Existence and uniqueness theorem for the one-dimensional backwards two-body problem of electrodynamics*

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The equations of motion, based on the retarded Liénard-Wiechert potentials, for two charged particles of like sign in classical electrodynamics are considered. The two charged particles are assumed to be moving along the x axis. No forces are considered except those produced by the two particles themselves. The "radiation reaction" is not considered. Because of the finite speed of propagation of electromagnetic effects, the equations of motion form a system of delay-differential equations. In general, for such a system one needs to specify the past trajectories of the two charged particles over a period of time to determine a unique solution in the future. But one generally expects that *instantaneous* values of position and velocity should suffice to determine a unique solution in a physical problem. A simple example shows that this should *not* be expected in the case of delay-differential equations. Consider the equation $x' = -x(t-\pi/2)$ and $x(0)=0$. Then $x = a\sin t$ is a solution for any a. Uniqueness fails to be true even for such a simple equation. However, for the one-dimensional two-body problem of classical electrodynamics, Driver and Zhdanov have proved under certain restrictions on the given instantaneous positions and velocities of the particles that a unique solution is determined. In this paper, we shall give an existence and uniqueness theorem which extends some of Zhdanov's results. If two particles are separated far enough and are moving toward each other but not too fast at $t=0$, then we shall show that their trajectories are uniquely determined both in the past and the future by their positions and velocities at $t=0$. The contraction mapping scheme is used in the proof.

I. INTRODUCTION

Using the retarded Liénard-Wiechert potentials, the equations of motion for the two-body problem of classical electrodynamics, in the one-dimensional case, can be written as a system of delay-differential equations where the time delays depend upon the unknown trajectories. The delays arise because of the finite speed of propagation of the electromagnetic effects.

We shall consider two particles of like sign moving along the x axis. No forces are considered except those produced by the two particles themselves.

Let $\tilde{x}(t)$ and $x(t)$ be the positions of the two particles on the x axis at observer time t, relative to some inertial reference frame, with

$$\tilde{x}(t) < x(t).$$

Then the equations of motion of the two-body problem are

$$\frac{v'}{(1-v^2)^{3/2}} = \frac{b[1+\tilde{v}(t-\tau)]}{\tau^2[1-\tilde{v}(t-\tau)]}, \tag{1a}$$

$$\frac{\tilde{v}'}{(1-\tilde{v}^2)^{3/2}} = -\frac{\tilde{b}[1-v(t-\tilde{\tau})]}{\tilde{\tau}^2[1+v(t-\tilde{\tau})]}, \tag{1b}$$

where

$$c\tau = |x - \tilde{x}(t-\tau)|, \tag{2a}$$

$$c\tilde{\tau} = |x(t-\tilde{\tau}) - \tilde{x}|, \tag{2b}$$

$$x' = cv, \tag{3a}$$

$$\tilde{x}' = c\tilde{v}, \tag{3b}$$

b, $\tilde{b}$, and c (the speed of light) are positive constants, and $x, \tilde{x}, x', \tilde{x}', v, \tilde{v}, v', \tilde{v}', \tau, \tilde{\tau}$ denote $x(t)$, $\tilde{x}(t), x'(t), \tilde{x}'(t), v(t), \tilde{v}(t), v'(t), \tilde{v}'(t), \tau(t), \tilde{\tau}(t)$. A prime denotes d/dt.

Some authors include the Dirac "radiation-reaction" terms in the equations of motion. However, it has been proved that these terms lead to some paradoxical results.[1,2] So in the present paper these terms are excluded.

For a delay-differential equation, the usual formulation of the problem is as follows: One specifies a rather arbitrary "initial function" on an "initial interval" and seeks an extension into the future so that the delay-differential equation is satisfied there. For Eqs. (1a)–(3b), Driver proved that rather arbitrary continuously differentiable "initial trajectories" of the particles will uniquely determine the trajectories of the particles in the future.[3,4]

Another approach to the equations of motion of this electrodynamic problem can also be taken. One might specify the positions and velocities of the two particles at $t=0$ and demand Eqs. (1a)–(3b) to be satisfied for all time (both in the past and the future). This has been conjectured to be a well-posed problem.[5,6]

However, consider for example the following scalar linear equation with a single delay:

$$x' = -x(t-\pi/2)$$

and $x(0)=0$. Then $x=a\sin t$ will be a solution for any constant a. Thus uniqueness fails in such a simple case. Indeed even if all the values of the derivatives of x are given at $t=0$, the equation still has infinitely many solutions.[7]

Therefore, for the much more complicated delay-differential equations of the two-body problem, we must be careful not to jump to conclusions about the existence and uniqueness of the trajectories when only instantaneous values of the positions and velocities are given. This then motivates the "backwards problem" for Eqs. (1a)–(3b), which can be stated as follows:

Problem P. Let $x_0, \tilde{x}_0, v_0, \tilde{v}_0$ be given numbers satisfying $\tilde{x}_0 < x_0$, $|v_0|<1$, $|\tilde{v}_0|<1$. We seek functions, x and $\tilde{x}$, on $(-\infty, 0]$ with continuous first derivatives such that $\tilde{x}(0)=\tilde{x}_0$, $x(0)=x_0$, $\tilde{x}'(0)=c\tilde{v}_0$, $x'(0)=cv_0$, $\tilde{x}<x$ on $(-\infty,0]$, $|\tilde{x}'|<c$, $|x'|<c$ on $(-\infty,0]$, and (1a)–(3b) are satisfied on $(-\infty,0]$.

Such a pair of functions, x and $\tilde{x}$, will be called a *solution* of problem P. The solution is said to be *unique* if any two solutions associated with the same "point data" (x_0, $\tilde{x}_0$, v_0, and $\tilde{v}_0$) agree with each other.

We shall give an existence and uniqueness theorem for problem P. The question of a unique solution existing also in the future then becomes an "initial-function" problem which has been treated.[3,4]

However, before presenting the rather complicated proof, we consider a very simple artificial example which will illustrate the basic idea. Consider this scalar linear delay-differential equation

$$x'=a(t)\,x(t-1) \text{ for } t\leq 0$$

with (E)

$$x(0)=x_0, \text{ where } x_0\in R\ .$$

If $a(t)$ is continuous and

$$\int_{-\infty}^{0} |a(\eta)|\,d\eta = r<1\ ,$$

then a unique bounded solution exists on $(-\infty, 0]$.

This was proved by Doss and Nasr[8] as follows. First define a complete metric space

$$A=\{f: f\in C(-\infty,0] \text{ and } f \text{ is bounded}\}$$

with the metric

$$d(f,g)=\sup_{s\leq 0}|f(s)-g(s)| \text{ for } f,g\in A\ .$$

We next define a mapping T of A into A. For $f\in A$, let us consider the ordinary differential equation

$$x'=a(t)f(t-1) \text{ for } t\leq 0$$

with

$$x(0)=x_0\ .$$

A unique solution, x, exists to this ordinary differential equation. Let

$$T(f)=x \text{ for } f\in A\ .$$

Then one finds

$$|T(f(t))|=|x(t)|\leq \sup_{s\leq 0}|f(s)| \quad \text{for } t\leq 0.$$

So $T(f)\in A$. Moreover, for any $f_1, f_2\in A$

$$d(T(f_1), T(f_2))\leq r d(f_1, f_2)\ .$$

By the contraction mapping theorem, the existence of a unique bounded solution to (E) then follows.

Let us consider two special cases to illustrate the above result.[9]

1. Let $a(t)=2te^{2t-1}$ in (E). Then applying the above, one can see that a unique bounded solution exists on $(-\infty,0]$. However, in this case one easily sees that there is also an unbounded solution, $x(t)=x_0e^{t^2}$.

2. Let

$$a(t)=\begin{cases} 0 \text{ for } t<-1 \\ -1-t \text{ for } -1\leq t\leq 0 \end{cases}$$

in (E). Again, the above result guarantees the existence and uniqueness of a bounded solution on $(-\infty,0]$. Moreover, this time there can be no unbounded solutions since $a(t)=0$ for $t<-1$. Thus (E) has a unique solution on $(-\infty,0]$.

Returning now to the electrodynamics equations, Driver proved that problem P has a unique solution when $v_0<\tilde{v}_0$ and $(\tilde{v}_0-v_0)(x_0-\tilde{x}_0)$ is large enough.[7] Travis extended the existence assertion to the case where $v_0\leq\tilde{v}_0$.[10] However, Travis did not prove uniqueness.

Recently Zhdanov, considering the special case of symmetric motion of two identical charged particles (i.e., the case when $\tilde{x}=-x$), proved that a unique solution exists when v_0^2+b/x_0 is sufficiently small.[11,12] Considering the symmetric motion, we can see that the results of Driver and Zhdanov overlap, but neither one of them contains the other.

It is the purpose of this paper to obtain an existence and uniqueness theorem for problem P, in the unsymmetric case with $v_0\leq 0\leq\tilde{v}_0$. The contraction mapping scheme is used in the proof.

II. EXISTENCE AND UNIQUENESS THEOREM

The following results about solutions of (2a) and (2b) were proved by Driver.[7]

Lemma. Let x and $\tilde{x}$ be two continuously differentiable functions with $\tilde{x}<x$ and $-uc\leq x'\leq 0$, $0\leq\tilde{x}'\leq uc$ on $(-\infty,0]$ for some $u\in(0,1)$. Then unique solutions of (2a), (2b) exist on $(-\infty,0]$ if and

only if $\tau(0)=\tau_0$, $\bar\tau(0)=\bar\tau_0$ satisfy (2a), (2b), respectively, at $t=0$, and

$$\tau' = \frac{v-\bar v(t-\tau)}{1-\bar v(t-\tau)}, \tag{4a}$$

$$\bar\tau' = \frac{-\bar v + v(t-\bar\tau)}{1+v(t-\bar\tau)} \tag{4b}$$

for $t\le 0$. Moreover, $t-\tau$ and $t-\bar\tau$ are strictly increasing and

$$\frac{x-\bar x}{c} \le \tau \le \frac{x-\bar x}{c(1-u)},$$

$$\frac{x-\bar x}{c} \le \bar\tau \le \frac{x-\bar x}{c(1-u)}$$

for $t\le 0$.

Now the main results—the existence and uniqueness theorem for problem P.

Theorem. There exists a number $a>0$ such that problem P has a unique solution whenever

$$\frac{\max(b,\bar b)c}{x_0-\bar x_0} \le a,$$

where x_0, $\bar x_0$, v_0, and $\bar v_0$ are given numbers such that $-\frac12 < v_0 \le 0 \le \bar v_0 < \frac12$, $\bar x_0 < x_0$, and

$$\max\left[\frac{v_0^2}{(1-4v_0^2)^{1/2}}, \frac{\bar v_0^2}{(1-4\bar v_0^2)^{1/2}}\right] \le \frac{\max(b,\bar b)c}{x_0-\bar x_0}.$$

Proof. Suppose that x_0, $\bar x_0$, v_0, and $\bar v_0$ are given numbers satisfying $-\frac12 < v_0 \le 0 \le \bar v_0 < \frac12$, $\bar x_0 < x_0$, and

$$\max\left[\frac{v_0^2}{(1-4v_0^2)^{1/2}}, \frac{\bar v_0^2}{(1-4\bar v_0^2)^{1/2}}\right] \le \frac{\max(b,\bar b)c}{x_0-\bar x_0}.$$

Let us choose $u\in(0,1)$ such that

$$\frac{u^2}{(1-u^2)^{1/2}} - \frac{u^2}{(4-u^2)^{1/2}} = \frac{4\max(b,\bar b)c}{x_0-\bar x_0}.$$

Then

$$\frac{4v_0^2}{(1-4v_0^2)^{1/2}} \le \frac{4\max(b,\bar b)c}{x_0-\bar x_0} \le \frac{u^2}{(1-u^2)^{1/2}}.$$

From the facts that $v_0\le 0$ and the function, $y/(1-y)^{1/2}$, is increasing we get

$$-\tfrac12 u \le v_0 \le 0.$$

Similarly we find that

$$0 \le \bar v_0 \le \tfrac12 u.$$

1. *Construction of a complete metric space*. The set

$$A=\left\{(f,\bar f): f,\bar f\in C(-\infty,0],\ -u\le f\le 0,\ 0\le \bar f\le u,\ 0\le f(t_1)-f(t_2)\le \frac{2bc^2(t_1-t_2)}{(1-u)(x_0-\bar x_0)^2},\right.$$
$$\left.\text{and } \frac{-2\bar b c^2(t_1-t_2)}{(1-u)(x_0-\bar x_0)^2} \le \bar f(t_1)-\bar f(t_2)\le 0 \text{ for any } t_1,t_2\in(-\infty,0] \text{ with } t_1>t_2\right\}$$

is a complete metric space with metric

$$d[(f,\bar f),(g,\bar g)] = \sup_{s\le 0}|f(s)-g(s)| + \sup_{s\le 0}|\bar f(s)-\bar g(s)|$$

for $(f,\bar f), (g,\bar g)\in A$. This can be proved by showing that A is a nonempty closed subset of $C(-\infty,0]\times C(-\infty,0]$. In other words, let $(f_n,\bar f_n)$ be any sequence in A which converges to $(f,\bar f)$ in the metric d. By the definition of the metric d, we know $(f_n,\bar f_n)$ converges uniformly to $(f,\bar f)$. It is then a straightforward calculation to show that $(f,\bar f)\in A$ also. Note that $(0,0)\in A$, thus A is nonempty.

2. *Proof that $(v,\bar v)\in A$ for any solution of problem P.*

Let $x,\bar x$ be a solution of problem P corresponding to $x_0,\bar x_0,v_0,\bar v_0$ given in the theorem. Let $v=x'/c$ and $\bar v=\bar x'/c$.

From (1a), (1b) we find that v is increasing and $\bar v$ is decreasing. So

$$v\le v_0\le 0 \text{ and } \bar v\ge \bar v_0\ge 0 \quad \text{for } t\le 0.$$

First, we are going to show that

$$-u\le v\le 0 \quad \text{for } t\le 0.$$

We have either

$$-u/2\le v\le 0 \quad \text{for } t\le 0$$

or

$$v(t_1)=-u/2 \quad \text{for some } t_1\le 0.$$

In the first case we are done. In the second case, we find that

$$v\ge -u/2 \quad \text{for } t\ge t_1$$

and

$v \leq -u/2$ for $t \leq t_1$.

Integrating (1a), we get

$$\frac{v}{(1-v^2)^{1/2}} = \frac{-u}{(4-u^2)^{1/2}} - \int_t^{t_1} \frac{b}{\tau^2(\eta)} \frac{1+\tilde{v}(\eta-\tau(\eta))}{1-\tilde{v}(\eta-\tau(\eta))} d\eta \quad \text{for } t \leq t_1 .$$

From the lemma, we find

$$\tau' = \frac{v - \tilde{v}(t-\tau)}{1-\tilde{v}(t-\tau)} \quad \text{for } t \leq 0 .$$

Therefore, from the facts that $\tau > 0$, $\tilde{v} \geq 0$ for $t \leq 0$ and $v \leq -u/2$ for $t \leq t_1$, we have

$$\tau' = \frac{v - \tilde{v}(t-\tau)}{1-\tilde{v}(t-\tau)} \leq \frac{-u}{2[1-\tilde{v}(t-\tau)]} < 0 \quad \text{for } t \leq t_1 .$$

So

$$-\frac{2\tau'}{u} \geq \frac{1}{1-\tilde{v}(t-\tau)} > 0 \quad \text{for } t \leq t_1 .$$

So

$$\frac{v}{(1-v^2)^{1/2}} \geq \frac{-u}{(4-u^2)^{1/2}} + \int_t^{t_1} \frac{2b}{\tau^2(\eta)} \frac{1+\tilde{v}(\eta-\tau(\eta))}{u} \tau'(\eta) d\eta$$

$$\geq \frac{-u}{(4-u^2)^{1/2}} - \frac{4b}{u\tau(t_1)} \geq -\left[\frac{u}{(4-u^2)^{1/2}} + \frac{4\max(b,\tilde{b})c}{u(x_0-\tilde{x}_0)}\right] = \frac{-u}{(1-u^2)^{1/2}} \quad \text{for } t \leq t_1 .$$

Hence, we have proved that

$$-u \leq v \leq 0 \quad \text{for } t \leq 0 .$$

Similar proofs can be used to show that

$$0 \leq \tilde{v} \leq u .$$

From the lemma we know that

$$\frac{x-\tilde{x}}{c} \leq \tau \leq \frac{x-\tilde{x}}{c-cu} .$$

Since

$$\frac{x-\tilde{x}}{c} = \frac{x_0-\tilde{x}_0}{c} - \int_t^0 [v(\eta)-\tilde{v}(\eta)]\, d\eta ,$$

and together with (1a), it is clear that

$$0 \leq v(t_1) - v(t_2) \leq \frac{2bc^2(t_1-t_2)}{(1-u)(x_0-\tilde{x}_0)^2}$$

for $t_1, t_2 \in (-\infty, 0]$ with $t_1 > t_2$. Similar proofs can be used for $\tilde{v}$. Thus we have shown that

$$(x'/c, \tilde{x}'/c) \in A .$$

3. *Definition of a mapping T from A to A.* For any $(f, \tilde{f}) \in A$, we consider the system of ordinary differential equations

$$\frac{v'}{(1-v^2)^{3/2}} = \frac{b}{\tau^2} \frac{1+\tilde{f}(t-\tau)}{1-\tilde{f}(t-\tau)} , \tag{5a}$$

$$\frac{\tilde{v}'}{(1-\tilde{v}^2)^{3/2}} = \frac{-\tilde{b}}{\tilde{\tau}^2} \frac{1-f(t-\tilde{\tau})}{1+f(t-\tilde{\tau})} , \tag{5b}$$

$$\tau' = \frac{v - \tilde{v} + \tilde{f} - \tilde{f}(t-\tau)}{1-\tilde{f}(t-\tau)} , \tag{6a}$$

$$\tilde{\tau}' = \frac{v - \tilde{v} - f + f(t-\tilde{\tau})}{1+f(t-\tilde{\tau})} \tag{6b}$$

for $t \leq 0$, where $f, \tilde{f}$ denote $f(t)$, $\tilde{f}(t)$. And for $t=0$

$$v(0)=v_0, \quad \tilde{v}(0)=\tilde{v}_0 , \quad \tau(0)=\tau_0, \quad \tilde{\tau}(0)=\tilde{\tau}_0 , \tag{7}$$

where τ_0, $\tilde{\tau}_0$ are the unique solutions of the following two equations respectively:

$$y = \frac{x_0-\tilde{x}_0}{c} + \int_{-y}^0 \tilde{f}(\eta) d\eta ,$$

$$y = \frac{x_0-\tilde{x}_0}{c} - \int_{-y}^0 f(\eta) d\eta .$$

This is a system of ordinary differential equations with data given at $t=0$ and a Lipschitz continuous right-hand side. So there is a unique solution $v, \tilde{v}, \tau, \tilde{\tau}$ to this system. And we define an operator T on A by

$$T(f, \tilde{f}) = (v, \tilde{v}) .$$

First we shall prove that $T(f,\tilde{f}) \in A$. We begin by showing that $\tau > 0$ and $\tilde{\tau} > 0$ for $t \leq 0$. From (6a), we find

$$\tau' = v - \tilde{v} + \tilde{f} - \tilde{f}(t-\tau)(1-\tau') ,$$

where τ' stands for $\tau'(t)$. Integrating both sides, we get

$$\tau(0)-\tau=\int_t^0[v(\eta)-\tilde v(\eta)]d\eta+\int_t^0\tilde f(\eta)d\eta-\int_{t-\tau}^{-\tau(0)}\tilde f(\eta)d\eta$$
$$=\int_t^0[v(\eta)-\tilde v(\eta)]d\eta-\int_{t-\tau}^{t}\tilde f(\eta)d\eta+\int_{-\tau(0)}^{0}\tilde f(\eta)d\eta.$$

Using the value of $\tau(0)$, we find

$$\tau-\int_{t-\tau}^{t}\tilde f(\eta)d\eta=\frac{x_0-\tilde x_0}{c}-\int_t^0[v(\eta)-\tilde v(\eta)]\,d\eta. \quad (8a)$$

Note that the right-hand side is positive and the left-hand side is an increasing function of τ and equals zero at $\tau=0$. So

$\tau>0$ for $t\leqslant 0$

is uniquely determined, and

$$\frac{x_0-\tilde x_0}{c}-\int_t^0[v(\eta)-\tilde v(\eta)]\,d\eta\leqslant\tau\leqslant\frac{1}{1-u}\left\{\frac{x_0-\tilde x_0}{c}-\int_t^0[v(\eta)-\tilde v(\eta)]d\eta\right\}. \quad (9a)$$

Similarly from (6b) and the value of $\tilde\tau(0)$, we find

$$\tilde\tau+\int_{t-\tilde\tau}^{t}f(\eta)d\eta=\frac{x_0-\tilde x_0}{c}-\int_t^0[v(\eta)-\tilde v(\eta)]d\eta. \quad (8b)$$

So

$\tilde\tau>0$ for $t\leqslant 0$,

and

$$\frac{x_0-\tilde x_0}{c}-\int_t^0[v(\eta)-\tilde v(\eta)]\,d\eta\leqslant\tilde\tau\leqslant\frac{1}{1-u}\left\{\frac{x_0-\tilde x_0}{c}-\int_t^0[v(\eta)-\tilde v(\eta)]\,d\eta\right\}. \quad (9b)$$

Next, from the definition of the set A, we find that for every $(f,\tilde f)\in A$, f is an increasing function and $\tilde f$ is a decreasing function. Then the proof that $T(f,\tilde f)$ belongs to A is quite similar to the proof that $(x'/c,\tilde x'/c)$ belongs to A for any solution $x,\tilde x$ of problem P.

$T(A)$ is thus a subset of A. Furthermore, $T(A)$ is closed. To show that $T(A)$ is closed, one considers an arbitrary sequence $T(f_n,\tilde f_n)$ in $T(A)$ which converges to $(g,\tilde g)\in A$ and shows that $(g,\tilde g)=T(f,\tilde f)\in T(A)$, where f_n, $\tilde f_n$ converge to $f,\tilde f$, respectively.

We have proved that for any solution $x,\tilde x$ of problem P associated with the $x_0,\tilde x_0,v_0,\tilde v_0$ given in the theorem, $(x'/c,\tilde x'/c)$ belongs to A. It is then clear that $(x'/c,\tilde x'/c)$ is a fixed point of T and therefore belongs to $T(A)$. On the other hand, for any fixed point $(f,\tilde f)$ of T on $T(A)$ let $x'=cf$, $\tilde x'=c\tilde f$ with $x(0)=x_0$, $\tilde x(0)=\tilde x_0$. Using the lemma, it is then easy to show that $x,\tilde x$ is a solution of problem P associated with $x_0,\tilde x_0,v_0,\tilde v_0$. Therefore, we can say that finding a solution to problem P is equivalent to finding a fixed point of T in $T(A)$.

4. *Some estimates to be used in proving that T is a contraction mapping.* The estimates in this section are obtained following the method used by Zhdanov.[11]

For any solution $v,\tilde v,\tau,\tilde\tau$ of system (5a)–(7), let

$$x=x_0-\int_t^0 cv(\eta)d\eta$$

and

$$\tilde x=\tilde x_0-\int_t^0 c\tilde v(\eta)d\eta.$$

We shall consider the restriction of T on $T(A)$.

For any two $(f_1,\tilde f_1)$ and $(f_2,\tilde f_2)$ belonging to $T(A)$, let $v_1,\tilde v_1,\tau_1,\tilde\tau_1$ and $v_2,\tilde v_2,\tau_2,\tilde\tau_2$ be the corresponding unique solution of the system of (5a)–(7). Then we have

$$T(f_1,\tilde f_1)=(v_1,\tilde v_1),$$
$$T(f_2,\tilde f_2)=(v_2,\tilde v_2).$$

From (8a), we find

$$|\tau_1-\tau_2|=\left|\int_t^0[v_1(\eta)-\tilde v_1(\eta)]\,d\eta-\int_t^0[v_2(\eta)-\tilde v_2(\eta)]\,d\eta-\int_{t-\tau_1}^{t}\tilde f_1(\eta)d\eta+\int_{t-\tau_2}^{t}\tilde f_2(\eta)d\eta\right|,$$

$$|\tau_1-\tau_2|\leqslant\left[\sup_{t\leqslant\eta\leqslant0}|v_1(\eta)-v_2(\eta)|+\sup_{t\leqslant\eta\leqslant0}|\tilde v_1(\eta)-\tilde v_2(\eta)|\right](-t)+\tau_1\sup_{\eta\leqslant0}|\tilde f_1(\eta)-\tilde f_2(\eta)|+|\tau_1-\tau_2|u.$$

So

$$|\tau_1-\tau_2|\leqslant\frac{1}{1-u}\left\{\left[\sup_{t\leqslant\eta\leqslant0}|v_1(\eta)-v_2(\eta)|+\sup_{t\leqslant\eta\leqslant0}|\tilde v_1(\eta)-\tilde v_2(\eta)|\right](-t)+\tau_1\sup_{\eta\leqslant0}|\tilde f_1(\eta)-\tilde f_2(\eta)|\right\}. \quad (10a)$$

Similarly we can prove

$$|\bar{\tau}_1-\bar{\tau}_2|\leq\frac{1}{1-u}\left\{\left[\sup_{t\leq\eta\leq 0}|v_1(\eta)-v_2(\eta)|+\sup_{t\leq\eta\leq 0}|\tilde{v}_1(\eta)-\tilde{v}_2(\eta)|\right](-t)+\bar{\tau}_2\sup_{\eta\leq 0}|f_1(\eta)-f_2(\eta)|\right\}. \tag{10b}$$

Second, from (5a) and (5b) we find

$$\int_t^0\frac{b\,d\eta}{\tau^2(\eta)}\leq\frac{u}{(1-u^2)^{1/2}} \tag{11a}$$

and

$$\int_t^0\frac{\tilde{b}\,d\eta}{\bar{\tau}^2(\eta)}\leq\frac{u}{(1-u^2)^{1/2}}. \tag{11b}$$

Third, also from (5a) and (9a) we have

$$\frac{v'}{(1-v^2)^{3/2}}\geq\frac{b}{\tau^2}\geq\frac{bc^2(1-u)^2}{(x-\tilde{x})^2}.$$

Similarly, from (5b) and (9b), we have

$$\frac{\tilde{v}'}{(1-\tilde{v}^2)^{3/2}}\leq\frac{-\tilde{b}c^2(1-u)^2}{(x-\tilde{x})^2}.$$

Let $k=\min(b,\tilde{b})c(1-u)^2(1-u^2)^{1/2}$. We find

$$\frac{d}{dt}\left[\frac{1}{(1-v^2)^{1/2}}+\frac{1}{(1-\tilde{v}^2)^{1/2}}+\frac{k}{(x-\tilde{x})(1-u^2)^{1/2}}\right]$$
$$=v\left[\frac{v'}{(1-v^2)^{3/2}}-\frac{ck}{(x-\tilde{x})^2(1-u^2)^{1/2}}\right]+\tilde{v}\left[\frac{\tilde{v}'}{(1-\tilde{v}^2)^{3/2}}+\frac{ck}{(x-\tilde{x})^2(1-u^2)^{1/2}}\right]\leq 0\quad\text{for } t\leq 0.$$

So we get

$$\frac{1}{(1-v^2)^{1/2}}+\frac{1}{(1-\tilde{v}^2)^{1/2}}+\frac{k}{(x-\tilde{x})(1-u^2)^{1/2}}\geq\frac{1}{(1-v_0^{\,2})^{1/2}}+\frac{1}{(1-\tilde{v}_0^{\,2})^{1/2}}+\frac{k}{(x_0-\tilde{x}_0)(1-u^2)^{1/2}}.$$

Subtracting 2 from both sides, we get

$$\frac{v^2}{(1-v^2)^{1/2}[1+(1-v^2)^{1/2}]}+\frac{\tilde{v}^2}{(1-\tilde{v}^2)^{1/2}[1+(1-\tilde{v}^2)^{1/2}]}+\frac{k}{(x-\tilde{x})(1-u^2)^{1/2}}$$
$$\geq\frac{1-(1-v_0^{\,2})^{1/2}}{(1-v_0^{\,2})^{1/2}}+\frac{1-(1-\tilde{v}_0^{\,2})^{1/2}}{(1-\tilde{v}_0^{\,2})^{1/2}}+\frac{k}{(x_0-\tilde{x}_0)(1-u^2)^{1/2}}\equiv\frac{E^2}{(1-u^2)^{1/2}},\quad\text{where } E>0.$$

So

$$\frac{v^2}{(1-u^2)^{1/2}}+\frac{\tilde{v}^2}{(1-u^2)^{1/2}}\geq\frac{E^2}{(1-u^2)^{1/2}}-\frac{k}{(x-\tilde{x})(1-u^2)^{1/2}}.$$

Since $v\leq 0$ and $\tilde{v}\geq 0$,

$$v^2-2\tilde{v}v+\tilde{v}^2\geq E^2-k/(x-\tilde{x})\geq 0.$$

Therefore

$$v-\tilde{v}\leq-[E^2-k/(x-\tilde{x})]^{1/2}.$$

Let

$$w=[E^2-k/(x-\tilde{x})]^{1/2}.$$

Then

$$\frac{dw}{dt}=\frac{k(x'-\tilde{x}')}{2[E^2-k/(x-\tilde{x})]^{1/2}(x-\tilde{x})^2}.$$

So

$$\frac{2kw'}{(E^2-w^2)^2c}\leq-1.$$

Integrating both sides with respect to t, we find

$$\int_w^{w_0}\frac{2k\,d\eta}{(E^2-\eta^2)^2c}\leq t,$$

where $w_0=[E^2-k/(x_0-\tilde{x}_0)]^{1/2}$. Integrating by the method of partial fractions, we obtain

$$\int_w^{w_0}\frac{2k\,d\eta}{(E^2-\eta^2)^2c}=\frac{1}{cE^2}\left[(x_0-\tilde{x}_0)\left(E^2-\frac{k}{x_0-\tilde{x}_0}\right)^{1/2}-(x-\tilde{x})\left(E^2-\frac{k}{x-\tilde{x}}\right)^{1/2}\right]$$
$$+\frac{k}{cE^3}\ln\frac{(x_0-\tilde{x}_0)^{1/2}\{E+[E^2-k/(x_0-\tilde{x}_0)]^{1/2}\}}{(x-\tilde{x})^{1/2}\{E+[E^2-k/(x-\tilde{x})]^{1/2}\}}.$$

So

$$\frac{-t}{x-\tilde{x}} \leq \frac{1}{cE} - \frac{k}{cE^3(x-\tilde{x})}\left(\tfrac{1}{2}\ln\frac{x_0-\tilde{x}_0}{x-\tilde{x}} + \ln\tfrac{1}{2}\right) \leq \frac{1}{cE} + \frac{k}{cE^3}\left[\tfrac{1}{2}\frac{\ln[(x-\tilde{x})/(x_0-\tilde{x}_0)]}{(x_0-\tilde{x}_0)(x-\tilde{x})/(x_0-\tilde{x}_0)} + \frac{\ln 2}{x_0-\tilde{x}_0}\right]$$

$$\leq \frac{1}{cE} + \frac{k}{cE^3}\left[\frac{\ln 2}{x_0-\tilde{x}_0} + \frac{1}{2e(x_0-\tilde{x}_0)}\right] = M. \qquad (12)$$

[Note: $M = O(u^{-1})$ as $u \to 0$, so uM is bounded as $u \to 0$.]

5. *Proof that T is a contraction mapping on T(A)*. From (5a), we find

$$|v_1 - v_2| \leq \left|\frac{v_1}{(1-v_1^2)^{1/2}} - \frac{v_2}{(1-v_2^2)^{1/2}}\right| \leq \int_t^0 \left|\frac{b[1+\tilde{f}_1(\eta-\tau_1(\eta))]}{\tau_1^2(\eta)[1-\tilde{f}_1(\eta-\tau_1(\eta))]} - \frac{b[1+\tilde{f}_2(\eta-\tau_2(\eta))]}{\tau_2^2(\eta)[1-\tilde{f}_2(\eta-\tau_2(\eta))]}\right| d\eta.$$

Using the triangle inequality and the inequality

$$\left|\frac{d}{dy}\left(\frac{1+y}{1-y}\right)\right| \leq \frac{2}{(1-u)^2} \quad \text{for } |y| \leq u < 1,$$

we find

$$|v_1 - v_2| \leq \int_t^0 \left\{\frac{2b}{(1-u)^2\tau_1^2(\eta)}\left[|\tilde{f}_1(\eta-\tau_1(\eta)) - \tilde{f}_2(\eta-\tau_1(\eta))| + |\tilde{f}_2(\eta-\tau_1(\eta)) - \tilde{f}_2(\eta-\tau_2(\eta))|\right]\right.$$
$$\left. + \frac{b(1+u)}{1-u}\left[\frac{1}{\tau_1^2(\eta)\tau_2(\eta)} + \frac{1}{\tau_1(\eta)\tau_2^2(\eta)}\right]|\tau_1(\eta) - \tau_2(\eta)|\right\} d\eta.$$

Now, using the fact that $(f_2, \tilde{f}_2) \in T(A)$, we get

$$|v_1 - v_2| \leq \int_t^0 \frac{2b\, d[(f_1,\tilde{f}_1),(f_2,\tilde{f}_2)]}{(1-u)^2\tau_1^2(\eta)}\, d\eta$$
$$+ \frac{2b}{1-u}\int_t^0 \left(\left\{\frac{2\tilde{b}c^2}{(1-u)^2\tau_1^2(\eta)[x_0-\tilde{x}_0 - c\int_\eta^0(f_2-f_2)]^2}\right.\right.$$
$$\left.\left. + \frac{1}{\tau_1^2(\eta)\tau_2(\eta)} + \frac{1}{\tau_1(\eta)\tau_2^2(\eta)}\right\}|\tau_1(\eta) - \tau_2(\eta)|\right) d\eta.$$

From (10a), we find

$$|v_1 - v_2| \leq \int_t^0 \frac{2b\, d[(f_1,\tilde{f}_1),(f_2,\tilde{f}_2)]}{(1-u)^2\tau_1^2(\eta)}\, d\eta$$
$$+ \frac{2b}{(1-u)^2}\int_t^0 \left\{\frac{2\tilde{b}c^2}{(1-u)^2}\,\frac{1}{\tau_1^2(\eta)}\,\frac{1}{[x_0-\tilde{x}_0 - c\int_\eta^0(f_2-\tilde{f}_2)]^2} + \frac{1}{\tau_1^2(\eta)\tau_2(\eta)} + \frac{1}{\tau_1(\eta)\tau_2^2(\eta)}\right\}$$
$$\times\left\{\left[\sup_{\eta\leq s\leq 0}|v_1(s) - v_2(s)| + \sup_{\eta\leq s\leq 0}|\tilde{v}_1(s) - \tilde{v}_2(s)|\right](-\eta)\right.$$
$$\left. + d[(f_1,\tilde{f}_1),\ (f_2,\tilde{f}_2)]\min(\tau_1(\eta),\tau_2(\eta))\right\} d\eta.$$

Then, from (11a), we get

$$|v_1 - v_2| \leq \frac{2\,d[(f_1,\tilde{f}_1),(f_2,\tilde{f}_2)]}{(1-u)^2}\left[\frac{3u}{(1-u^2)^{1/2}} + \frac{2cu\tilde{b}}{(1-u^2)^{1/2}(x_0-\tilde{x}_0)(1-u)^4}\right]$$
$$+ \frac{2b}{(1-u)^2}\int_t^0 \left(\left\{\frac{2\tilde{b}c^2}{(1-u)^2}\,\frac{1}{\tau_1(\eta)}\,\frac{1}{[x_0-\tilde{x}_0 - c\int_\eta^0(f_2-f_2)]^2} + \frac{1}{\tau_1^2(\eta)} + \frac{1}{\tau_2^2(\eta)}\right\}\right.$$
$$\left.\times\left[\sup_{\eta\leq s\leq 0}|v_1(s) - v_2(s)| + \sup_{\eta\leq s\leq 0}|\tilde{v}_1(s) - \tilde{v}_2(s)|\right]\max\left(\frac{-\eta c}{x_1(\eta)-\tilde{x}_1(\eta)}, \frac{-\eta c}{x_2(\eta)-\tilde{x}_2(\eta)}\right)\right) d\eta.$$

Now, applying (12), we get

$$|v_1 - v_2| \leq \frac{2u}{(1-u)^2(1-u^2)^{1/2}}\left[3 + \frac{2c\tilde{b}}{(1-u)^4(x_0-\tilde{x}_0)}\right] d[(f_1, \tilde{f}_1), (f_2, \tilde{f}_2)]$$
$$+ \frac{2}{(1-u)^2}\int_t^0 G(\eta)cM\left[\sup_{\eta\leq s\leq 0}|v_1(s) - v_2(s)| + \sup_{\eta\leq s\leq 0}|\tilde{v}_1(s) - \tilde{v}_2(s)|\right] d\eta,$$

where

$$G(\eta) = \frac{2\tilde{b}c^2}{(1-u)^2}\frac{1}{\tau_1(\eta)}\frac{b}{[x_0 - \tilde{x}_0 - c\int_\eta^0 (f_2 - \tilde{f}_2)]^2} + \frac{b}{\tau_1^{\,2}(\eta)} + \frac{b}{\tau_2^{\,2}(\eta)}.$$

Similarly, we find

$$|\tilde{v}_1 - \tilde{v}_2| \leq \frac{2u}{(1-u)^2(1-u^2)^{1/2}}\left[3 + \frac{2cb}{(1-u)^4(x_0-\tilde{x}_0)}\right] d[(f_1, \tilde{f}_1), (f_2, \tilde{f}_2)]$$
$$+ \frac{2}{(1-u)^2}\int_t^0 \tilde{G}(\eta)cM\left[\sup_{\eta\leq s\leq 0}|v_1(s) - v_2(s)| + \sup_{\eta\leq s\leq 0}|\tilde{v}_1(s) - \tilde{v}_2(s)|\right] d\eta,$$

where

$$\tilde{G}(\eta) = \frac{2bc^2}{(1-u)^2}\frac{1}{\tilde{\tau}_1(\eta)}\frac{\tilde{b}}{[x_0 - \tilde{x}_0 - c\int_\eta^0 (f_2 - \tilde{f}_2)]^2} + \frac{\tilde{b}}{\tilde{\tau}_1^{\,2}(\eta)} + \frac{\tilde{b}}{\tilde{\tau}_2^{\,2}(\eta)}.$$

So

$$\sup_{t\leq s\leq 0}|v_1(s) - v_2(s)| + \sup_{t\leq s\leq 0}|\tilde{v}_1(s) - \tilde{v}_2(s)|$$
$$\leq \frac{4u}{(1-u)^2(1-u^2)^{1/2}}\left[3 + \frac{2c\max(b,\tilde{b})}{(1-u)^4(x_0-\tilde{x}_0)}\right] d[(f_1, \tilde{f}_1), (f_2, \tilde{f}_2)]$$
$$+ \frac{2}{(1-u)^2}\int_{-\infty}^0 [G(\eta) + \tilde{G}(\eta)]cM\left[\sup_{\eta\leq s\leq 0}|v_1(s) - v_2(s)| + \sup_{\eta\leq s\leq 0}|\tilde{v}_1(s) - \tilde{v}_2(s)|\right] d\eta.$$

So by Gronwall's lemma, we find

$$d[(v_1, \tilde{v}_1), (v_2, \tilde{v}_2)] \leq \frac{4u}{(1-u)^2(1-u^2)^{1/2}}\left[3 + \frac{2c\max(b,\tilde{b})}{(1-u)^4(x_0-\tilde{x}_0)}\right]\exp\left\{\frac{2}{(1-u)^2}\int_{-\infty}^0 [G(\eta) + \tilde{G}(\eta)]cM\,d\eta\right\}$$
$$\times d[(f_1, \tilde{f}_1), (f_2, \tilde{f}_2)].$$

Since $\int_{-\infty}^0 [G(\eta) + \tilde{G}(\eta)]cM\,d\eta$ is bounded there exists a number $a > 0$ such that whenever $\max(b, \tilde{b})c/(x_0 - \tilde{x}_0) \leq a$, u will be small enough to make the coefficient of $d[(f_1, \tilde{f}_1), (f_2, \tilde{f}_2)]$ in the above inequality less than 1. Therefore T is a contraction mapping on a complete metric space $T(A)$. So T has a unique fixed point in $T(A)$. Equivalently we proved that problem P has a unique solution. Q. E. D.

Remarks. (a) We conjecture that the assertion of the theorem would be true for the more general case where $-1 < v_0 \leq \tilde{v}_0 < 1$ (instead of $-1 < v_0 \leq 0 \leq \tilde{v}_0 < 1$). One might be tempted to follow the method used by Travis—choosing a new reference frame which travels with the speed, say $\tilde{v}_0$, relative to the original inertial reference frame, in order to reduce the problem to the case treated in the theorem. However, simultaneous events in one inertial system are not necessarily simultaneous in another. Thus in the new reference frame we would have the particle on the right with a non-positive velocity given at one instant and the particle on the left with a non-negative velocity given at another instant. Then one would have to solve a harder problem of finding a unique solution of system (5a)–(6b) for which the values of functions v, τ are given at one instant and $\tilde{v}$, $\tilde{\tau}$ are given at another instant.

This boundary-value problem remains to be studied.

(b) We found the value of a very small. One conjecture is that some estimates better than those given in part 4 of the proof might give a larger a.

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[1]C. J. Eliezer, Proc. Cambridge Philos. Soc. 39, 173 (1943).
[2]D. K. Hsing and R. D. Driver, Univ. of Rhode Island, Dept. of Math. Technical Report No. 61, 1975 (unpublished).
[3]R. D. Driver, Ann. Phys. (N.Y.) 21, 122 (1963).
[4]R. D. Driver and M. J. Norris, Ann. Phys. (N.Y.) 42, 347 (1967).
[5]H. van Dam and E. P. Wigner, Phys. Rev. 138, 1576 (1965).
[6]H. van Dam and E. P. Wigner, Phys. Rev. 142, 838 (1966).
[7]R. D. Driver, Phys. Rev. 178, 2051 (1969).
[8]S. Doss and S. K. Nasr, Am. J. Math. 75, 713 (1953).
[9]A. B. Nersesjan, Akad. Nauk Armjan. SSR Dokl. 36, 193 (1963).
[10]S. P. Travis, Phys. Rev. D 11, 292 (1975).
[11]V. I. Zhdanov, Internat. J. Theor. Phys., 15, 157 (1976).
[12]V. I. Zhdanov, in proceedings of the Conference on the Theory and Applications of Differential Equations with a Deviating Argument, Kiev, USSR, 1975, Abstracts of Reports, pp. 91, 92 (unpublished).