{3.43}$$

However, from the *difference* between (3.36) and (3.29), we learn that

$$\int dx(\phi_n^{(0)*}\phi_n^{(1)} + \phi_n^{(1)*}\phi_n^{(0)}) = -\int dx\int \frac{dk}{2\pi}\,|\eta_k(x)|^2 , \tag{3.44}$$

and therefore $\phi_n^{(1)}$, which was needed in the preceding subsection, can be eliminated from the present calculation, as was shown in I. The remainder of the calculation is as given in I.

C. WKB quantization from time-dependent variational principle

As was the case for one-dimensional quantum mechanics, the final method is closest in spirit to the path-integral approach. In Eq. (3.9) or in the more convenient approximate version (3.17), the choice of time $t=0$ was made. Instead we consider an arbitrary time and define

$$\phi_n(x,t)=\frac{1}{2\pi}\int dp' e^{-ip'x}\langle n-1(p')|\psi(0,t)|n(0)\rangle$$

$$=\frac{1}{2\pi}\int dp' e^{-ip'x}\exp[-iE_{n,n-1}t+ip'^2t/2(n-1)]\times\langle n-1(p')|\psi(0,0)|n(0)\rangle. \quad (3.45)$$

One sees thus that

$$\left[E_n-E_{n-1}+\frac{1}{2(n-1)}\partial_x{}^2\right]\phi_n(x,t)=i\partial_t\phi_n(x,t). \quad (3.46)$$

We can therefore write in place of (3.17)

$$\begin{aligned}i\partial_t\phi_n(x,t)=&-\tfrac{1}{2}\partial_x{}^2\phi_n(x,t)\\&-K\left(1-\frac{1}{n}\right)|\phi_n(x,t)|^2\phi_n(x,t)\\&+2K\int\frac{dk}{2\pi}|\eta_k(x,t)|^2\phi_n(x,t)\\&+K\int\frac{dk}{2\pi}\chi_k(x,t)\eta_k(x,t)\phi_n^*(x,t),\end{aligned} \quad (3.47)$$

where $\chi_k(x,t), \eta_k(x,t)$, which are also modified in an obvious way, have the phases necessary to maintain time translation invariance.

Equation (3.47) can be derived from the stationary property of the action

$$I_n=\int_0^{T_n}dt\int dx\,\phi_n^*(x,t)i\,\partial_t\phi_n(x,t)-\int_0^{T_n}dt\,E_n[\phi_n(x,t),\chi_k(x,t),\dots]. \quad (3.48)$$

Here E_n is the functional given in Eq. (3.19) and T_n—to be determined—is defined by the periodicity condition

$$\omega_nT_n=-2\pi \quad (\omega_n<0). \quad (3.49)$$

We consider the variation utilized in the principle of least action for a conservative system, i.e., fixed energy but varying final time:

$$\begin{aligned}&\delta\phi_n(\pm\infty,t)=0,\\&\delta\phi_n(x,0)=0,\\&\delta\phi_n(x,T_n)=-\partial_t\phi(x,t)\big|_{t=T_n}\Delta T_n\\&\qquad=-\dot{\phi}(x,T_n)\Delta T_n.\end{aligned} \quad (3.50)$$

Standard textbook manipulations for this variation yield

$$\Delta(I_n+E_nT_n)=0, \quad (3.51)$$

in consequence of (3.47), and conversely the requirement (3.51) implies (3.47).

Utilizing (3.51), it is straightforward to derive (3.37) again. We have [cf. (3.48)]

$$\frac{d}{dn}(I_n+E_nT_n)=\frac{\partial}{\partial n}\bigg|_{\phi^*,\phi,T}(I_n+E_nT_n)$$

$$=\frac{\partial}{\partial n}I_n-2\pi=-T_n\frac{\partial E_n}{\partial n}-2\pi=-2\pi, \quad (3.52)$$

since we have previously calculated $(\partial E_n/\partial n)$ and shown it to vanish. Incorporating the *established value* of the integration constant, we have

$$I_n+E_nT_n=-2\pi(n-\tfrac{1}{3}), \quad (3.53)$$

which is easily unraveled to (3.37).

Before going on, it would be well to summarize the results of this section. We have found that a WKB quantization condition could be derived from either a time-independent or a time-dependent variational principle.

The achievement of a satisfactory WKB condition requires the association of a variationally determined relation with the expectation value of a simple operator in a semiclassical state. This condition was then used to obtain the bound-state spectrum. Independently the same spectrum can be obtained (but was not) with the same input from the energy self-consistency condition. By combining the WKB condition with energy self-consistency, the simplest possible derivation results.

IV. SINE-GORDON MODEL: WEAK-COUPLING LIMIT

A. WKB quantization from time-independent variational principle

We next study the Hamiltonian

$$H(t)=\int dx[\tfrac{1}{2}(\partial_t\phi(x,t))^2+\tfrac{1}{2}(\partial_x\phi(x,t))^2+\tfrac{1}{2}m_0{}^2\phi^2-\tfrac{1}{24}\lambda\phi^4] \quad (4.1)$$

and commutation relations

$$[\phi(x,t),\partial_t\phi(y,t)]=i\delta(x-y). \quad (4.2)$$

If $|n(p)\rangle$ is the bound state ("breather") with momentum p, we utilize the amplitude

$$\phi_{n'n}(x)=\int\frac{dp'}{2\pi}\langle n'(p')|\phi(x,0)|n(0)\rangle. \quad (4.3)$$

In Appendix C 1, it is shown that for $|n-n'|\ll n$ or n', the amplitude (4.3) satisfies in the tree approximation, except for corrections of relative order n^{-2}, the equation

$$\left[(E_n-E_n')^2+\frac{d^2}{dx^2}-m_0{}^2\right]\phi_{n'n}(x)=-\tfrac{1}{6}\lambda\sum_{n''n'''}\phi_{n'n''}(x)\phi_{n''n'''}(x)\phi_{n'''n}(x), \quad (4.4)$$

where E_n is the energy of the state $|n(0)\rangle$. In II, we have, furthermore, shown that in the weak-coupling limit, it suffices for a calculation which contains the first quantum corrections to consider only the amplitudes $\phi_{n\pm1,n}$ and the "small vibration" amplitudes (see II and Appendix C 2) $\chi_k(x), \eta_k(x)$ quite analogous to those considered for the NLSE.

A further simplification compared to the considerations of II occurs if we notice that the theory takes its most convenient form in terms of the amplitude

$$\phi_{n+1/2,n-1/2}(x) = \phi_{n-1/2,n+1/2}(x) \equiv \phi_n(x), \qquad (4.5)$$

which is an analytic continuation in n. Up to the one-loop approximation, it is "shown" in Appendix C 1 that $\phi_n(x)$ is given as the solution of the equation

$$[\omega_n^2 + \partial_x^2 - m_0^2 + \tfrac{1}{2}\lambda\phi_n^2(x)]\phi_n(x) = \tfrac{1}{4}\lambda \int \frac{dk}{2\pi}[2|\chi_k(x)|^2 + 2|\eta_k(x)|^2 + \chi_k(x)\eta_k(x) + \chi_k^*(x)\eta_k^*(x)]\phi_n(x), \qquad (4.6)$$

where once again

$$\omega_n \equiv \frac{dE_n}{dn}, \qquad (4.7)$$

and omitted terms are at least of order n^{-1} compared to the one-loop terms. To leading order $\chi_k(x)$ and $\eta_k(x)$ satisfy the equations

$$E_k^2\chi_k = (-\partial_x^2 + m^2)\chi_k - \tfrac{1}{2}\lambda\phi_n^2(2\chi_k + \eta_k^*), \qquad (4.8)$$

$$(-E_k^2 + 2\omega_n^2)\eta_k^* = (-\partial_x^2 + m^2)\eta_k^* - \tfrac{1}{2}\lambda\phi_n^2(\chi_k + 2\eta_k^*), \qquad (4.9)$$

where $E_k^2 = k^2 + m^2$.

The next step is to display a variational principle from which (4.6), (4.8), and (4.9) may be deduced. In the present instance, the stationary functional is the obvious choice, namely the expectation value of the Lagrangian L, computed to the one-loop approximation. We have[20]

$$\delta L_n = \delta(C_n - E_n) = 0, \qquad (4.10)$$

where (L is size of system, not to be confused with the Lagrangian)

$$C_n = \left\langle n(0)\right| \int dx(\partial_t\phi(x,0))^2 \left|n(0)\right\rangle \Big/ L - \epsilon_0$$

$$\cong 2\omega_n^2\left[\int dx(\phi_n^2(x)) + \sum_k \int dx|\eta_k(x)|^2\right] + \sum_k \int dx\, E_k^2[|\chi_k(x)|^2 - |\eta_k(x)|^2] - \epsilon_0. \qquad (4.11)$$

and

$$E_n = \langle n(0)|H|n(0)\rangle/L$$

$$= \int dx\left\{(\omega_n^2 + m^2)\phi_n^2(x) + \left[\frac{d}{dx}\phi_n(x)\right]^2 - \tfrac{1}{4}\lambda\phi_n^4(x)\right\} - \epsilon_0$$

$$+ \sum_k \int dx\left[\tfrac{1}{2}E_k^2|\chi_k|^2 + \tfrac{1}{2}m^2|\chi_k|^2 + \frac{1}{2}\left|\frac{d}{dx}\chi_k\right|^2 + (2\omega_n^2 - E_k^2)\tfrac{1}{2}|\eta_k|^2 + \tfrac{1}{2}m^2|\eta_k|^2 + \frac{1}{2}\left|\frac{d}{dx}\eta_k\right|^2\right.$$

$$\left. - \tfrac{1}{4}\lambda\phi_n^2(x)(2|\chi_k|^2 + 2|\eta_k|^2 + \chi_k\eta_k + \chi_k^*\eta_k^*)\right] - \delta m^2 \int dx\,\phi_n^2(x). \qquad (4.12)$$

ϵ_0 is a vacuum subtraction constant and we have explicitly indicated the mass renormalization.

Understanding and correct utilization of the variational principle (4.10) must be preceded by some comments on the finiteness of the various ingredients. The first observation is that the expression for L_n (from which any vacuum subtraction constants are totally absent) is perfectly finite: First of all, $\phi_n(x)$, Eq. (4.24) below is finite and integrable. Second, derivation of the explicit one-loop corrections depending on χ_k and η_k^* shows that we should evaluate the x integral first, followed by the summation over k (for finite L and finite cutoff). In Appendix C 4, we show that from (4.8) and (4.9), when sufficient care is exercised, we can derive the identity

$$0=\sum_k \int dx \Big[E_k^2(|\chi_k|^2-|\eta_k|^2)+2\omega_n^2|\eta_k|^2 -m^2(|\chi_k|^2+|\eta_k|^2)-\left|\frac{d}{dx}\chi_k\right|^2-\left|\frac{d}{dx}\eta_k\right|^2 +\tfrac{1}{2}\lambda\phi_n^2(x)(2|\chi_k|^2+2|\eta_k|^2+\chi_k\eta_k+\chi_k^*\eta_k^*)\Big] +\delta m^2\int dx\,\phi_n^2(x). \tag{4.13}$$

Since this expression contains all the possibly divergent pieces of L_n, we thus are assured that L_n is finite.

The derivation of (4.13) given in Appendix C 4 starts from a convergent double integral and "spreads out" the terms into divergent pieces. When we decompose L_n into the difference shown in (4.10)-(4.12), which is a convenience for the purpose at hand, we lose control of this spreading out; it is by no means obvious *a priori* that the use of a single subtraction constant ϵ_0 is sufficient to render E_n and C_n finite, nor is it obvious from a superficial examination of these expressions. An inability to show that they are finite would, however, lead to a collapse of the entire approach.

The proof that E_n as defined in (4.12) is finite follows from an energy self-consistency requirement studied in Appendix C 4. There we describe the derivation of the relation (for which it suffices to ignore the motion of the heavy particle)

$$\langle n,k^{(+)}|H|n,k^{(+)}\rangle-\langle n|H|n\rangle\equiv E_k=\int dx\Big[E_k^2(|\chi_k|^2-|\eta_k|^2)+2\omega_n^2|\eta_k|^2+m^2(|\chi_k|^2+|\eta_k|^2)+\left|\frac{d}{dx}\chi_k\right|^2+\left|\frac{d}{dx}\eta_k\right|^2 -\tfrac{1}{2}\lambda\phi_n^2(x)(2|\chi_k|^2+2|\eta_k|^2+\chi_k\eta_k+\chi_k^*\eta_k^*)\Big]. \tag{4.14}$$

Equation (4.14) may be viewed as providing the correct normalization of the scattering solution for finite L. If indeed (4.14) is substituted into (4.12) the latter becomes

$$E_n=E_n^{(0)}[\phi_n]+\sum_k \tfrac{1}{2}E_k-\epsilon_0-\delta m^2\int dx\,\phi_n^2(x) \equiv E_n^{(0)}[\phi_n]+\text{“}\sum_k \tfrac{1}{2}E_k\text{”}, \tag{4.15}$$

where $E_n^{(0)}[\phi_n]$ consists of the finite terms dependent only on the classical amplitude ϕ_n—the "classical" energy of the field ϕ_n. Furthermore, the renormalized zero-point sum, first evaluated by Dashen, Hasslacher, and Neveu (DHN), has been shown to be finite.

By averaging (4.13) and the sum over k of (4.14) we derive

$$\sum_k \tfrac{1}{2}E_k-\delta m^2\int dx\,\phi_n^2(x) =\sum_k\int dx[E_k^2(|\chi_k|^2-|\eta_k|^2)+2\omega_n^2|\eta_k|^2], \tag{4.16}$$

which converts C_n into the finite expression

$$C_n=2\omega_n^2\int dx\,\phi_n^2(x)+\text{“}\sum_k \tfrac{1}{2}E_k\text{”}. \tag{4.17}$$

It is also useful to notice that the combination of (4.15) and (4.17) gives a useful expression for L_n, namely

$$L_n=2\omega_n^2\int dx\,\phi_n^2(x)-E_n^{(0)}[\phi_n]. \tag{4.18}$$

We are finally prepared to apply the variational principle in the form of the condition

$$\frac{dL_n}{dn}=\left.\frac{\partial L_n}{\partial n}\right|_{\phi,\chi,\eta,\ldots} \tag{4.19}$$

The left-hand side is evaluated with the help of (4.17),

$$\frac{dL_n}{dn}=\frac{dC_n}{dn}-\omega_n=\frac{d}{dn}(4.17)-\omega_n. \tag{4.20}$$

The right-hand side is evaluated from (4.18) as

$$\frac{\partial L_n}{\partial n}=2\omega_n\frac{d\omega_n}{dn}\int dx\,\phi_n^2(x). \tag{4.21}$$

Combining (4.20) and (4.21), we find upon rearrangement the condition

$$1=\frac{d}{dn}\left[2\omega_n\int dx\,\phi_n^2(x)+\frac{1}{\omega_n}\text{“}\sum_k \tfrac{1}{2}E_k\text{”}\right], \tag{4.22}$$

which integrates to [cf. (4.17)]

$$C_n=\omega_n n. \tag{4.23}$$

Here the constant of integration has properly been set equal to zero because C_n has been *defined* to vanish for $n=0$.

To utilize (4.11) or (4.17), we need the solution for $\phi_n(x)$ derived in Appendix C 3, namely

$$\phi_n(x)=\frac{2m\sin\alpha}{(\lambda')^{1/2}}\frac{1}{\cosh z}+\frac{m}{(\lambda')^{1/2}}\frac{1}{n}\frac{\sin\alpha}{\cosh^3 z}, \tag{4.24}$$

where we have set

$$\omega_n=m\cos\alpha,\quad z=(m\sin\alpha)x, \tag{4.25}$$

$$\lambda' = \lambda(1 - \lambda/8\pi m^2)^{-1}. \qquad (4.26)$$

With the help of (4.24), C_n is evaluated in the weak-coupling limit, $\alpha \ll 1$, in Appendix C 5. We obtain the condition

$$\frac{16m^2}{\lambda'}\alpha + \tfrac{2}{3} = n, \qquad (4.27)$$

or

$$\alpha = \frac{\lambda'}{16m^2}(n - \tfrac{2}{3}) = \alpha_0 + (\alpha_1/n),$$
$$\alpha_0 = (\lambda' n/16m^2), \quad \alpha_1 = -\tfrac{2}{3}\alpha_0. \qquad (4.28)$$

With the aid of (4.28) we shall rewrite (4.24) as

$$\phi_n(x) = \frac{2m\sin\alpha_0}{(\lambda')^{1/2}}\frac{1}{\cosh z} + \phi_n^{(1)}(z)$$
$$= \phi_n^{(0)}(\lambda', z) + \phi_n^{(1)}(z), \qquad (4.29)$$

where $\phi_n^{(1)}$ contains both the contribution from α_1 and the second term of (4.24). The significance of this decomposition is that if we write

$$C_n = C_n^{(0)} + C_n^{(1)}, \qquad (4.30)$$

where

$$C_n^{(0)} = 2\omega_n^2 \int dx\, \phi_n^{(0)2}(\lambda', z) = n\omega_n, \qquad (4.31)$$

then

$$C_n^{(1)} = 0, \qquad (4.32)$$

the analog of (3.44), is the condition which *determines* α_1.

B. WKB quantization from energy self-consistency

Following the lesson learned from the NLSE, we now go to the energy self-consistency condition in order to benefit from the information (4.31) and (4.32). Writing Eq. (4.15) in the form

$$E_n = E_n^{(0)}[\phi_n] + E_n^{(1)} , \qquad (4.33)$$

the vacuum sum $E_n^{(1)}$ may actually be written as

$$E_n^{(1)} = C_n^{(1)} - 2\omega_n^2 \int (\phi_n^{(0)}\phi_n^{(1)} + \phi_n^{(1)}\phi_n^{(0)})\,dx . \qquad (4.34)$$

The first term of (4.33) is treated with the help of the decomposition (4.29) and the lowest-order equation of motion for $\phi_n^{(0)}$. We thus find in analogy with a similar calculation in the last section,[21]

$$E_n^{(0)}[\phi_n] = E_n^{(0)}[\phi_n^{(0)}(\lambda', z)]$$
$$+ 2\omega_n^2 \int (\phi_n^{(0)}\phi_n^{(1)} + \phi_n^{(1)}\phi_n^{(0)})\,dx . \qquad (4.35)$$

Adding (4.34) and (4.35) and recalling (4.32) yields

$$E_n = E_n^{(0)}[\phi_n^{(0)}(\lambda', z)]$$
$$= \frac{16m^3}{\lambda'}\sin\frac{\lambda' n}{16m^2} , \qquad (4.36)$$

in agreement with previous results.[2,22]

We shall be content here with this single derivation of the spectrum. We shall not go into the time-dependent variational principle either, but shall give the corresponding derivation in the next section.

V. SINE-GORDON MODEL: COMPLETE CALCULATION

We turn then to the Hamiltonian

$$H(t) = \int dx[\tfrac{1}{2}(\partial_t\phi(x,t))^2 + \tfrac{1}{2}(\partial_x\phi(x,t))^2]$$
$$+ m_0^2\frac{m^2}{\lambda}\left(1 - \cos\frac{\sqrt{\lambda}}{m}\phi\right) , \qquad (5.1)$$

with the associated field equation

$$(-\partial_t^2 + \partial_x^2)\phi(x,t) - \frac{m_0^2 m}{\sqrt{\lambda}}\sin\frac{\sqrt{\lambda}}{m}\phi(x,t) = 0 . \qquad (5.2)$$

The properties of the bound states under study are completely determined up to the one-loop level by two functions. The first is the "breather,"

$$Y^{(0)}(x,\omega,\theta) = \frac{4m}{\sqrt{\lambda}}\tan^{-1}\left(\frac{\tan\alpha\cos\theta}{\cosh z}\right) , \qquad (5.3)$$

where

$$\omega = m\cos\alpha , \quad z = (m\sin\alpha)x , \qquad (5.4)$$

which is a solution of (5.2) considered as a classical equation, with $\theta = \omega t$. As explained in II and implied repeatedly by the prëvious sections of this paper, $Y^{(0)}$ is a generating function, through its Fourier series in θ, of the matrix elements of the field operator $\phi(x)$ among the bound states $|n\rangle$, in a limit in which all recoil is ignored. The same arguments as were applicable in the preceding section show that ignoring recoil cannot introduce an error greater than $O(n^{-2})$. We have

$$\frac{1}{2\pi}\int_0^{2\pi} d\theta\, e^{-i\nu\theta}Y^{(0)}(x,\omega(n),\theta)$$
$$\cong \langle n - \tfrac{1}{2}\nu|\,\phi(x)\,|n + \tfrac{1}{2}\nu\rangle , \qquad (5.5)$$

where the notation implies that the relation

$$\omega = \omega(n) \qquad (5.6)$$

has been obtained from a suitable quantization condition, as developed, for example, in this section.

We are, in fact, interested in a function $Y(x,\omega,\theta)$ which corrects $Y^{(0)}$ up to the one-loop approximation. The procedure for deriving an equation for this function is the same as that used at the beginning of the preceding section. The function Y still has the significance (5.5) but now obeys the equation

$$\omega^2\partial_\theta{}^2Y-\partial_x{}^2Y+\frac{m_0{}^2m}{\sqrt{\lambda}}\sin\left(\frac{\sqrt{\lambda}}{m}Y\right)-\tfrac{1}{2}m_0{}^2\frac{\sqrt{\lambda}}{m}\sum_k|\psi_k|^2\sin\frac{\sqrt{\lambda}}{m}Y=0\ . \tag{5.7}$$

Here $\psi_k^\dagger(x,n,\theta)$, the second of the functions required in this section, is defined in analogy with (5.5),

$$\frac{1}{2\pi}\int_0^{2\pi}d\theta\, e^{-i\nu\theta}\psi_k^\dagger(x,n,\theta)\cong\langle n-\nu\,|\,\phi(x)\,|\,n,k^{(+)}\rangle\ , \tag{5.8}$$

and the state $|n,k^{(+)}\rangle$ is the scattering state of a meson incident on the bound state $|n\rangle$. As shown in II, $\psi_k^\dagger$ satisfies the equation

$$E_k{}^2\psi_k^\dagger+2i\omega E_k\partial_\theta\psi_k^\dagger-\omega^2\partial_\theta{}^2\psi_k^\dagger+\partial_x{}^2\psi_k^\dagger-m_0{}^2\left(\cos\frac{\sqrt{\lambda}}{m}Y\right)\psi_k^\dagger=0\ , \tag{5.9}$$

which was, of course, derived from (5.2) by the arguments given in II. In Appendix D1 we describe the derivation of $\psi_k^\dagger(x,n,\theta)$ from a suitable multi-soliton classical solution. We find[22]

$$(2E_kL)^{1/2}\exp(-ikx)\psi_k^\dagger(x,n,\theta)=\tfrac{1}{2}(1+e^{2i\delta})+\frac{1}{2}\frac{\cosh z\,\sinh z\,(e^{2i\delta}-1)}{D}-\tfrac{1}{2}\tan^2\alpha\cos^2\theta\frac{(1+e^{2i\delta})}{D}-\frac{1}{2}\frac{\tan^2\alpha\cos\theta e^{i\delta}}{D}\left(\rho e^{-i\theta}+\frac{1}{\rho e^{-i\theta}}\right)\ , \tag{5.10}$$

where

$$D=\cosh^2z+\tan^2\alpha\cos^2\theta\ , \tag{5.11}$$

$$\rho=-\frac{E-m\cos\alpha}{E+m\cos\alpha}, \tag{5.12}$$

and

$$\tan\delta=\frac{2\nu}{\nu^2-1},\quad \nu=k/m\sin\alpha\ . \tag{5.13}$$

It is important to note two properties of the function $\psi_k^\dagger(x,n,\theta)$:

$$\lim_{k\to\infty}\psi_k^\dagger(x,n,\theta)(2E_kL)^{1/2}\exp(-ikx)=1\ , \tag{5.14}$$

$$\lim_{L\to\infty}\int dx\,|\,\psi_k(x,n,\theta)\,|^2=(2E_k)^{-1}\ . \tag{5.15}$$

The equations of motion (5.7) and (5.9) can be derived from a variational principle of the "time-independent" variety,[20]

$$\delta_Y(C_n-E_n)=\delta_{\psi_k}(C_n-E_n)=0\ , \tag{5.16}$$

where

$$E_n=\frac{1}{2\pi}\int_0^{2\pi}d\theta\int dx\left[\tfrac{1}{2}\omega^2(\partial_\theta Y)^2+\tfrac{1}{2}(\partial_xY)^2-\frac{m_0m^2}{\lambda}\left(\cos\frac{\sqrt{\lambda}}{m}Y-1\right)\right]+\sum_k\int_0^{2\pi}\frac{d\theta}{2\pi}\int dx\left\{\tfrac{1}{2}E_k{}^2\,|\,\psi_k\,|^2+\tfrac{1}{2}i\omega E_k[(\partial_\theta\psi_k^\dagger)\psi_k-\psi_k^\dagger\partial_\theta\psi_k]+\tfrac{1}{2}\omega^2\,|\,\partial_\theta\psi_k\,|^2+\tfrac{1}{2}\,|\,\partial_x\psi_k\,|^2+\tfrac{1}{2}m_0{}^2\left(\cos\frac{\sqrt{\lambda}}{m}Y\right)|\,\psi_k\,|^2\right\}-\epsilon_0\ , \tag{5.17}$$

$$C_n=\frac{1}{2\pi}\int_0^{2\pi}d\theta\int dx\,\omega^2(\partial_\theta Y)^2+\sum_k\int_0^{2\pi}\frac{d\theta}{2\pi}\int dx\left\{E_k{}^2\,|\,\psi_k\,|^2+i\omega E_k[(\partial_\theta\psi_k^\dagger)\psi_k-\psi_k^\dagger\partial_\theta\psi_k]+\omega^2\,|\,\partial_\theta\psi_k\,|^2\right\}-\epsilon_0\ . \tag{5.18}$$

Again we must establish the finiteness of L_n, E_n, and C_n. These results are proved by considering the generalizations of Eqs. (4.13)–(4.17), which can be derived by the arguments of Appendixes C and D, but will be stated here without further ado. From (5.9), we derive the generalization of (4.13), namely

$$0=\sum_k \int_0^{2\pi} \frac{d\theta}{2\pi} \int dx \Big\{ E_k{}^2 |\psi_k|^2 + i\omega E_k[(\partial_\theta \psi_k^\dagger)\psi_k - \psi_k^\dagger \partial_\theta \psi_k] + \omega^2 |\partial_\theta \psi|^2 - \tfrac{1}{2}m^2|\psi_k|^2 - \tfrac{1}{2}|\partial_x\psi_k|^2 - \tfrac{1}{2}m^2\Big(\cos\frac{\sqrt{\lambda}}{m}Y-1\Big)|\psi_k|^2\Big\}$$
$$-(\delta m^2)\frac{m^2}{\lambda}\int_0^{2\pi}\int dx\Big(\cos\frac{\sqrt{\lambda}}{m}Y-1\Big)\ . \tag{5.19}$$

This is easily seen to guarantee the finiteness of $L_n = C_n - E_n$.

The generalization of (4.14) is

$$E_k = \int_0^{2\pi}\frac{d\theta}{2\pi}\int dx\Big\{E_k{}^2|\psi_k|^2 + i\omega E_k[(\partial_\theta\psi_k^\dagger)\psi_k - \psi_k^\dagger\partial_\theta\psi_k] + \omega^2|\partial_\theta\psi_k|^2 + \tfrac{1}{2}m^2|\psi_k|^2 + \tfrac{1}{2}|\partial_x\psi_k|^2 + \tfrac{1}{2}m^2\Big(\cos\frac{\sqrt{\lambda}}{m}Y-1\Big)|\psi_k|^2\Big\}\ , \tag{5.20}$$

which, once again, provides the correct normalization of ψ_k for finite normalization volume L. From (5.20), we deduce the energy in the form

$$E_n = E_n^{(0)}[Y] + E_n^{(1)}\ , \tag{5.21}$$

$$E_n^{(0)} = \int_0^{2\pi}\frac{d\theta}{2\pi}\int dx\Big[\tfrac{1}{2}\omega^2(\partial_\theta Y)^2 + \tfrac{1}{2}(\partial_x Y)^2 - \frac{m^4}{\lambda}\Big(\cos\frac{\sqrt{\lambda}}{m}Y-1\Big)\Big]\ , \tag{5.22}$$

$$E_n^{(1)} = \tfrac{1}{2}\sum_k E_k - \epsilon_0 + (\delta m^2)\frac{m^2}{\lambda}\int_0^{2\pi}\frac{d\theta}{2\pi}\int dx\Big(\cos\frac{\sqrt{\lambda}}{m}Y-1\Big)\ . \tag{5.23}$$

Using the results

$$\int_0^{2\pi}\frac{d\theta}{2\pi}\int dx\Big(\cos\frac{\sqrt{\lambda}}{m}Y-1\Big) = -\frac{8}{m}\sin\alpha\ ,$$
$$\delta m^2 = -\frac{\lambda}{8\pi}\int\frac{dk}{E_k}\ , \tag{5.24}$$

one verifies that the last term of (5.23) has the same value as its weak-coupling limit, thus guaranteeing the finiteness of $E_n^{(1)}$ and consequently of E_n.

Finally, by combining (5.19) and (5.20), we have

$$\sum_k \tfrac{1}{2}E_k + (\delta m^2)\frac{m^2}{\lambda}\int_0^{2\pi}d\theta\int dx\Big(\cos\frac{\sqrt{\lambda}}{m}Y-1\Big)$$
$$=\sum_k\int_0^{2\pi}\frac{d\theta}{2\pi}\int dx[E_k{}^2|\psi_k|^2 + i\omega E_k[(\partial_\theta\psi_k^\dagger)\psi_k - \psi_k^\dagger\partial_\theta\psi_k] + \omega^2|\partial_\theta\psi_k|^2]\ , \tag{5.25}$$

and this guarantees that C_n has the form

$$C_n = \int_0^{2\pi}\frac{d\theta}{2\pi}\int dx\,\omega^2(\partial_\theta Y)^2 + E_n^{(1)} \tag{5.26}$$

and is finite.

The proof that

$$C_n = n\omega_n \tag{5.27}$$

goes through precisely as in the preceding section for the time-independent variational principle.

We also give the time-dependent version of our proof. This time-dependent version involves the action which can be written as

$$I_n = T_n(C_n - E_n)\ , \tag{5.28}$$

where T_n is the period: $\omega_n T_n = 2\pi$. By a change of variable, $\theta = \omega_n t$, we have

$$I_n=\int_0^{T_n}dt\int dx\left[\tfrac{1}{2}(\partial_t Y)^2-\tfrac{1}{2}(\partial_x Y)^2+m_0^2\left(\frac{m^2}{\lambda}\right)\left(\cos\frac{\sqrt{\lambda}}{m}Y-1\right)\right]$$
$$+\sum_k\int_0^{T_n}dt\int dx\left\{\tfrac{1}{2}|\partial_t\psi_k|^2+iE_k[(\partial_t\psi_k^\dagger)\psi_k-\psi_k^\dagger\partial_t\psi_k]+\tfrac{1}{2}E_k^2|\psi_k|^2-\tfrac{1}{2}|\partial_x\psi_k|^2-\tfrac{1}{2}m^2\left(\cos\frac{\sqrt{\lambda}}{m}Y\right)|\psi_k|^2\right\}. \quad (5.29)$$

We then derive the equations of motion from the principle of least action

$$\Delta(I_n+E_nT_n)=0\ , \quad (5.30)$$

where the variations vanish at spatial infinity, and at $t=0$, but

$$\delta Y(x,n,T_n)=-\partial_{T_n}Y(x,n,T_n)\Delta T_n\ , \quad (5.31)$$

$$\delta\psi_k(x,n,T_n)=-\partial_{T_n}\psi_k(x,n,T_n)\Delta T_n\ . \quad (5.32)$$

The quantization condition follows from the equation

$$\frac{d}{dn}(I_n+E_nT_n)=\frac{\partial}{\partial n}(I_n+E_nT_n)$$
$$=\omega_nT_n=2\pi\ , \quad (5.33)$$

where the partial derivative is to be computed at fixed Y, $\psi^\dagger$, ψ, and T_n. (This entails $\partial_n I_n=0$.) The quantization condition becomes

$$I_n+E_nT_n=T_nC_n=2\pi n \quad (5.34)$$

provided that we have been careful to define C_n so that $C_0=0$ *and* C_n finite for any n.

Let us apply (5.34) in the form

$$\omega_n n=\frac{1}{2\pi}\int_0^{2\pi}d\theta\int dx[\omega_n^2(\partial_\theta Y)^2]+E_n^{(1)}. \quad (5.35)$$

In lowest order, we have

$$\frac{1}{2\pi}\int_0^{2\pi}d\theta\int dx\,\omega^2(\partial_\theta Y)^2=n\omega\,. \quad (5.36)$$

This yields the well-known condition

$$\frac{16m^2}{\lambda}\alpha=n\,. \quad (5.37)$$

To carry the calculation to the one-loop level, we require the solution of (5.7). This is obtained in Appendix D2 and its structure discussed there. In particular, it is seen that the renormalization $\lambda\to\lambda'$, which played as essential role in the preceding section is associated only with the first Fourier coefficient of the full solution.

We next apply (5.35). Comparison with (4.27) assures us that the result must be of the form

$$\frac{16m^2}{\lambda'}\alpha+\tfrac{2}{3}+f(\alpha)=n\,, \quad (5.38)$$

where $f(\alpha)$ is $O(\alpha^2)$ for small α and is discussed in Appendix D3. The structure of this result allows us to carry through the remainder of the argument very much in parallel with the discussion which follows (4.28). Thus we find again that

$$\alpha=\alpha_0+(\alpha_1/n)\,, \quad (5.39)$$

where α_0 and α_1 are both of order unity and only α_1 differs from the weak-coupling value.

We must finally compute the energy utilizing the results of the quantization procedure. Substituting the solution of Appendix D2 in Eqs. (5.21)–(5.23), we have

$$E_n^{(0)}[Y^{(0)}+Y^{(1)}]\cong E_n^{(0)}[Y^{(0)}]$$
$$+2\omega^2\int_0^{2\pi}\frac{d\theta}{2\pi}\int dx(\partial_\theta Y^{(0)})(\partial_\theta Y^{(1)})$$
$$\cong\frac{16m^3}{\lambda}\sin\alpha-\frac{2m}{\pi}\sin\alpha$$
$$+\tfrac{2}{3}m+mO(\alpha^2)\,. \quad (5.40)$$

On the other hand, as already calculated in Appendix C5,

$$E_n^{(1)}=mO(\alpha^2)\,. \quad (5.41)$$

Since the sum of the $O(\alpha^2)$ terms in (5.40) and (5.41) precisely defines the $f(\alpha)$ in (5.38) and writing α in the form (5.39) one sees that the sum of (5.40) and (5.41) becomes

$$E_n=\frac{16m^2}{\lambda'}\sin\frac{\lambda' n}{16m^2}\,, \quad (5.42)$$

as previously derived.

We have thus shown by application to a variety of examples the availability, at least up to the one-loop level, of a consistent WKB quantization scheme within the framework of conventional canonical quantization in the Heisenberg picture.

We see no difficulty of principle in extending these calculations to higher order. We shall also show in subsequent work how similar techniques can be applied to scattering problems.

ACKNOWLEDGMENTS

The authors thank Dr. F. Krejs for his participation in the initial stages of this work. Support for this work was given to A. K. in part by the U. S. Energy Research and Development Administration under Contract No. AT(E11-1) 3071 Theoretical, and to A. W. in part by the National Science Foundation.

APPENDIX A

We show that if $A(x)$ is any polynomial in x, then

$$\langle n|A(x)|n+\nu\rangle+\langle n|A(x)|n-\nu\rangle=2(A(x))_\nu[1+O(n^{-2})], \tag{A1}$$

where $A(x)_\nu$ is the νth Fourier component of $A(x)$, defined as follows: First let $A(x)$ be $x^{\alpha+1}$, α an integer. Then

$$(x^{\alpha+1})_\nu=\sum_{\nu_1\nu_2\cdots\nu_\alpha}\langle n-\tfrac{1}{2}\nu_1|x|n+\tfrac{1}{2}\nu_1\rangle\langle n-\tfrac{1}{2}(\nu_2-\nu_1)|x|+\tfrac{1}{2}(\nu_2-\nu_1)\rangle$$
$$\times\langle n-\tfrac{1}{2}(\nu_3-\nu_2)|x|n+\tfrac{1}{2}(\nu_3-\nu_2)\rangle\cdots\langle n-\tfrac{1}{2}(\nu-\nu_\alpha)|n|n+\tfrac{1}{2}(\nu-\nu_\alpha)\rangle$$
$$\equiv\sum_{\nu_1\cdots\nu_\alpha}x_{\nu_1}x_{\nu_2-\nu_1}x_{\nu_3-\nu_2}\cdots x_{\nu-\nu_\alpha}. \tag{A2}$$

The polynomial $(A(x))_\nu$ is the sum of terms of form (A2).

Proof. Let $A(x)=xB(x)$. Then

$$\langle n|A|n+\nu\rangle+\langle n|A|n-\nu\rangle=\sum_{\nu_1}\langle n|x|n-\nu_1\rangle\langle n-\nu_1|B|n+\nu\rangle+\sum\langle n|x|n+\nu_1\rangle\langle n+\nu_1|B|n-\nu\rangle. \tag{A3}$$

Next we expand

$$\langle n|x|n\pm\nu_1\rangle=x_{\nu_1}\pm\tfrac{1}{2}\nu_1\partial_n x_{\nu_1},\ \langle n\mp\nu_1|B|n\pm\nu\rangle=\langle n|B|n\pm(\nu+\nu_1)\rangle\mp\nu_1\partial_n B_{\nu+\nu_1}. \tag{A4}$$

After substituting (A4) into (A3) the first point to notice is that the sum of terms which are linear in ∂_n and therefore nominally $O(n^{-1})$ cancels. All arguments, of course, presuppose that a sum over any index ν_i converges within a range of values $|\nu_i|\ll n$.

We thus find, using $x_{\nu_1}=x_{-\nu_1}$,

$$\langle n|A|n+\nu\rangle+\langle n|A|n-\nu\rangle=\sum_{\nu_1}x_{\nu_1}[\langle n|B|n+\nu+\nu_1\rangle+\langle n|B|n-(\nu+\nu_1)\rangle]$$
$$=\sum_{\nu_1\nu_2}x_{-\nu_1}x_{-\nu_2}[\langle n|C|n+\nu+\nu_1+\nu_2\rangle+\langle n|C|n-(\nu+\nu_1+\nu_2)\rangle].$$

Here we have written $B=xC$ and noticed that in the sum over ν_1 each term presents the same problem as the original one analyzed from (A3) forward. We have only to continue the argument until C has been fully analyzed to reach the conclusion (A1).

APPENDIX B: NONLINEAR SCHRÖDINGER EQUATION

1. *Equation of motion to one loop*

We derive Eq. (3.9). Consider

$$\int\frac{dp'}{2\pi}\langle n-1(p')|[\psi(x,0),H]|n(0)\rangle=\int\frac{dp'}{2\pi}\left[E_n-E_{n-1}-\frac{p'^2}{2(n-1)}\right]e^{-ip'x}\langle n-1(p')|\psi(0)|n(p)\rangle$$
$$=\left[(E_n-E_{n-1})+\frac{1}{2(n-1)}\frac{d^2}{dx^2}\right]\phi_n(x). \tag{B1}$$

The second term of (B1) combines with the usual "kinetic energy" to yield the reduced mass factor $n/(n-1)$ apparent in (3.9). Next consider the interaction term in the tree approximation,

$$K\int\frac{dp'}{2\pi}\frac{dp''}{2\pi}\frac{dp'''}{2\pi}e^{-ip'x}\langle n-1(p')|\psi^\dagger(0)|n-2(p'')\rangle\langle n-2(p'')|\psi(0)|n-1(p''')\rangle\langle n-1(p''')|\psi(0)|n(0)\rangle. \tag{B2}$$

This is easily evaluated using Galilean invariance in the form (3.6) and the inverse of (3.8) to yield the third term of (3.9).

To obtain the one-loop contribution we shall consider a typical contribution in some detail. In this typical contribution as well as in the remaining ones we encounter the amplitudes

$$\eta_{nk}(x) = \int \frac{dp'}{2\pi}\, e^{-ip'x}\left\langle (n-2)\left(-k+p'\,\frac{(n-2)}{(n-1)}\right), k+\frac{p'}{n-1}\,|\,\psi(0)\,|\,n(0)\right\rangle$$

$$\cong \int \frac{dp'}{2\pi}\, e^{-ip'x}\langle (n-2)(-k+p'), k\,|\,\psi(0)\,|\,n(0)\rangle\,, \tag{B3}$$

$$\chi_{nk}(x) = \int \frac{dp'}{2\pi}\, e^{-ip'x}\langle n(p')\,|\,\psi(0)\,|\,n(-k), k\rangle\,. \tag{B4}$$

In (B3), the second writing neglects terms of relative order n^{-1}. We then consider a contribution to the interaction

$$K \int \frac{dk}{2\pi}\frac{dp'}{2\pi}\frac{dp''}{2\pi}\frac{dp'''}{2\pi}\, e^{-ip'x}\left\langle n-1(p')\,|\,\psi^{+}(0)\,|\,(n-3)\left[-k+p''\left(\frac{n-3}{n-2}\right)\right], k+\frac{p''}{n+2}\right\rangle$$

$$\times\left\langle n-2\left[-k+p''\left(\frac{n-3}{n-2}\right)\right], k+\frac{p''}{n-2}\,|\,\psi(0)\,|\,n-1(p''')\right\rangle\langle n-1(p''')\,|\,\psi(0)\,|\,n(0)\rangle\,. \tag{B5}$$

If we utilize Galilean invariance in a typical form

$$\left\langle n-3\left[-k+p''\left(\frac{n-3}{n-2}\right)\right], k+\frac{p''}{n+2}\,|\,\psi(0)\,|\,n-1(p''')\right\rangle$$

$$=\left\langle n-3\left[-k+p''\left(\frac{n-3}{n-2}\right)-p'''\left(\frac{n-3}{n-1}\right)\right], k+\frac{p''}{n+2}-\frac{p'''}{n-1}\,|\,\psi(0)\,|\,n-1(0)\right\rangle.$$

and keep everywhere only the leading terms in $n-1$, we recognize that we may apply the faltung theorem for Fourier transforms and obtain for (B5)

$$(\text{B5}) = K\int \frac{dk}{2\pi}\,|\,\eta_{nk}(x)\,|^2\phi_n(x)\,. \tag{B6}$$

This is one of three terms contributing to the one-loop approximation. The others are obtained similarly. Equations (3.10) and (3.11) satisfied in leading order by these amplitudes have been obtained in I as Eq. (4.12) and the solutions given in Eq. (4.18).

We remark that in carrying through the above considerations, it was not necessary to use time $t=0$. A common arbitrary "spectator" time could have been utilized in all derivations.

2. *An extended variational principle*

Here we show that the variational principle (3.21) can be modified so as to provide a stationary property with respect to variations of $\chi_k, \chi_k^*, \eta_k, \eta_k^*$ as well as with respect to variations of ϕ and ϕ^*. Toward this end, we first rewrite E_n with the help of special examples of the completeness relation, also utilized in I,

$$\int \frac{dk}{2\pi}\,|\,\eta_k(x)\,|^2 = \int \frac{dk}{2\pi}\,|\,\chi_k(x)\,|^2 - \delta(0) + \frac{2}{n}\,|\,\phi_n^{(0)}(x)\,|^2\,, \tag{B7}$$

$$\int \frac{dk}{2\pi}\left|\frac{d}{dx}\eta_k(x)\right|^2 = \int \frac{dk}{2\pi}\left|\frac{d}{dx}\chi_k(x)\right|^2 - \lim_{x\to y}\frac{d}{dx}\frac{d}{dy}\delta(x-y)$$

$$+(6/n)\left|\frac{d}{dx}\phi_n^{(0)}(x)\right|^2 - (2/n)\frac{d}{dx}\left[\phi_n^{(0)}(x)\frac{d}{dx}\phi_n^{(0)}(x)\right]. \tag{B8}$$

With the help of (B2), Eq. (3.19) can be rewritten

$$E_n = \tfrac{1}{2}\int dx\left|\frac{d}{dx}\phi_n(x)\right|^2 - \tfrac{1}{2}K\left(1-\frac{1}{n}\right)\int dx[\,|\,\phi_n(x)\,|^2]^2$$

$$+\tfrac{1}{4}\int \frac{dk}{2\pi}\int dx\left[\left|\frac{d}{dx}\eta_k(x)\right|^2 + \left|\frac{d}{dx}\chi_k(x)\right|^2\right] - \tfrac{1}{4}\int dx\int \frac{dk}{2\pi}k^2 + \frac{3}{2n}\int dx\left|\frac{d}{dx}\phi_n^{(0)}(x)\right|^2$$

$$-K\int \frac{dk}{2\pi}\int dx[\,|\,\eta_k(x)\,|^2 + |\,\chi_k(x)\,|^2]\,|\,\phi_n(x)\,|^2 + K\delta(0)\int |\,\phi_n(x)\,|^2 dx$$

$$-\frac{2K}{n}\int |\,\phi_n^{(0)}(x)\,|^2\,|\,\phi_n(x)\,|^2 - \tfrac{1}{2}K\int \frac{dk}{2\pi}\int dx[\phi_n(x)^2\chi_k^*(x)\eta_k^*(x) + \text{c.c.}]\,. \tag{B9}$$

To have a variational principle for the new amplitudes, the constraint C_n must be augmented to a new quantity D_n,

$$D_n = (E_n - E_{n-1}) \int dx\, |\phi_n(x)|^2 - \frac{1}{2(n-1)} \int dx \left| \frac{d}{dx}\phi_n(x)\right|^2 + \omega_n \int dx \int \frac{dk}{2\pi} |\eta_k(x)|^2 + \int dx \int \frac{dk}{2\pi} \tfrac{1}{4}k^2[|\chi_k(x)|^2 - |\eta_k(x)|^2]. \quad \text{(B10)}$$

With the aid of (3.9), (3.10), and (3.11), we find

$$\delta_{\phi^*}(D_n - E_n) = \delta_{\eta^*}(D_n - E_n) = \delta_{\chi_k^*}(D_n - E_n) = 0. \quad \text{(B11)}$$

The extended variational principle is particularly convenient for the evaluation of $(\partial E/\partial n)|_{\phi,\phi^*}$ required in (3.31). If we calculate this derivative from (B9) we must differentiate explicitly $\phi_n^{(0)}$ (but not ϕ_n), n, χ_k, and η_k. The terms not involving χ_k and η_k (and their conjugates) turn out to cancel to the order required:

$$\left.\frac{\partial E_n}{\partial n}\right|_{\phi,\phi^*,\chi,\chi^*,\eta,\eta^*} = 0. \quad \text{(B12)}$$

The terms involving χ_k, η_k are readily evaluated using the equations of motion for these quantities, or what is equivalent, (B11). We thus find

$$\left.\frac{\partial E_n}{\partial n}\right|_{\phi,\phi^*} = \omega_n \frac{d}{dn} \int dx \int \frac{dk}{2\pi} |\eta_k(x)|^2 + \int \frac{dk}{2\pi} \tfrac{1}{4} k^2 \frac{d}{dn} \int dx [|\chi_k(x)|^2 - |\eta_k(x)|^2]$$
$$= \omega_n \frac{d}{dn} \int dx \int \frac{dk}{2\pi} |\eta_k(x)|^2, \quad \text{(B13)}$$

since the *norm*

$$\int dx [|\chi_k(x)|^2 - |\eta_k(x)|^2] = 1 \quad \text{(B14)}$$

is independent of n. A direct evaluation yields

$$\int dx \int \frac{dk}{2\pi} |\eta_k(x)|^2 = \tfrac{1}{3}, \quad \text{(B15)}$$

and therefore

$$\left.\frac{\partial E_n}{\partial n}\right|_{\phi,\phi^*} = 0, \quad \text{(B16)}$$

which simplifies the considerations of Sec. III.

3. Solution to the equation of motion

We shall solve Eq. (3.17) to first order. If we define

$$\omega_{\text{eff}} = (E_n - E_{n-1})(n-1)/n \quad \text{(B17)}$$

and choose $\phi_n(x)$ to be real, the equation to be solved can be written to sufficient accuracy as

$$\omega_{\text{eff}}\phi_n(x) + \frac{1}{2}\frac{d^2}{dx^2}\phi_n(x) + K\phi_n^3(x) = (2K/n)[\phi_n^{(0)}(x)]^3 - K\int \frac{dk}{2\pi}[2|\eta_k(x)|^2 + \chi_k(x)\eta_k(x)]\phi_n^{(0)}(x). \quad \text{(B18)}$$

To evaluate the inhomogeneous term, we have

$$\int \frac{dk}{2\pi} 2|\eta_k(x)|^2 = \frac{\sqrt{2}}{2}(|\omega_{\text{eff}}|)^{1/2} \frac{1}{\cosh^4 z} = 2\int \frac{dk}{2\pi}\chi_k(x)\eta_k(x), \quad \text{(B19)}$$

$$z = (2|\omega_{\text{eff}}|)^{1/2} x. \quad \text{(B20)}$$

With the substitutions

$$\phi(x) = \frac{(|\omega_{\text{eff}}|)^{1/2}}{K}\beta(x), \quad \text{(B21)}$$

$$\beta(x) = \beta^{(0)}(x) + [\beta^{(1)}(x)\sqrt{2}/n], \quad \text{(B22)}$$

$$\beta^{(0)}(z) = \sqrt{2}/\cosh z, \quad \text{(B23)}$$

we find that $\beta^{(1)}(z)$ satisfies the equation

$$-\beta^{(1)} + \frac{d^2}{dz^2}\beta^{(1)} + \frac{6}{\cosh^2 z}\beta^{(1)} = \frac{4}{\cosh^3 z} - \frac{3}{\cosh^5 z}, \quad \text{(B24)}$$

and has the solution

$$\beta^{(1)}(z) = \frac{1}{2}\frac{1}{\cosh^3 z}, \quad \text{(B25)}$$

or

$$\phi_n(z) = (2|\omega_{\text{eff}}|/K)^{1/2}(\cosh z)^{-1} + (|\omega_{\text{eff}}|/2Kn^2)^{1/2}(\cosh z)^{-3}. \quad \text{(B26)}$$

Inserting (B25) into the quantization condition

$$\int dx\, |\phi_n(x)|^2 = n - \tfrac{1}{3}, \quad \text{(B27)}$$

evaluating the elementary integrals, and expanding to first order, we find

$$\frac{2}{K}(2|\omega_{eff}|)^2 \simeq n-1\,, \tag{B28}$$

which yields (3.39) when we expand (B17).

APPENDIX C: WEAK-COUPLING VERSION OF SINE-GORDON EQUATION

1. Accuracy of the tree approximation

From the equation of motion

$$-[[\phi,H],H]-\partial_x^2\phi+m^2\phi-\tfrac{1}{6}\lambda\phi^3=0\,, \tag{C1}$$

utilizing the definition (4.3), we obtain the equation

$$\{[E_n-E_{n'}(\hat{p})]^2+\partial_x^2-m^2\}\phi_{nn'}(x) = -\tfrac{1}{6}\lambda\int\frac{dp}{2\pi}e^{-ipx}\langle n'(p)|\phi(0)|n'(p')\rangle\langle n''(p')|\phi(0)|n''(p'')\rangle\langle n''(p'')|\phi(0)|n(0)\rangle\,. \tag{C2}$$

Since

$$E_{n'}(\hat{p})=(E_{n'}+\hat{p}^2)^{1/2}\simeq n'm+\frac{\hat{p}^2}{2n'm}=n'm\left[1+O\left(\frac{1}{n'^2}\right)\right], \tag{C3}$$

$$p=-i\partial_x\sim m\,, \tag{C4}$$

we may to first-order terms replace $E_{n'}(\hat{p})$ by $E_{n'}$. The proof that we may, to the same order of accuracy, replace the right-hand side of (C2) by the right-hand side of (4.4) also involves little more than the recognition of (C4), after suitable introduction everywhere of the inverse of (4.3). Arguments of this type are given in the Appendixes of II.

To reach Eq. (4.6), our real starting point, we must (i) consider only the equations for $\phi_{n,n\pm1}$ and expand these about ϕ_n defined in (4.5). By considering the average of the two equations for $n'=n\pm1$, we learn that the "tree" contribution in (4.6) is accurate to $O(n^{-2})$. The remaining $O(n^{-1})$ terms are obtained from the one-loop contributions, which were calculated in II.

2. Absorption amplitudes

The loop contributions are expressed in terms of the quantities

$$\chi_{n,k}(x)\equiv\langle n|\phi(x)|n,k\rangle\,, \tag{C5}$$

$$\eta^*_{n,k}(x)=\langle n|\phi(x)|n-2,k\rangle\,, \tag{C6}$$

which as shown in II, satisfy the equations

$$\nu^2\chi_\nu=-\partial_z^2\chi_\nu-\frac{4}{\cosh^2 z}\chi_\nu-\frac{2}{\cosh^2 z}\eta^*_\nu\,, \tag{C7}$$

$$(-2-\nu^2)\eta^*_\nu=-\partial_z^2\eta^*_\nu-\frac{4}{\cosh^2 z}\eta^*_\nu-\frac{2}{\cosh^2 z}\chi_\nu\,, \tag{C8}$$

where

$$\nu=(k/m\sin\alpha),\ z=m(\sin\alpha)x\,. \tag{C9}$$

Equations (C7) and (C8) are the dimensionless form of (3.10) and (3.11) and have as solutions

$$(2E_kL)^{1/2}\chi_\nu=\frac{e^{i\nu z}}{\nu^2-1-2i\nu}\left(\nu^2-1+2i\nu\tanh z+\frac{1}{\cosh^2 z}\right), \tag{C10}$$

$$(2E_kL)^{1/2}\eta^*_\nu=\frac{e^{i\nu z}}{\nu^2-1-2i\nu}\,\frac{1}{\cosh^2 z}\,. \tag{C11}$$

The normalization reads

$$\lim_{L\to\infty}\int dx[|\chi_\nu(x)|^2-|\eta_\nu(x)|^2]=(2E_k)^{-1}\,. \tag{C12}$$

3. Solution to the equation of motion

We present here the solution of Eq. (4.6). If m^2 is the renormalized mass, $m^2=m_0^2+\delta m^2$, the right-hand side of this equation is $\frac{1}{4}\lambda\phi\mathcal{G}$, where $\mathcal{G}$ is the integral

$$\begin{aligned}\mathcal{G}&=\int\frac{dk}{2\pi}(2|\chi_k|^2+2|\eta_k|^2+\chi_k\eta_k+\chi_k^*\eta_k^*)+\delta m^2\\&=\frac{1}{\cosh^2 z}\left[-\frac{1}{\pi}\tan^2\alpha-\left(1-\frac{2\alpha}{\pi}\right)\tan\alpha+\left(\frac{1}{2}-\frac{\alpha}{\pi}\right)\tan^3\alpha\right]\\&\quad+\frac{3}{2}\,\frac{1}{\cosh^4 z}\left[\frac{1}{\pi}\tan^2\alpha+\left(\frac{1}{2}-\frac{\alpha}{\pi}\right)\tan\alpha(1-\tan^2\alpha)\right].\end{aligned} \tag{C13}$$

Keeping only the leading terms in the weak-coupling limit in which we set $\tan\alpha\cong\alpha\cong\sin\alpha$, we have

$$\mathcal{G}\cong\frac{1}{\pi}\,\frac{\sin^2\alpha}{\cosh^2 z}-\frac{\sin\alpha}{\cosh^2 z}+\frac{3}{4}\,\frac{\sin\alpha}{\cosh^4 z}\,. \tag{C14}$$

The first term of (C14) may be incorporated into the zero-order equation if one makes the coupling-constant replacement

$$\lambda \to \lambda' = \lambda[1-(\lambda/8\pi m^2)]^{-1} . \tag{C15}$$

The substitution

$$\phi = \frac{\sqrt{2}\, m \sin\alpha}{(\lambda')^{1/2}} \beta(z), \quad z = m(\sin\alpha)x , \tag{C16}$$

allows (4.6) to be written in the dimensionless version

$$\left(\frac{d^2}{dz^2} - 1 + \beta^2\right)\beta = -\frac{\lambda}{4m^2\sin\alpha}\frac{\beta}{\cosh^2 z} + \frac{3\lambda}{16m^2\sin\alpha}\frac{\beta}{\cosh^4 z} \cong -\frac{4}{n}\frac{\beta}{\cosh^2 z} + \frac{3}{n}\frac{\beta}{\cosh^4 z} , \tag{C17}$$

where the lowest-order quantization condition (4.21) has been utilized. The expansion

$$\beta = \beta^{(0)} + \frac{2}{n}\beta^{(1)}, \quad \beta^{(0)} = \frac{\sqrt{2}}{\cosh z} \tag{C18}$$

leads to the equation

$$\left(\frac{d^2}{dz^2} - 1 + \frac{6}{\cosh^2 z}\right)\beta^{(1)} = \frac{4}{\cosh^3 z} - \frac{3}{\cosh^5 z} , \tag{C19}$$

which is solved by

$$\beta^{(1)} = \frac{1}{2}\frac{1}{\cosh^3 z} . \tag{C20}$$

The total result is then given in Eq. (4.24).

4. Canceling divergences

We describe the derivation of Eqs. (4.13) and (4.14). Equation (4.13) is derived from (4.8) and and (4.9). Because χ_k is not square integrable, we modify the usual procedure which would be to multiply (4.8) by χ_k^*, (4.9) by η_k, integrate and add. Instead we multiply (4.8) by the complex conjugate of

$$\xi_k = \chi_k - e^{ikx}(2E_k L)^{-1/2} \tag{C21}$$

but otherwise proceed as usual. With this change, all integrals are absolutely convergent and we can interchange orders of integration if we choose. Elimination of ξ_k by means of (C21) accounts for the last term of (4.13).

The derivation of Eq. (4.14) is somewhat lengthier. It consists of a careful evaluation of the difference of expectation values which defines E_k with the consequent recognition that *most* contributions cancel in the difference. To illustrate the argument consider the difference—in the "fixed source" limit—

$$\langle n, k^{(\dagger)} | [\phi(x)]^2 | n, k^{(\dagger)} \rangle \langle n | \phi^2(x) | n \rangle . \tag{C22}$$

We have (using discrete momenta as well)

$$\begin{aligned} \langle n, k | [\phi(x)]^2 | n, k\rangle &\cong |\langle n, k|\phi(x)|n, k\rangle|^2 + \sum_{k'\neq k} |\langle n, k|\phi(x)|n, k, k'\rangle|^2 + |\langle n, k|\phi(x)|n\rangle|^2 \\ &\quad + |\langle n, k|\phi(x)|n, k, k\rangle|^2 + \text{corresponding terms in which } n \text{ changes by 2} \\ &\cong |\langle n|\phi(x)|n\rangle|^2 + \sum_{k'} |\langle n|\phi(x)|n, k'\rangle|^2 \\ &\quad + 2|\langle n|\phi(x)|n, k\rangle|^2 + \text{terms in which } n \text{ changes by 2}, \end{aligned} \tag{C23}$$

keeping disconnected pieces as the major contribution. The first two terms cancel upon subtraction in (C22). Thus altogether

$$\begin{aligned} (\text{C22}) &= 2|\langle n|\phi|n, k\rangle|^2 + 2|\langle n|\phi|n-2, k\rangle|^2 \\ &= 2|\chi_k|^2 + 2|\eta_k|^2 . \end{aligned} \tag{C24}$$

This argument accounts specifically for the term of (4.14) proportional to m^2. Corresponding arguments yield the remaining terms.

5. The quantization condition

We evaluate C_n, given by Eq. (4.17). First of all, using Eq. (4.24), we find to the required accuracy

$$2\omega_n^2 \int dx\, \phi_n^2(x) = \omega_n\left(\frac{16m^2}{\lambda'} + \frac{2}{3}\right) . \tag{C25}$$

The remaining term in C_n is the zero-point sum, which as been evaluated by DHN and in II,

$$\text{“}\sum_k \tfrac{1}{2}E_k\text{”} = -\frac{2m}{\pi}\sin\alpha - \frac{2m}{\pi}\cos\alpha\left(\frac{\pi}{2} - \alpha\right) + m \cong \frac{m}{2}\sin^2\alpha + \frac{2m}{\pi}\frac{\sin^3\alpha}{6} . \tag{C26}$$

The value given by DHN and adopted in II does not contain the last term ($+m$) because in previous calculations ϵ_0, the vacuum subtraction, has been augmented by that amount. The arguments used in Sec. IV to obtain the energy show that this change has no effect on the final result. Here we see that the zero-point sum does not contribute in the weak-coupling limit.

APPENDIX D: SINE-GORDON EQUATION

1. Absorption amplitudes

We will now calculate the function $\psi_k^\dagger(x, n, \theta)$, whose Fourier componenet are the matrix elements for absorption of one meson of momentum k by a bound state $|n\rangle$ according to (5.8). Because this function satisfies the linearized sine-Gordon equation (5.9) it may be obtained from linearizing a particular four-soliton solution consisting of one doublet with period $(2\pi/m)(1+\epsilon^2)^{1/2}$ and momentum k which is used as a probe. This calculation is fully explained in Appendix C of DHN but will be repeated here in order to correct a misprint in the solution.

The unlinearized solution to the sine-Gordon equation is

$$\Phi = \frac{4m}{\lambda}\tan^{-1}\left[\frac{G}{F}\right], \tag{D1}$$

where

$$F(x,t) = \sum_{\mu_i=(0,1)}^{4(e)} \exp\left(\sum_{i<j}^{4} B_{ij}\mu_i\mu_j + \sum_{j=1}^{4}\mu_j x_j\right), \tag{D2}$$

$$G(x,t) = \sum_{\mu_i=(0,1)}^{4(o)} \exp\left(\sum_{i<j}^{4} B_{ij}\mu_i\mu_j + \sum_{j=1}^{4}\mu_j x_j\right). \tag{D3}$$

The summations here are over all combinations of $\mu_1=0,1$, $\mu_2=0,1$, $\mu_3=0,1$, $\mu_4=0,1$ subject to the constraint that their sum be even (e) or odd (o). Space and time enter through

$$x_i = k_i x + \beta_i t + \gamma_i, \tag{D4}$$

where

$$k_i = (1-v_i^2)^{-1/2}, \tag{D5}$$

$$\beta_i = v_i(1-v_i^2)^{-1/2}, \tag{D6}$$

and the velocities are defined by

$$v_i = \frac{i}{\epsilon}, \quad v_2 = \frac{-i}{\epsilon},$$
$$v_3 = \frac{\eta v + i}{\eta + iv}, \quad v_4 = \frac{\eta v - i}{\eta - iv}. \tag{D7}$$

The constants γ_i are related to the periods by

$$e^{\gamma_1} = e^{\gamma_2} = \epsilon, \tag{D8}$$

$$e^{\gamma_3} = e^{\gamma_4} = \eta. \tag{D9}$$

The functions B_{ij} are given by

$$\exp(B_{ij}) = \frac{(k_i - k_j)^2 - (\beta_i - \beta_j)^2}{(k_i + k_j)^2 - (\beta_i + \beta_j)^2}. \tag{D10}$$

To evaluate F and G to first order in η requires keeping only terms linear in e^{x_3} or e^{x_4}. We may then set $\eta=0$ in v_3 and v_4. Consequently $x_3 = x_4$ in this limit. Evaluating the B_{ij} with $\eta=0$ gives

$$F + \delta F = 1 + \cos^2\alpha\, e^{x_1+x_2} + 2\rho e^{+i\delta} e^{x_1+x_3} + \frac{2e^{i\delta}}{\rho} e^{x_2+x_3},$$
$$G + \delta G = e^{x_1} + e^{x_2} + 2e^{x_3} + 2\cot^2\alpha\, e^{2i\delta} e^{x_1+x_2+x_3}. \tag{D11}$$

Substituting for the x_i gives

$$F = 2e^z\cosh z$$
$$\delta F = 2\eta\tan\alpha\, e^z e^{i\delta}\left(\rho e^{-i\theta} + \frac{e^{i\theta}}{\rho}\right) e^{i(kx-Et)},$$
$$G = 2\tan\alpha\, e^z\cos\theta,$$
$$\delta G = 2\eta(1 + e^{2i\delta}e^{2z})e^{i(kx-Et)}, \tag{D12}$$

where $\theta = mt\cos\alpha$, $z = mx\sin\alpha$, ρ and δ are given by (5.12) and (5.13). These results are in agreement with Appendix C of DHN except for a few factors of 2 and a crucial minus sign in δF.

The linearized form of (D1) is then

$$\varphi + \delta\varphi = \frac{4m}{\lambda}\tan^{-1}(G/F) + \frac{4m}{\sqrt{\lambda}}\frac{F\delta G - G\delta F}{F^2 + G^2}. \tag{D13}$$

The first correction is thus the solution

$$\psi_k^\dagger(x, n, \theta)e^{-iEt} \propto \frac{F\delta G - G\delta F}{F^2 + G^2}. \tag{D14}$$

The properly normalized solution is

$$(2E_k L)^{1/2}\psi_k^\dagger(x,n,\theta)e^{-iEt} = \frac{1}{2D}\left[\cosh z(e^{-z} + e^{2i\delta}e^z) - \tan^2\alpha\cos\theta e^{i\delta}\left(\rho e^{-i\theta} + \frac{e^{i\theta}}{\rho}\right)\right]e^{i(kx-Et)}, \tag{D15}$$

which may be rewritten as (5.10).

2 Exact solution to the one-loop equation

We now proceed to solve (5.7), the sine-Gordon equation with one-loop corrections. The equation may be written

$$(\partial_x{}^2-\partial_t{}^2)Y=\frac{m_0{}^2 m}{\sqrt{\lambda}}\sin\left(\frac{\sqrt{\lambda}}{m}Y\right)-\tfrac{1}{2}m\sqrt{\lambda}\sum_k|\psi_k|^2\sin\left(\frac{\sqrt{\lambda}}{m}Y\right), \tag{D16}$$

with $m_0{}^2$ replaced by m^2 in the last term. The first step is to calculate the source term. The square of (5.10) is

$$L|\psi_k|^2=\frac{1}{2E_k}\left\{1+\left(\frac{\cosh^2\alpha\sinh^2 z}{D^2}-1\right)\sin^2\delta+\frac{\tan^4\alpha\cos^4\theta}{D^2}[(A+\cos\delta)^2+B^2\tan^2\theta]\right.$$
$$\left.-\frac{2\tan^2\alpha\cos^2\theta}{D}(A+\cos\delta)\cos\delta-\frac{2\cosh z\sinh z\tan^2\alpha\cos^2\theta}{D^2}(B\sin\delta)\tan\theta\right\}, \tag{D17}$$

where

$$A=\tfrac{1}{2}(\rho+1/\rho),\qquad B=\tfrac{1}{2}(\rho-1/\rho). \tag{D18}$$

All dependence on k is contained in A, B, and δ. Carrying out the momentum integration gives

$$L\int_{-\infty}^{\infty}\frac{dk}{2\pi}|\psi_k|^2=\int_{-\infty}^{\infty}\frac{dk}{4\pi E}+\frac{-(2/\pi)g^2+a}{D}+\frac{(2/\pi)g^4+bg^2}{D^2}, \tag{D19}$$

where

$$g=\tan\alpha\cos\theta,\qquad D=\cosh^2 z+g^2. \tag{D20}$$

Thus all time dependence occurs through g. The constants a and b are explicitly

$$a=\frac{1}{\pi}\tan^2\alpha-\left(\frac{1}{2}-\frac{\alpha}{\pi}\right)\frac{\sin\alpha}{\cos^3\alpha},\qquad b=\frac{1}{\pi}(1-\tan^2\alpha)+\left(\frac{1}{2}-\frac{\alpha}{\pi}\right)\frac{1}{\sin\alpha\cos^3\alpha}. \tag{D21}$$

The logarithmically divergent part of (D19) precisely renormalizes the bare mass in (D16) to give

$$m^2=m_0{}^2-\frac{\lambda}{8\pi}\int_{-\infty}^{\infty}\frac{dk}{E}. \tag{D22}$$

The next step is to set $Y=Y^{(0)}+Y^{(1)}$ where

$$Y^{(0)}=\frac{4m}{\sqrt{\lambda}}\tan^{-1}\left(\frac{g}{\cosh z}\right) \tag{D23}$$

is the solution to (D16) in the absence of the source (but with the renormalized mass). The equation for $Y^{(1)}$ is then

$$(\partial_x{}^2-\partial_t{}^2)Y^{(1)}=m^2\cos\left(\frac{\sqrt{\lambda}}{m}Y^{(0)}\right)Y^{(1)}-\tfrac{1}{2}m\sqrt{\lambda}\left[\frac{(-2/\pi)g^2+a}{D}+\frac{(2/\pi)g^4+bg^2}{D^2}\right]\times\sin\left(\frac{\sqrt{\lambda}}{m}Y^{(0)}\right). \tag{D24}$$

If one notes that

$$\cos\left(\frac{\sqrt{\lambda}}{m}Y^{(0)}\right)=1-\frac{8g^2}{D}+\frac{8g^4}{D^2},\qquad \sin\left(\frac{\sqrt{\lambda}}{m}Y^{(0)}\right)=\left(\frac{1}{D}-\frac{2g^2}{D^2}\right)4g\cosh z, \tag{D25}$$

then it is not surprising that $Y^{(1)}$ should be expressible as a linear combination of D^{-1} and D^{-2} with time-dependent coefficients. Indeed, the exact solution is

$$Y^{(1)}(x,t)=-\frac{(1/4\pi)(\sqrt{\lambda}/m)g\cosh z}{D}+\frac{(\sqrt{\lambda}/4m)[(2/\pi)g^3+bg]\cosh z}{D^2}. \tag{D26}$$

Such a simple solution to (D24) is possible only because of the special form of the source term. In particular, the relation

$$\frac{a}{\sin^2\alpha}+b=2/\pi$$

between the constants a and b is essential.

3. Quantization condition

In order to examine the sum of $Y^{(0)}$ and $Y^{(1)}$ it is helpful to expand both in a power series in g. We find

$$Y^{(0)}+Y^{(1)}=\frac{4m}{\lambda}g\left(\frac{A_1}{\cosh z}+\frac{B_1}{\cosh^3 z}\right)-\tfrac{1}{3}g^3\left(\frac{A_3}{\cosh^5 z}+\frac{B_3}{\cosh^7 z}\right)+\cdots, \tag{D27}$$

where

$$A_1=1-\frac{\lambda}{16\pi m^2},$$
$$A_3=1-\frac{9\lambda}{16\pi m^2},\ldots,A_{2k+1}=1-\frac{(2k+1)^2\lambda}{16\pi m^2},$$
$$B_1=\frac{\lambda}{16m^2}b, \tag{D28}$$
$$B_3=\frac{6\lambda}{16m^2}b,\ldots,B_{2k+1}=\frac{(k+1)(2k+1)\lambda}{16m^2}b.$$

Note that the expansion of $Y^{(0)}$ alone corresponds to $A_i=1$, $B_i=0$. Because $b\approx(2\sin\alpha)^{-1}$ in (D21), it follows that

$$B_{2k+1}=\frac{(k+1)(2k+1)}{2n}[1+O(\sin^2\alpha)]. \tag{D29}$$

Note that the renormalization $\lambda\to\lambda'$ (of the energy) occurs only for the first Fourier coefficient:

$$\frac{1}{\sqrt{\lambda}}A_1=\frac{1}{(\lambda')^{1/2}}.$$

With the aid of (D27) and (D28) we are finally in a position to compute the "classical" term of the quantization condition (5.35) to the one-loop level. We have

$$\omega\int_{-\infty}^{\infty}dx\int_0^{2\pi}\frac{d\theta}{2\pi}(\partial_\theta Y)^2 \cong\frac{16m^2}{\lambda}\alpha+2\cot\alpha\int_{-\infty}^{\infty}dz\int_0^{2\pi}\frac{d\theta}{2\pi}(\partial_\theta Y^{(0)})(\partial_\theta Y^{(1)}). \tag{D30}$$

For the integral, we obtain by an expansion in $\tan\alpha$

$$\int_{-\infty}^{\infty}dz\int_0^{2\pi}\frac{d\theta}{2\pi}\partial_\theta Y^{(0)}\partial_\theta Y^{(1)}=\tfrac{1}{3}\tan\alpha-\frac{1}{\pi}\tan^2\alpha+\tfrac{1}{5}\tan^3\alpha+\cdots,$$

and thus

$$(\text{D30})\cong\frac{16m^2}{\lambda'}\alpha+\tfrac{2}{3}+\tfrac{2}{5}\alpha^2+O(\alpha^3). \tag{D31}$$

This is to be compared with Eq. (C25), from which it differs first in the term $\frac{2}{5}\alpha^2$

---

[1]It is beyond the scope of this paper to consider recent developments in the gauge theories with soliton (monopole) and instanton solutions and the intriguing hints they contain about the real world.

[2]R. F. Dashen, B. Hasslacher, and A. Neveu, Phys. Rev. D 10, 4130 (1974); 11, 1324 (1975), referred to as DHN.

[3]J. L. Gervais and B. Sakita, Phys. Rev. D 11, 2943 (1975).

[4]C. Callan and D. Gross, Nucl. Phys. B93, 29 (1975).

[5]V. Korepin and L. Faddeev, Teor. Mat. Fiz. 25, 147 (1975) [Theor. Math. Phys. 25, 1039 (1976)].

[6]N. Christ and T. D. Lee, Phys. Rev. D 12, 1606 (1975).

[7]E. Tomboulis, Phys. Rev. D 12, 1678 (1975).

[8]J. Goldstone and R. Jackiw, Phys. Rev. D 11, 1486 (1975).

[9]A. Klein and F. Krejs, Phys. Rev. D 12, 3112 (1975); 13, 3282 (1976); 13, 3295 (1976) (the latter are referred to as I and II in the present work).

[10]For a beautiful review of this and the more recent developments in gauge theories, see R. Jackiw, Rev. Mod. Phys. 49, 681 (1977).

[11]H. de Vega, Nucl. Phys. B115, 411 (1976).

[12]J. L. Gervais and A. Jevicki, Nucl. Phys. B110, 113 (1976).

[13]M. T. Jaekel, Nucl. Phys. B118, 506 (1977).

[14]S. Y. Lee and A. Gavrielides, Purdue Univ. report (unpublished).

[15]J. Honerkamp, M. Schlindwein, and A. Wiesler, Nucl. Phys. B121, 531 (1977).

[16]An alternative and more detailed account of the material of this section has been published by A. Klein, J. Math. Phys. 19, 292 (1978).

[17]This point is discussed in the first entry of Ref. 9 and in some more detail in C. T. Li, A. Klein, and F. Krejs, Phys. Rev. D 12, 2311 (1975).

[18]For a recent discussion related to the present context, see R. Jackiw and G. Woo, Phys. Rev. D 12, 1643 (1975).

[19]The connection between number (or charge) conservation and Bohr-Sommerfeld quantization was noted previously by I. Ventura and G. C. Marques, Phys. Lett. 64B, 43 (1976), and C. Montonen, Nucl. Phys. B112, 349 (1976). The present section extends the remarks of these papers by noting below that, in effect, this relationship must be utilized (at least in disguised form) to fully establish a quantization condition which includes quantum corrections. In effect this relationship is needed to establish the appropriate connection formula.

[20]As we show below L_n is well defined, but C_n and E_n require a vacuum subtraction, necessarily the same for each. Even with the subtraction, the finiteness of C_n and E_n is not superficially evident, but requires careful consideration of the passage to the $L\to\infty$ limit. The essential steps are given in the ensuing discussion.

[21]The first term of (4.35) is not strictly correct since we still have λ, not λ', in the quartic interaction term. The difference does not, however, contribute in the weak-coupling limit, since it is of order $(\lambda/m^2)\sin^2\alpha$ compared to the main term in the result.

[22]The argument leading to this result in II is inadequate and should be replaced by the discussion given here.