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Rotating nonspherical masses and their effects on the precession of a gyroscope

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The formula due to Lense and Thirring for the dragging of locally inertial frames is generalized, in the limit of weak stationary fields, to any arbitrary axisymmetric spinning body. We determine the multipole structure of the off-diagonal terms in the metric resulting from the mass current of the body. For a rotating mass divided into nearly spherical strata of equal density, the relativistic multipole moments are related to the deviation from spherical symmetry in a quite simple manner. We apply our results to an oblate spheroid stratified into slightly flattened ellipsoids. Thus a realistic geophysical model enables us to evaluate the contribution of the earth's nonsphericity to the Lense-Thirring precession of a gyroscope. The magnitude of this contribution does not exceed the experimental error for observations covering a span of about one year.

I. INTRODUCTION

The gyroscope experiment, proposed by Schiff[1] and presently conducted at Stanford University by Everitt, Fairbank, and their co-workers,[2] is a test of the general theory of relativity sensitive to the off-diagonal potentials resulting from the diurnal rotation of the earth, the so-called Lense-Thirring[3] terms in the metric tensor. In all the calculations of these terms, the earth is assumed to be spherically symmetric. However, this assumption is not realistic and it has been already shown by O'Connell,[4] Barker and O'Connell,[5] and Wilkins[6] that the quadrupole moment of the earth gives a measurable contribution to the precession of the spin of a gryoscope when the earth's rotation is neglected. So our purpose in this work is to determine the additional contribution due to the deviation of the central body from spherical symmetry when the rotation is taken into account.

We assume the central body to be axisymmetric and to spin around its axis with a small angular velocity ω. In Sec. II, the linearized Einstein equations are solved in the first-order approximation with respect to ω. It is shown that the off-diagonal potentials arising from the rotation involve in fact a single function H to be determined. Then we get, for the dragging of the inertial frames by an axisymmetric rotating mass, a formula generalizing the Lense-Thirring formula.

In Sec. III, we perform the multipole expansion of H and we write the relativistic multipole coefficients K_n involved in this development in a convenient form for further comparison with the Newtonian 2^n-pole moments. Then we derive the corresponding expansion of the angular velocity of inertial axes. The computational details are given in the appendixes.[7] We may note that Krause[8] and Martin[9] used similar expansions, although they did not generalize the Lense-Thirring formula of precession.

In an earlier work,[10] we determined the terms K_n for a spinning heterogeneous sphere by making use of the orthogonality properties of the spherical harmonics. Therefore, only nonspherical configurations are considered here.

Section IV is concerned with rotating bodies made of nearly spherical layers of uniform density. The expressions of the multipole moments are quite simplified when one neglects terms of higher order than the first with respect to the deviation of each layer from spherical symmetry. A detailed treatment is given for a spheroid stratified into slightly flattened ellipsoids of revolution. The coefficient K_2 is shown to be then the only significant relativistic multipole moment. Moreover, our theory yields an expression of K_2 close to that of the Newtonian quadrupole moment J_2.

These latter results are applied to the terrestrial spheroid in Sec. V. The coefficient K_2 is estimated from a classical model of the earth. We can thus discuss the influence of the internal layering of the earth on the Lense-Thirring precession of an orbiting gyroscope.

II. STRUCTURE OF THE GRAVITATIONAL FIELD

Let us consider an isolated axisymmetric body, slowly spinning with a uniform angular velocity $\vec{\omega}$. The gravitational field is assumed to be weak and stationary, so that the metric components may be written in some quasi-Galilean coordinate system $x^0 = ct$, $x^i = (x, y, z)$:

$$g_{\mu\nu}(x^i) = \eta_{\mu\nu} + h_{\mu\nu}(x^i) ,\quad \{\mu,\nu\} = \{0,1,2,3\},\ i = \{1,2,3\},$$

where $\eta_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$ and $h_{\mu\nu}$ are small quantities. Moreover, let us impose the gauge conditions

$$\eta^{ij}\partial_i h_{jk} - \tfrac{1}{2}\partial_k h = 0 , \quad h = \eta^{\rho\sigma} h_{\rho\sigma} . \tag{1}$$

Then, the linearized Einstein equations are

$$\nabla^2 h_{\mu\nu} = \begin{cases} 0 \text{ outside the matter} \\ 2\kappa(T_{\mu\nu} - \tfrac{1}{2} T\eta_{\mu\nu}) \text{ inside the matter ,} \end{cases} \tag{2}$$

where $T_{\mu\nu}$ is the energy-momentum tensor, $T = \eta^{\rho\sigma}T_{\rho\sigma}$, and κ is the Einstein constant $\kappa = 8\pi G/c^2 = 1.87 \times 10^{-26}$ m kg^{-1} (c is the speed of light and G the Newtonian constant of gravitation).

The angular velocity $\vec{\omega}$ is directed along the z axis. If one neglects the pressure and terms of second order in $(\omega r_0/c)^2$,[11] where r_0 is a characteristic size of the body, the nonvanishing components of the energy-momentum tensor are, for a volume distribution of proper density ρ,

$$T_{00} = \rho , \quad T_{01} = \rho(\omega y/c) , \quad T_{02} = -\rho(\omega x/c) . \tag{3}$$

The matter is supposed to be distributed in a finite number of domains $\mathfrak{D}_i$ limited by closed Liapunov surfaces[12] S_i. In each domain $\mathfrak{D}_i$, the density ρ is assumed to admit a gradient satisfying a uniform Hölder condition. However, ρ or its first derivatives may be discontinuous across the surfaces S_i. Under these assumptions, there is one and only one system of solutions of the field equations (2) such that the gravitational potentials satisfy the Lichnerowicz axioms, i.e.:

(a) They are continuously differentiable everywhere.

(b) They have piecewise continuously differentiable partial derivatives of second order.

(c) They are regular at infinity [$|rh_{\mu\nu}|$ and $|r^2\partial_i h_{\mu\nu}|$ are bounded when $r = (x^2+y^2+z^2)^{1/2}$ becomes larger and larger].

Replacing $T_{\mu\nu}$ by (3), we have for the nonvanishing $h_{\mu\nu}$

$$h_{00} = h_{ii} = -\frac{\kappa}{4\pi} U , \quad U = \int_{\mathfrak{D}} \frac{\rho(\vec{r}')}{R} d\tau , \tag{4}$$

$$h_{01} = -\frac{\kappa\omega}{2\pi c} \int_{\mathfrak{D}} \frac{\rho(\vec{r}')y'}{R} d\tau , \quad h_{02} = \frac{\kappa\omega}{2\pi c} \int_{\mathfrak{D}} \frac{\rho(\vec{r}')x'}{R} d\tau , \tag{5}$$

where $\mathfrak{D}$ is the union of the domains $\mathfrak{D}_i$, $d\tau$ is the volume element of the Euclidean 3-space at the point (x',y',z'), and $R = |\vec{r} - \vec{r}'|$, $\vec{r}$ and $\vec{r}'$ denoting, respectively, the position vectors of (x,y,z) and (x',y',z').

Here, ρ is a time-independent axisymmetric function. Hence the relations $\sum_\alpha \partial_\alpha T_{\alpha\beta} = 0$ hold for the tensor (3), or equivalently the conservation law of the mass current

$$\vec{\nabla}\cdot(\rho\vec{v}) = 0 , \quad \vec{v} = \vec{\omega} \times \vec{r} . \tag{6}$$

As a consequence, the gauge conditions (1) are verified by the functions $h_{\mu\nu}$.[13] Now, put

$$\vec{h} = h_{01}\frac{\partial}{\partial x} + h_{02}\frac{\partial}{\partial y} . \tag{7}$$

From a mathematical point of view, this 3-vector may be identified with the magnetic vector potential created by a steady axisymmetric distribution of electric currents $\vec{j} = (\kappa/2\pi c)\rho\vec{v}$. But it is well known that the orbits of such a vector field are circles around the z axis. Hence, if we define ξ and φ by

$$x = \xi \cos\varphi , \quad y = \xi \sin\varphi , \quad \xi \geq 0 , \quad 0 \leq \varphi < 2\pi ,$$

we may write

$$\vec{h} = \frac{\kappa\omega}{2\pi c} H \frac{\partial}{\partial\varphi} ,$$

where H is a function which depends only upon ξ and z. So, passing again to the quasi-Galilean coordinates, we get

$$h_{01}(x,y,z) = -\frac{\kappa}{2\pi}\frac{\omega y}{c} H(\xi,z) ,$$
$$h_{02}(x,y,z) = \frac{\kappa}{2\pi}\frac{\omega x}{c} H(\xi,z) . \tag{8}$$

Thus, the off-diagonal potentials due to the rotation of the central body can be constructed with a unique axisymmetric function. In order to obtain an integral expression for H, let us differentiate h_{01} with respect to y and suppose the current point is in the xOz plane. Thus, we get from (8)

$$\frac{\partial}{\partial y} h_{01}(\xi, 0, z) = -\frac{\kappa\omega}{2\pi c} H(\xi,z) .$$

This relation defines $H(\xi,z)$ on the z axis itself. Now, let us replace h_{01} by expression (8). We may differentiate under the summation sign, since h_{01} is a Newtonian potential. So we get

$$H(\xi,z) = \int_{\mathfrak{D}} \frac{\rho y'^2}{R^3}\Big|_{\substack{x=\xi \\ y=0}} d\tau = \int_{\mathfrak{D}} \rho \frac{\xi'^2 \sin^2(\varphi - \varphi')}{R^3} d\tau . \tag{9}$$

On the z axis, this formula becomes

$$H(z) = \tfrac{1}{2} \int_{\mathfrak{D}} \rho \frac{\xi'^2}{[\xi'^2 + (z-z')^2]^{3/2}} d\tau . \tag{10}$$

The off-diagonal potentials h_{01} and h_{02} induce a precession of the spin of a gyroscope, the rate of which is given by[14]

$$\vec{\Omega} = \tfrac{1}{2} c \vec{\nabla} \times \vec{h} . \tag{11}$$

From a mathematical point of view, $\vec{\Omega}$ is the magnetic induction associated with the vector potential

$\frac{1}{2}c\vec{h}$. Hence, in free space,

$$\nabla^2\vec{\Omega}=0\,. \tag{12}$$

The components of $\vec{\Omega}$ are, in the natural triad associated with (ξ,φ,z),

$$\Omega_\xi=-\frac{\kappa\omega}{4\pi}\,\xi\,\frac{\partial H}{\partial z}\,, \tag{13a}$$

$$\Omega_z=\frac{\kappa\omega}{4\pi}\left(2H+\xi\frac{\partial H}{\partial\xi}\right), \tag{13b}$$

$$\Omega_\varphi=0\,. \tag{13c}$$

Let $\vec{u}$ and $\vec{k}$ denote the unit vectors in the $\vec{r}$ and z directions, respectively. We can write

$$\vec{\Omega}=\frac{\kappa\omega}{4\pi}\left[-r\frac{\partial H}{\partial z}\vec{u}+(2H+\vec{r}\cdot\vec{\nabla}H)\vec{k}\right]. \tag{14}$$

On the z axis, this expression reduces to

$$\vec{\Omega}=\frac{\kappa}{2\pi}H(z)\vec{\omega}\,. \tag{15}$$

III. MULTIPOLE EXPANSIONS OF POTENTIALS AND OF $\vec{\Omega}$

Introduce the spherical coordinates (r,θ,φ) relative to O as origin; U and H can be expressed in terms of r and θ only. Let r_e be the radius of the smallest sphere around O containing the whole body (in practical cases, r_e is the equatorial radius). For any $r_1>r_e$, the Newtonian potential is represented in the region $r\geq r_1$ by the absolutely and uniformly convergent expansion[15]

$$U(r,\theta)=\frac{M}{r}\left[1-\sum_{n=1}^{\infty}J_n\left(\frac{r_e}{r}\right)^n P_n(\cos\theta)\right], \tag{16}$$

where M is the total mass, P_n is the Legendre polynomial of degree n, and

$$J_n=-\frac{1}{Mr_e^{\,n}}\int_{\mathfrak{D}}\rho r'^n P_n(\cos\theta')d\tau\,. \tag{17}$$

Similarly, a standard method of potential theory leads to a unique absolutely and uniformly convergent expansion for H in the region $r\geq r_1>r_e$ (see Appendix A):

$$H(r,\theta)=\frac{I}{2r^3}\left[1-\sum_{n=1}^{\infty}K_n\left(\frac{r_e}{r}\right)^n P'_{n+1}(\cos\theta)\right], \tag{18}$$

where I is the moment of inertia of the body about the z axis, and

$$K_n=\frac{2}{2n+3}\,\frac{Mr_e^2}{I}\,(L_n-J_{n+2})\,, \tag{19}$$

with

$$L_n=-\frac{1}{Mr_e^{\,n+2}}\int_{\mathfrak{D}}\rho r'^{n+2}P_n(\cos\theta')d\tau\,. \tag{20}$$

Now, Eqs. (12) and (13) enable us to derive from (18) the following expansions for the components Ω_ξ and Ω_z in the region $r\geq r_1>r_e$ (see Appendix B):

$$\Omega_\xi=\frac{GI\omega}{c^2r^3}\sin\theta\times\left[3\cos\theta-\sum_{n=1}^{\infty}(n+1)K_n\left(\frac{r_e}{r}\right)^n P'_{n+2}(\cos\theta)\right], \tag{21}$$

$$\Omega_z=\frac{GI\omega}{c^2r^3}\left[3\cos^2\theta-1-\sum_{n=1}^{\infty}(n+1)(n+2)K_n\left(\frac{r_e}{r}\right)^n P_{n+2}(\cos\theta)\right]. \tag{22}$$

Using the recurrence formula

$$(n+2)P_{n+2}(\cos\theta)=\cos\theta P'_{n+2}(\cos\theta)-P'_{n+1}(\cos\theta)\,,$$

(21) and (22) may be condensed in a single vectorial expression

$$\vec{\Omega}=\frac{GI\omega}{c^2r^3}\left\{3\cos\theta\,\vec{u}-\vec{k}-\sum_{n=1}^{\infty}(n+1)K_n\left(\frac{r_e}{r}\right)^n[P'_{n+2}(\cos\theta)\vec{u}-P'_{n+1}(\cos\theta)\vec{k}]\right\}. \tag{23}$$

The relativistic coefficients K_n are due to the heterogeneities and/or the nonsphericity of the spinning mass. In fact, when the central body is spherically symmetric, the density is a function $\rho(r)$, so that the L_n vanish for $n\geq1$ as the J_n do themselves, on account of the well-known orthogonality properties of the Legendre polynomials. Hence, the K_n are null in that case, and we get from (18) and (23), respectively,

$$H=\frac{I}{2r^3}\,,\quad \vec{\Omega}=\frac{GI\omega}{c^2r^3}(3\cos\theta\,\vec{u}-\vec{k})\,,\quad r>r_0\,, \tag{24}$$

where r_0 is the radius of the rotating sphere. These formulas correspond to the Lense-Thirring metric.[16]

IV. MULTIPOLE MOMENTS OF NEARLY SPHERICAL BODIES

Suppose now that the rotating mass is divided into infinitely thin strata of equal density ρ. More precisely, assume that the strata constitute a one-parameter family of closed surfaces of revolution about the z axis defined by the equations

$$r=r_mF(u,\mu)\,,\quad F(0,\mu)=0\,,\quad \mu=\cos\theta\,.$$

Here, the range of the parameter u is $0\leq u\leq 1$, $u=0$ corresponds to the origin O and $u=1$ to the external surface of the body; r_m is the mean radius, i.e., the radius of the sphere enclosing the same volume as the whole body.

Then the density is a function $\rho(u)$, assumed to be piecewise continuous on the interval $0\leq u\leq 1$. Moreover, we suppose that $F(u,\mu)$ possesses a continuous first derivative $\partial F/\partial u>0$. So F is a monotonically increasing function of u for any fixed value of μ, which agrees with the assumption of internal layering.

We can modify the integral expressions (17) and (20) for J_n and L_n, respectively, by passing from the variables (r,θ,φ) to (u,μ,φ).[17] Since $\partial F/\partial u>0$, the Jacobian of this transformation is admissible. Then a straightforward calculation gives

$$J_n=-\frac{3}{2(n+3)}\left(\frac{r_m}{r_e}\right)^n\frac{1}{\rho_m}\times\int_0^1\rho(u)\frac{d}{du}\int_{-1}^1F^{n+3}(u,\mu)P_n(\mu)d\mu du\,,\qquad(25)$$

$$L_n=-\frac{3}{2(n+5)}\left(\frac{r_m}{r_e}\right)^{n+2}\frac{1}{\rho_m}\times\int_0^1\rho(u)\frac{d}{du}\int_{-1}^1F^{n+5}(u,\mu)P_n(\mu)d\mu du\,,\qquad(26)$$

where ρ_m denotes the mean density of the spinning body.

The expressions for the mass and the moment of inertia about the z axis become, respectively,

$$M=\frac{2\pi}{3}r_m{}^3\int_0^1\rho(u)\frac{d}{du}\int_{-1}^1F^3(u,\mu)d\mu du\,,$$

$$I=\frac{4\pi}{15}r_m{}^5\int_0^1\rho(u)\frac{d}{du}\int_{-1}^1F^5(u,\mu)\times[P_0(\mu)-P_2(\mu)]d\mu du\,.$$

For a body layered into very nearly spherical concentric surfaces, the function $F(u,\mu)$ may be written as

$$F(u,\mu)=u[1+\epsilon\Phi(u,\mu)],\qquad(27)$$

where $\Phi(u,\mu)$ is a continuously differentiable function and ϵ a small dimensionless quantity. If terms of higher order than the first with respect to ϵ are neglected, the powers of $F(u,\mu)$ may be written as

$$F^k(u,\mu)=u^k[1+k\epsilon\Phi(u,\mu)]\,.$$

Then putting

$$\Phi_n(u)=\tfrac{1}{2}\int_{-1}^1\Phi(u,\mu)P_n(\mu)d\mu\qquad(28)$$

and using the orthogonality relations between the Legendre polynomials, we get from (25) and (26), respectively,

$$J_n=-3\epsilon\left(\frac{r_m}{r_e}\right)^n\frac{1}{\rho_m}\int_0^1\rho(u)\frac{d}{du}[u^{n+3}\Phi_n(u)]du\,,\qquad(29)$$

$$L_n=-3\epsilon\left(\frac{r_m}{r_e}\right)^{n+2}\frac{1}{\rho_m}\int_0^1\rho(u)\frac{d}{du}[u^{n+5}\Phi_n(u)]du\,.\qquad(30)$$

In order to simplify expressions of M and I, let us connect u to the mean radius r_m' of the corresponding layer by the relation $u=r_m'/r_m$. Then $\Phi_0(u)=0$ (Ref. 18) and

$$M=4\pi r_m{}^3\int_0^1\rho(u)u^2du\,,$$

$$I=\frac{8\pi}{3}r_m{}^5\int_0^1\rho(u)\left\{u^4-\epsilon\frac{d}{du}[u^5\Phi_2(u)]\right\}du\,.$$

If such a quasispherical body is homogeneous, put

$$\Phi(\mu)=\Phi(1,\mu)\,,\quad \Phi_n=\Phi_n(1)\,.$$

The integration of (29) and (30) gives

$$J_n=-3\epsilon\left(\frac{r_m}{r_e}\right)^n\Phi_n\,,\quad L_n=\left(\frac{r_m}{r_e}\right)^2J_n\,.\qquad(31)$$

Substituting these expressions into (19), we get K_n in terms of Newtonian 2^n-pole moments[19]

$$K_n=\frac{5}{2n+3}\left[J_n-\left(\frac{r_e}{r_m}\right)^2J_{n+2}\right]\Big/\left[1+\frac{5}{3}\left(\frac{r_e}{r_m}\right)^2J_2\right]\approx\frac{5}{2n+3}(J_n-J_{n+2})\,.\qquad(32)$$

Now let us apply the preceding results to a spheroid stratified into ellipsoidal shells, the flattening of which is a small function $\alpha(u)$. If r_mu is the mean radius of the layer parametrized by u, one may write

$$\epsilon\Phi(u,\mu)=-\tfrac{2}{3}\alpha(u)P_2(\mu)\,.$$

Using the orthogonality relations satisfied by the P_n, we deduce from (29) and (30) that the only nonvanishing J_n and L_n are those where $n=2$:

$$J_2=\frac{2}{5}\left(\frac{r_m}{r_e}\right)^2\frac{1}{\rho_m}\int_0^1\rho(u)\frac{d}{du}[u^5\alpha(u)]du\,,\qquad(33)$$

$$L_2=\frac{2}{5}\left(\frac{r_m}{r_e}\right)^4\frac{1}{\rho_m}\int_0^1\rho(u)\frac{d}{du}[u^7\alpha(u)]du\,.\qquad(34)$$

Therefore, all the relativistic coefficients K_n are null, except K_2:

$$K_2=\frac{2}{7}\frac{Mr_e{}^2}{I}L_2\,.\qquad(35)$$

According to the present views in geophysics and seismology, such a stratified ellipsoid constitutes a good model of the earth. So we are now in a position to evaluate the influence of K_2 on the behavior of a gyroscope in orbit around the earth.

V. APPLICATION TO THE GYROSCOPE EXPERIMENT

It is usually assumed that each layer of equal density within the earth is like an equipotential surface under the combined influence of the Newtonian attraction and the centrifugal acceleration due to the axial rotation. Under this condition of equilibrium, the flattening $\alpha(u)$ is connected with the density distribution by the Clairaut equation.[20] Moreover, it is natural to suppose that the density increases with the depth, so that $d\rho/du \leq 0$. Then, it can be shown from the Clairaut equation that the flattening is a monotonically increasing function of u. Therefore, J_2 and K_2 are positive and have the same order of magnitude. The most recent determination of the quadrupole moment J_2 is[21]

$$J_2 = (1\,082.64 \pm 0.01) \times 10^{-6}\,. \tag{36}$$

On the other hand, an accurate value of the relativistic coefficient K_2 can be obtained when the density and the flattening are known for a sufficiently great number of layers. Let u_i, $0 \leq i \leq n$ be a finite set of real numbers such that $u_0 = 0$, $u_i < u_{i+1}$, $u_n = 1$. From $\rho(u) > 0$, $d\rho/du < 0$, $\alpha(u) > 0$, and $d\alpha/du > 0$, we derive the inequalities

$$\sum_{n=0}^{i=n-1} \rho(u_{i+1} - 0)A_i \leq \int_0^1 \rho(u)\frac{d}{du}[u^7\alpha(u)]du \leq \sum_{i=1}^{i=n} \rho(u_i + 0)A_i\,,$$

where

$$A_i = \int_{u_i}^{u_{i+1}} \frac{d}{du}[u^7\alpha(u)]du = u_{i+1}{}^7\alpha(u_{i+1}) - u_i{}^7\alpha(u_i)\,.$$

To form an estimate of the lower and the upper bounds of the above integral, we use the density law and the flattening computed by Bullen and Bullard from seismological data.[22] Substituting the obtained values into (34) and (35) we find

$$0.861 \times 10^{-3} < K_2 < 0.887 \times 10^{-3}\,.$$

In the following we shall adopt the mean value

$$K_2 = 0.874 \times 10^{-3} \tag{37}$$

to be compared with the value of J_2.

From (23), we get for $\vec{\Omega}$ at a point of latitude $\psi = \pi/2 - \theta$

$$\vec{\Omega} = \frac{GI\omega}{c^2r^3}\left\{3\sin\psi\left[1 - \tfrac{5}{2}K_2\left(\frac{r_e}{r}\right)^2(7\sin^2\psi - 3)\right]\vec{u} - \left[1 - \tfrac{9}{2}K_2\left(\frac{r_e}{r}\right)^2(5\sin^2\psi - 1)\right]\vec{k}\right\}. \tag{38}$$

The orbit selected for the gyroscope experiment is a slightly distorted circular polar orbit. The corresponding equations of motion can be derived from the classical Lagrangian of the earth-gyroscope system[23]:

$$r(t) = a\left[1 + \tfrac{1}{4}J_2\left(\frac{r_e}{a}\right)^2\cos 2nt\right], \tag{39}$$

$$\psi(t) = nt + \tfrac{1}{8}J_2\left(\frac{r_e}{a}\right)^2\sin 2nt\,, \tag{40}$$

where n denotes the mean motion of the satellite and a the mean radius of the orbit. It is clear that the equatorial radius a_e of such a trajectory is

$$a_e = a\left[1 + \tfrac{1}{4}J_2\left(\frac{r_e}{a}\right)^2\right].$$

Inserting now expressions (39) and (40) in (38) and integrating over a period, we get the average of $\vec{\Omega}$:

$$\langle\vec{\Omega}\rangle = \frac{GI\vec{\omega}}{2c^2a^3}\left[1 + \tfrac{3}{8}(4J_2 - 9K_2)\left(\frac{r_e}{a}\right)^2\right], \tag{41}$$

where terms of higher order than the first with respect to J_2 and K_2 are neglected. Thus, we find for the correction of the Lense-Thirring secular term $\langle\vec{\Omega}_{\rm LT}\rangle = GI\vec{\omega}/2c^2a^3$ the following:

$$\Delta\langle\vec{\Omega}\rangle = \tfrac{3}{8}(4J_2 - 9K_2)\left(\frac{r_e}{a}\right)^2\left\langle\vec{\Omega}_{\rm LT}\right\rangle\,. \tag{42}$$

In order to evaluate $\langle\Omega_{\rm LT}\rangle$, let us decompose this quantity as follows:

$$\frac{GI\omega}{2c^2a^3} = \frac{GM}{2c^2}\frac{r_e^2}{a^3}\frac{I-A}{Mr_e^2}\frac{I}{I-A}\,\omega\,, \tag{43}$$

where A denotes the earth's moment of inertia about an equatorial axis. For the speed of light and the earth's angular velocity of rotation, we take, respectively, $c = (299\,792.5 \pm 0.1) \times 10^3$ m sec^{-1} and $\omega = 4.746\,682\,47 \times 10^8$ (sec of arc)/(sidereal year).[24] The values of the gravity factor GM and of the equatorial radius are, respectively, $(398\,600.5 \pm 0.3) \times 10^9$ m^3 sec^{-2} (Ref. 21) and $6\,378\,140 \pm 16$ m.[21] The quantity $(I-A)/Mr_e^2$ is the quadrupole moment J_2 given by (36) and $I/(I-A)$ is the reciprocal of the dynamical ellipticity of the earth. Slightly different values of $(I-A)/I$ are found by various authors. We take $I/(I-A) = 305.5 \pm 0.5$.[25] We thus get

$$\langle\Omega_{\rm LT}\rangle = (43.88 \pm 0.07) \times 10^{-3}\ \text{(sec of arc)/(sidereal year)}$$

for a quasicircular polar orbit 300 miles = 482.8032 km above the earth at the equator. Note that the uncertainty on $\langle\Omega_{LT}\rangle$ is mainly due to the uncertainty on the quantity $I/(I-A)$. For such an orbit, the correction (42) is

$$\Delta\langle\Omega\rangle = -0.05\times10^{-3} \text{ (sec of arc)/year}.$$

A measurement accurate to 0.001 (sec of arc)/year seems feasible by use of the London moment-readout technique. So the contribution of the flattening and of the internal structure of the earth to the Lense-Thirring term is about twenty times smaller than what is hoped can be measured.

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APPENDIX A: EXPANSION OF H

It follows from (2) and (8) that Hx is harmonic function in free space. Therefore, Hx may be developed as

$$Hx = H(r,\theta) r\sin\theta\cos\varphi = \sum_{n=0}^{\infty} \frac{H_n}{r^{n+2}} P^1_{n+1}(\cos\theta)\cos\varphi,$$

where the H_n are constants and $P^1_{n+1}(\cos\theta)$ denotes the associated Legendre function $\sin\theta P'_{n+1}(\cos\theta)$. The above expansion is absolutely and uniformly convergent in the region $r \geqslant r_1$ for any $r_1 > r_e$ (r_e is defined in Sec. III). Now, dividing each member by $r\sin\theta\cos\varphi$, we obtain

$$H(r,\theta) = \sum_{n=0}^{\infty} \frac{H_n}{r^{n+3}} P'_{n+1}(\cos\theta).$$

In order to determine the coefficients H_n, let the current point be on the positive part of the z axis. Then $P'_{n+1}(\cos\theta)$ takes the value $\frac{1}{2}(n+1)(n+2)$, so that

$$H(z) = \tfrac{1}{2}\sum_{n=0}^{\infty} \frac{(n+1)(n+2)H_n}{z^{n+3}}. \tag{44}$$

But (10) may be written as

$$H(z) = \tfrac{1}{2}\int_{\mathfrak{D}} \rho \frac{r'^2 \sin^2\theta'}{(z^2 - 2zr'\cos\theta' + r'^2)^{3/2}}\, d\tau.$$

When $z \geqslant r_1 > r_e$, we have $r' \leqslant r_e < z$ and we may expand $(z^2 - 2zr'\cos\theta' + r'^2)^{-3/2}$ into the uniformly convergent series[26]

$$(z^2 - 2zr'\cos\theta' + r'^2)^{-3/2} = \sum_{n=0}^{\infty} \frac{r'^n}{z^{n+3}} P'_{n+1}(\cos\theta').$$

Substituting this expression in the integral expression of $H(z)$ and permuting the summation symbols, we get

$$H_n = \frac{1}{(n+1)(n+2)} \int_{\mathfrak{D}} \rho r'^{n+2} \sin^2\theta' P'_{n+1}(\cos\theta') d\tau.$$

For $n=0$, we have

$$H_0 = \tfrac{1}{2}\int_{\mathfrak{D}} \rho r'^2 \sin^2\theta'\, d\tau = \frac{I}{2},$$

where I is the moment of inertia about the z axis. Using the recurrence relation[27]

$$(2n+3)\sin^2\theta P'_{n+1}(\cos\theta) = (n+1)(n+2)[P_n(\cos\theta) - P_{n+2}(\cos\theta)],$$

we find

$$H_n = \frac{1}{2n+3}\int_{\mathfrak{D}} \rho r'^{n+2}[P_n(\cos\theta') - P_{n+2}(\cos\theta')] d\tau. \tag{45}$$

Let us put now

$$K_n = -2H_n/I r_e^{\,n}$$

and compare with (17) the expression of K_n obtained after replacing H_n by (45). We easily get (19) and (20).

APPENDIX B: EXPANSION OF $\vec{\Omega}$

Expression (13a), for Ω_ξ may be written as

$$\Omega_\xi = -\frac{\kappa\omega}{4\pi} r\sin\theta \frac{\partial H}{\partial z}.$$

The function Hx is harmonic in a vacuum. So $x(\partial H/\partial z)$ has the same property, since $x(\partial H/\partial z) = (\partial/\partial z)(Hx)$. Moreover this quantity depends upon φ as Hx does itself. Therefore, $\partial H/\partial z$ may be expanded as H in any region $r \geqslant r_1 > r_e$:

$$\frac{\partial H}{\partial z} = \sum_{n=0}^{\infty} \frac{H'_n}{r^{n+3}} P'_{n+1}(\cos\theta).$$

In order to determine the coefficients H'_n, let us write the above expansion on the positive part of the z axis and compare it with the series obtained by differentiating (44) term by term with respect to z. We get $H'_0 = 0$ and $H'_{n+1} = -(n+1)H_n$. Then, we derive from (13a)

$$\Omega_\xi = \frac{\kappa\omega}{4\pi r^3}\sin\theta \sum_{n=0}^{\infty} \frac{(n+1)H_n}{r^n} P'_{n+2}(\cos\theta),$$

which is equivalent to (21).

It follows from (12) that Ω_z is harmonic in a vacuum. Multiplying (44) by $\kappa\omega/2\pi$ we obtain the expression of Ω_z on the z axis as a power series in the reciprocal of z, since this quantity is related to $H(z)$ by (15). Then a classical theorem on the axisymmetric harmonic functions allows us to write immediately

$$\Omega_z = \frac{\kappa\omega}{4\pi r^3}\sum_{n=0}^{\infty} \frac{(n+1)(n+2)H_n}{r^n} P_{n+2}(\cos\theta).$$

This expansion is equivalent to (22).

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